Regularity of Solutions of Nonlinear Variational Inequalities with a Gradient Bound as Constraint

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The aim of this paper is to obtain regularity theorems for solutions of nonlinear variational inequalities with a gradient bound as constraint, and in particular to generalize a result of BREZIS & STAMPACCHIA ([3; Théorème III.1]). Our results will be applicable to the elastic-plastic torsion of a cylindrical bar with a multiply connected cross section (cf. [6] for a description of this problem).

Let $\Omega$ be a bounded multiply connected domain in $\mathbb{R}^N$, $N \geq 2$, having finitely many holes $\Omega_k$, $k=1, \ldots, n$, with respective boundaries $\Gamma_k = \partial \Omega_k$. The boundary of $\Omega$ is then the union of the disjoint family $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$. We assume moreover that $\partial \Omega$ is Lipschitz continuous and satisfies the following outward sphere condition: for any boundary point $x_0$ there is a ball $B$ of fixed radius $R$ such that the intersection of $\Omega$ and $B$ consists of the point $x_0$ alone.

We shall consider variational inequalities of the form

\begin{equation}
\langle Au + f, v - u \rangle \geq 0 \quad \forall v \in K,
\end{equation}

where

\[ K = \{v \in H^{1,\infty}(\Omega) : |Dv| \leq 1, v|_{\Gamma_k} = c_k, k = 0, \ldots, n\}, \]

and where the $c_k$ are given constants, $f$ is a function belonging to $L^p(\Omega)$, $1 < p < \infty$, and $A$ is a quasilinear differential operator in divergence form

\begin{equation}
A = -D^i(a_i(p))
\end{equation}

\footnote{Every open bounded set whose boundary is of class $C^2$ satisfies the outward sphere condition with some suitable $R$.}
whose coefficients satisfy the conditions

(i) \[ a_i \in C^1(\mathbb{R}^N) \]

and

(ii) \[ \frac{\partial a_i}{\partial p} \varepsilon^i \xi \geq 0 \quad \forall \xi \in \mathbb{R}^N. \]

If we assume that the convex set \( K \) is not empty, then the following result holds.

**Theorem.** Under the assumptions stated above, the variational inequality (*) has a solution \( u \in K \) such that

\[ A u \in L^p(\Omega). \]

Furthermore, if \( \partial \Omega \in C^{1,1} \) and if \( a_i \) is a coercive vector field, then \( u \in H^{2,1}(\Omega) \) provided \( p > N \).

**Proof.** The existence of a solution \( u \in K \) follows from well-known existence theorems for maximal monotone operators, since the convex set \( K \) is compact (cf. e.g. [3]). The crucial point is to show that the relation (2) is valid. To prove this assertion, it will be sufficient to demonstrate that the triple \( \{ u, K, A \} \) is \( J \)-compatible in the sense of [3; Théorème I.1]; namely, if \( J_p \) is the duality mapping from \( L^p(\Omega) \) to \( L^{p'}(\Omega) \) defined by

\[ J_p(v) = |v|^{p-2} v, \quad 1/p + 1/q = 1, \]

then for every \( \varepsilon > 0 \) we shall prove the existence of an element \( w_\varepsilon \) in \( L^p(\Omega) \), whose \( L^p \) norm is bounded independently of \( \varepsilon \), such that the equation

\[ u_\varepsilon + \varepsilon J_p(A u_\varepsilon + w_\varepsilon) = u \]

has a solution \( u_\varepsilon \in K \) with \( A u_\varepsilon \in L^p(\Omega) \). According to the results of BREZIS & STAMPACCHIA, \( A u \) then belongs to \( L^p(\Omega) \) too.

To prove (4) let us consider the monotone, hemicontinuous operator \( A_0 \) from \( H^{1,2}(\Omega) \) to \( H^{-1,2}(\Omega) \) defined by

\[ A_0 = - D^i(\tilde{a}_i(p)), \]

where \( \tilde{a}_i(p) = a_i(p) \) on the compact set \( |p| \leq 2 \). The existence of such an operator has been shown by BREZIS & STAMPACCHIA in [3]. We then define the multivalued operator \( \tilde{A} = A_0 + B \) through the assignments:

\[ A_\sigma = A_0 - \sigma A, \]

\[ B v = \mu \beta(v - \phi); \]

here \( \sigma \) is any positive number, \( \phi \) is any given element of \( K \), \( \mu \) is a positive constant to be determined later, and \( \beta \) is the following maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \)

\[ \beta(t) = \begin{cases} 
-1, & t \leq 0 \\
[1,1], & t = 0 \\
1, & t \geq 0.
\end{cases} \]

\( \tilde{A} \) maps elements of \( H^{1,2}(\Omega) \) onto subsets of \( H^{-1,2}(\Omega) \).
For later use we shall need the following result (compare [3; Lemme III.2]).

**Lemma 1 (Comparison Lemma).** Let $\Theta$ be a nondecreasing real function with $\Theta(0) = 0$. For $i = 1, 2$, let $F_i$, $\phi_i \in L^\infty(\Omega)$, and $u_i \in H^{1,2}(\Omega)$ be functions such that the relations

\[
0 \in A_{\mu} u_i + \mu \beta(u_i - \phi_i) + \Theta(u_i - F_i)
\]

and

\[
0 \in A_{\mu} u_2 + \mu \beta(u_2 - \phi_2) + \Theta(u_2 - F_2)
\]

hold in the sense of distributions. Then we have

\[
|u_2 - u_1| \leq \max\left(\sup_{\Omega} |u_2 - u_1|, \sup_{\Omega} |F_2 - F_1|, \sup_{\Omega} |\phi_2 - \phi_1|\right).
\]

**Proof.** (i) First we shall show that

\[
u_1 - u_2 \leq T = \max\left(\sup_{\Omega} (u_1 - u_2), \sup_{\Omega} (F_1 - F_2), \sup_{\Omega} (\phi_1 - \phi_2)\right)
\]

provided

\[
0 \leq A_{\mu} u_2 + \mu \beta(u_2 - \phi_2) + \Theta(u_2 - F_2)
\]

(i.e. provided there is an element in $A_{\mu} u_2 + \mu \beta(u_2 - \phi_2) + \Theta(u_2 - F_2)$ such that (12) is satisfied in the distributional sense).

For any $\varepsilon > 0$, we set

\[
\eta = \max(u_1 - u_2, T + \varepsilon) - (T + \varepsilon) \in H^{1,2}_D(\Omega).
\]

From (8) and (12) it is clear that

\[
0 \geq \langle A_{\mu} u_1 - A_{\mu} u_2, \eta \rangle + \mu \int_{\Omega} \{\beta(u_1 - \phi_1) - \beta(u_2 - \phi_2)\} \eta \, dx
\]

\[
+ \int_{\Omega} \{\Theta(u_1 - F_1) - \Theta(u_2 - F_2)\} \eta \, dx.
\]

Now in the set $\{u_1 - u_2 - T - \varepsilon \geq 0\}$ we have

\[
u_1 - F_1 > u_2 - F_2
\]

and

\[
u_1 - \phi_1 > u_2 - \phi_2.
\]

The last two integrals in (14) are consequently nonnegative by the definition of $\beta$ and $\Theta$; hence

\[
0 \geq \int_{\{u_1 - u_2 \geq T + \varepsilon\}} |D(u_1 - u_2)|^2 \, dx
\]

which implies the assertion (10).

(ii) The estimate (10) is obtained by permuting the indices in the first part of the proof.

Next we prove the crucial

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Lemma 2. Let $\Theta$ be a continuous, bounded, nondecreasing real function with $\Theta(0) = 0$. Then for any $F \in K$ there exists a solution $v \in K$ of the relation
\begin{equation}
0 \in A v + \Theta(v - F),
\end{equation}
provided $\mu$ is sufficiently large.

The proof will be given in three steps: First, (18) has a solution $v \in \phi + H^{1,2}_0(\Omega)$ (see the Appendix). Second, for any pair of points $x \in \Omega, x_0 \in \partial \Omega$, we shall show that
\begin{equation}
|v(x) - v(x_0)| \leq |x - x_0|.
\end{equation}
Then, with the help of the comparison lemma we conclude that
\begin{equation}
|v(x) - v(y)| \leq |x - y| \quad \forall x, y \in \Omega.
\end{equation}

To prove (19) we shall construct appropriate comparison functions $\delta^+$ and $\delta^-$ following HARTMANN & STAMPACCHIA [5; Lemma 10.1]. Let $x_0 \in f^*_k$. By assumption there is a ball with radius $R$ which touches $f^*_k$ in $x_0$. Without loss of generality we may assume that the center of the ball lies in the origin.

Now define
\begin{equation}
\delta_0(x) = |x| - R.
\end{equation}
One easily checks that
\begin{align*}
\delta_0(x) \geq \delta_0(x_0) = 0 & \quad \text{for } |x| \geq |x_0| = R, \\
D^i \delta_0 = \frac{x^i}{|x|}, \quad D^i D^j \delta_0 = \frac{\delta^{ij}}{|x|} = \frac{x^i x^j}{|x|^3}.
\end{align*}
Moreover, for any function $u \in K$ we have the estimate
\begin{equation}
|u(x) - c_k| \leq \inf_{y \in \Gamma_k} |x - y| \leq |x| - R = \delta_0(x) \quad \forall x \in \Omega.
\end{equation}
Now choose
\begin{equation}
\delta^+ = \delta_0 + c_k + \epsilon,
\end{equation}
where $\epsilon$ is any positive constant. The estimate (22) then implies
\begin{equation}
\Theta(\delta^+ - F) \geq \Theta(0) = 0
\end{equation}
and

$$\beta(\delta^+ - \phi) = 1.$$  

Hence,

$$\tilde{A} \delta^+ + \Theta(\delta^+ - F) \geq -D'(a_i(D \delta^+)) - \sigma D \delta^+ + \mu \geq c + \mu,$$

where $c$ depends only on the first derivatives of the $a_i$'s on the compact set $|p| = 1$, and on $R$, $N$, and diam $\Omega$.

If we choose $\mu$ sufficiently large we therefore get

$$\tilde{A} \delta^+ + \Theta(\delta^+ - F) \geq 0.$$

As we have shown in the first part of the proof of Lemma 1, it follows from (18) and (27) that

$$v \leq \delta^+,$$

or in other words

$$v(x) - v(x_0) = v(x) - c_k \leq |x| - |x_0| \leq |x - x_0| \quad \forall x \in \Omega, \forall x_0 \in \partial \Omega.$$

To prove (19), we set

$$\delta^- = -\delta_0 + c_k - \epsilon.$$

From (22) and the relation

$$\tilde{A} \delta^- + \Theta(\delta^- - F) \leq 0$$

we get by similar considerations

$$\delta^- \leq v.$$

Thus (19) is proved.

To complete the proof of Lemma 2, let $x_1, x_2 \in \Omega$, and make the definitions

$$h = x_2 - x_1, \quad \Omega_h = \Omega - h, \quad v_h(x) = v(x + h), \quad F_h(x) = F(x + h),$$

and

$$\phi_h(x) = \phi(x + h).$$

In the open set $\Omega = \Omega \cap \Omega_h$ we have

$$0 \in A_\sigma v + \mu \beta(v - \phi) + \Theta(v - F)$$

and

$$0 \in A_\sigma v_h + \mu \beta(v_h - \phi_h) + \Theta(v_h - F_h).$$

From the comparison lemma we now conclude that the inequality

$$|v_h - v| \leq \max\left(\sup_{\epsilon \in \epsilon} |v_h - v|, \sup_{\epsilon \in \epsilon} |F_h - F|, \sup_{\epsilon \in \epsilon} |\phi_h - \phi|\right) \leq |h|$$

holds in $\Omega$ (here $x \in \partial \Omega$ is equivalent that $x$ or $x + h$ belongs to $\partial \Omega$). Consequently we have

$$|v(x_1) - v(x_2)| \leq |x_1 - x_2| \quad \forall x_1, x_2 \in \Omega.$$

This completes the proof of Lemma 2.

The $J$-compatibility of $\{u, K, A\}$ now follows easily. According to Lemma 2, for any $\sigma > 0$ there exists a solution $v_\sigma \in K$ of the relation

$$0 \in A_\sigma v - \sigma Dv + \Theta(v - F) + \mu \beta(v - \phi).$$
Now we choose a sequence of values $\sigma$ tending to zero such that the corresponding functions $v_\sigma$ converge uniformly to some function $v \in K$ satisfying

\begin{equation}
0 \in A v + \Theta(v - F) + \mu \beta(v - \phi)
\end{equation}

(here we use the fact that $A_\omega + \Theta(-F)$ is a monotone, hemicontinuous operator and that $\beta(v_\sigma - \phi)$ converges weakly in $L^2(\Omega)$ to $\beta(v - \phi)$; see the Appendix for similar considerations).

For any $\varepsilon > 0$ let $\Theta$ be a function of the type indicated in Lemma 2, which agrees with the function

$t \rightarrow |t/\varepsilon|^{q-2} t/\varepsilon$

on the interval $[-C, C]$, $C > 2 \text{diam } \Omega$. Also let $F = u$ in Lemma 2. Then there are elements $u_\varepsilon \in K$ and $w_\varepsilon \in L^q(\Omega)$ such that

$$u_\varepsilon + \varepsilon J_p(A u_\varepsilon + w_\varepsilon) = 0,$$

where $w_\varepsilon$ is some element in $\mu \beta(u_\varepsilon - \phi)$. Thus the relation (2) is proved.

The final assertion of the theorem is well-known (cf. e.g. [4; Appendix]).

Appendix

To prove that

\begin{equation}
0 \in A v + \Theta(v - F) + \mu \beta(v - \phi)
\end{equation}

has a solution $v_\varepsilon \in \phi + H^{1,2}_0(\Omega)$, consider the regularized graph

\begin{equation}
\beta_\varepsilon(t) = \begin{cases} 
-1, & t \leq -\varepsilon \\
\varepsilon/t, & |t| \leq \varepsilon \\
1, & t \geq \varepsilon.
\end{cases}
\end{equation}

Since $\beta_\varepsilon$ is continuous, bounded, and nondecreasing, there exists a solution $v_\varepsilon$ of

\begin{equation}
0 = A v_\varepsilon + \Theta(v_\varepsilon - F) + \mu \beta_\varepsilon(v_\varepsilon - \phi).
\end{equation}

Since the lower order terms in (A.3) are bounded, we have the estimates

\begin{equation}
\|v_\varepsilon\|_{1,2,\Omega} \leq \text{const}(\sigma)
\end{equation}

and

\begin{equation}
\|v_\varepsilon\|_{2,p,\Omega} \leq \text{const}(\sigma, p, \Omega')
\end{equation}

for any $p > N$ and for all $\Omega' \subset \subset \Omega$.

Hence a subsequence of $v_\varepsilon$ (which we again call $v_\varepsilon$) converges weakly in $\phi + H^{1,2}_0(\Omega)$ and uniformly on compact subsets of $\Omega$ to some function $v$.

The crucial step in the proof that $v$ satisfies (A.1) is to show that $\beta_\varepsilon(v_\varepsilon - \phi)$ converges weakly in $L^2(\Omega)$ to some element of $\beta(v - \phi)$.

Since $\beta_\varepsilon$ is bounded, a subsequence of $\beta_\varepsilon(v_\varepsilon - \phi)$ converges weakly to some function $\gamma$ which has its range in the convex set $[-1, 1]$. If we could show that

\begin{equation}
\gamma(x) = \begin{cases} 
1 \text{ almost everywhere in } \{v - \phi > 0\} \\
-1 \text{ almost everywhere in } \{v - \phi < 0\}
\end{cases}
\end{equation}

then $\gamma$ would belong to $\beta(v - \phi)$. 
We shall only prove the first assertion of (A.6): Since \( v \) and \( \phi \) are continuous and \( G = \{ x \in \Omega : v - \phi > 0 \} \) is open. Let \( K \) be a compact subset of \( G \), then \( (v - \phi)_K \geq \varepsilon > 0 \) and
\[
(v_\varepsilon - \phi)_K \geq \varepsilon/2 \quad \forall 0 < \varepsilon \leq \varepsilon_0.
\]
On the other hand, for \( \varepsilon < \min(\tau/2, \varepsilon_0) \) we have
\[
\beta_\varepsilon(v_\varepsilon - \phi) = 1 \quad \text{in } K.
\]
The remainder of the proof now follows by standard techniques and will be omitted.

References