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Hypersurfaces of Prescribed Mean Curvature over Obstacles

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1. Introduction

Let Ω be a bounded domain in the euclidean space $\mathbb{R}^n$, $n \geqq 2$, with Lipschitz boundary $\partial\Omega$. We shall consider surfaces which are graphs of functions u defined on Ω having prescribed mean curvature $H = H(x, u)$ with the side condition that they should be bounded from below by an obstacle ψ. The case $H=0$ (minimal surfaces) has been discussed in detail by several authors, compare [6, 7, 12, 13, 17, 18, 20, 21, 24] of the references. Tomi [31] has also investigated parametric surfaces with variable H. More general variational problems with obstructions have been discussed in [9] and [10].

During the session on "Variationsrechnung", held from June 18th to June 24th, 1972 in Oberwolfach, Miranda showed that the functional

$$\Phi(v) := \int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\, dx + \int_\Omega \int_0^{v(x)} nH(x,t)\, dt\, dx + \int_{\partial\Omega} |v-f|\, d\mathscr{H}_{n-1}$$

where H is bounded measurable and $f \in L^1(\partial\Omega)$, has a minimum in the class $BV(\Omega) \cap \{v \geqq \psi\}$, provided that

$$H_0(\text{meas}\,(\Omega)/\omega_n)^{1/n} < 1 \quad \text{and} \quad \psi \in C^1(\bar{\Omega}),\ \psi|_{\partial\Omega} \leqq f,$$

where $H_0 := \sup|H|$ and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Let u be a solution of $\Phi(v) \to \min$. If one assumes $H(x,\cdot)$ to be non-decreasing, Miranda could show by a result due to Massari that there is an open subset G of $\Omega \times \mathbb{R}^n$ such that graph $(u) \cap G$ is a C^1 manifold of dimension $(n-1)$ and the Hausdorff measure $\mathscr{H}_s(\Omega \times \mathbb{R} - G) = 0$ if $s > n-7$. If one assumes moreover f to be continuous on $\partial\Omega$ and $n|H(x, f(x))| \leqq (n-1) \cdot$ mean curvature of $\partial\Omega$, then u is continuous on $\partial\Omega$ and $u_{|\partial\Omega} = f$.

We shall use a different approach in order to show that whenever the Dirichlet problem

$$(*, \tau) \qquad \begin{aligned} A u_\tau^* + n\tau H(x, u_\tau^*) &= 0 \\ u^*_{\tau|\partial\Omega} &= f, \end{aligned}$$

$$A v := -D^i(a^i(Dv)), \qquad a_i(p) := p^i(1+|p|^2)^{-\frac{1}{2}},$$

has a C^1 solution u_τ^* for every $\tau\in[0,1]$ with C^2 boundary values f, where H is Lipschitz in (x,t) and nondecreasing in t, then the analogical constrained probelm

$$(**)\qquad \begin{aligned}&\langle Au+nH(x,u),\ v-u\rangle\geqq 0, \quad \forall v\in\mathbf{K},\\ &\mathbf{K}:=\{v\in H^{1,\infty}(\Omega):\ v\geqq\psi, v_{|\partial\Omega}=f\},\end{aligned}$$

has a solution $u\in\mathbf{K}$ for every Lipschitz obstacle ψ provided that $\partial\Omega\in C^2$, $f\in C^2(\partial\Omega)$, and $\psi_{|\partial\Omega}\leqq f$.

Here $H^{1,\infty}(\Omega)$ means the set of all Lipschitz functions on $\bar\Omega$ and $\langle\cdot,\cdot\rangle$ denotes the pairing between $H_0^{1,\infty}(\Omega):=\{v\in H^{1,\infty}(\Omega):\ v_{|\partial\Omega}=0\}$ and its "dual space"

$$H^{-1,1}(\Omega):=\left\{f_0+\sum_{i=1}^n D^i f_i:\ f_0, f_i\in L^1(\Omega)\right\},$$

i.e.

$$\langle Au+nH(x,u),\ v-u\rangle=\int_\Omega a_i(Du)\,D^i(v-u)\,dx+\int_\Omega nH(x,u)\,(v-u)\,dx.$$

We shall prove furthermore that the solution u is in $C^{1,\alpha}(\bar\Omega)$ for all α, $0<\alpha<1$, if $\psi\in C^2(\bar\Omega)$.

Regularity and existence results are also obtained in the cases

$$\text{(A)}\qquad \psi\in C^{0,\alpha}(\bar\Omega),\quad f\in C^2(\partial\Omega),\quad \psi_{|\partial\Omega}\leqq f,$$

and

$$\text{(B)}\qquad \psi\in C^{0,1}(\Omega)\cap C^0(\bar\Omega),\quad f\in C^0(\partial\Omega),\quad \psi_{|\partial\Omega}<f.$$

Under the assumption (A) we obtain $u\in C^{0,\alpha}(\bar\Omega)$, while in the second case $u\in C^{0,1}(\Omega)\cap C^0(\bar\Omega)\cap H^{1,1}(\Omega)$.

In the following we shall use the standard terminology (compare [22] and [23]), e.g. $H^{m,p}(\Omega)$ means the Sobolev spaces (sometimes denoted by $W^{m,p}(\Omega)$), and $C^{m,\alpha}(\Omega)$ and $C^{m,\alpha}(\bar\Omega)$ are the spaces of functions whose derivatives of order m are Hölder continuous with exponent α, $0<\alpha<1$, in Ω and in $\bar\Omega$ respectively.

2. The Main Theorem

We shall first give a precise statement of our assumptions. Let us assume that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. We may furthermore assume that the given boundary values $f\in C^2(\partial\Omega)$ are the trace of a function $f\in C^2(\bar\Omega)$ and we set

$$c_0:=\sup_\Omega|f|,\quad c_1:=\sup_\Omega|Df|,\quad c_2:=\sup_\Omega|D^2f|.$$

Let K be an upper bound for the unsigned normal curvatures of the boundary surface and denote by H_{n-1} its mean curvature.

We suppose that the Dirichlet problem

$$(*, \tau) \qquad \begin{aligned} A u_\tau^* + n\tau H(x, u_\tau^*) &= 0 \\ u_{\tau|\partial\Omega}^* &= f \end{aligned}$$

has a solution $u_\tau^* \in C^1(\bar{\Omega})$ for every $\tau \in [0, 1]$ with a uniform gradient bound where

$$(1) \qquad H = H(x, t) \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}), \quad \frac{\partial H}{\partial t} \geqq 0.$$

Then the following holds.

Theorem 1. *The variational inequality*

$$(**) \qquad \begin{aligned} &\langle A u + n H(x, u),\, v - u \rangle \geqq 0, \quad \forall v \in \mathbf{K}, \\ &\mathbf{K} := \{v \in H^{1,\infty}(\Omega) \colon v \geqq \psi,\ v_{|\partial\Omega} = f\} \end{aligned}$$

has a solution $u \in \mathbf{K}$ *for every* $\psi \in H^{1,\infty}(\Omega)$ *with* $\psi_{|\partial\Omega} \leqq f$ *provided that*

$$(2) \qquad n|H(x, f(x))| \leqq (n-1)\, H_{n-1}(x), \quad \forall x \in \partial\Omega.$$

Proof. The proof consists of three steps: In Section 2.1 we derive *a priori estimates* for a solution $u \in \mathbf{K} \cap C^1(\bar{\Omega})$ under the *additional hypotheses* that $\psi \in C^2(\bar{\Omega})$ and $H \in C_c^2(\mathbb{R}^n \times \mathbb{R})$, namely

$$(3) \qquad u^* \equiv u_1^* \leqq u \leqq u^* + 2 \max\big(\sup_\Omega |u^*|,\ \sup_\Omega \psi\big),$$

$$(4) \qquad |Du|_{\partial\Omega} \leqq L \equiv K_1 \cdot \max(|D\psi|_\Omega,\ |Du^*|_\Omega,\ 1),$$

and

$$(5) \qquad |Du|_\Omega \leqq C \equiv K_2 \cdot \max(L, |D\psi|_\Omega)$$

where both constants depend only on $c_0, c_1, c_2, \sup|u^*|, \sup\psi, K$, $\sup|DH|$, and the structure of the coefficients of A. *Existence* is then proved in Section 2.2 by a *continuity method*, while the additional assumptions on ψ and H are removed in the Sections 2.3 and 2.4.

2.1. A Priori Estimates

From the variational inequality and the assumption $H \in C_s^2(\mathbb{R}^n \times \mathbb{R})$ it follows immediately that

$$(6) \qquad A u + n H(x, u) \geqq 0 \quad \text{in the distributional sense}$$

and

$$(7) \qquad \begin{aligned} A u + n H(x, u) &= 0 \quad \text{in } \{u > \psi\}, \\ u &\in C^3(\{u > \psi\}). \end{aligned}$$

The estimate (3) is now a consequence of the inequalities

$$Au + nH(x,u) - Au^* - nH(x,u^*) \geqq 0, \tag{8}$$

$$\begin{aligned} &A(u^*+c) + nH(x,u^*+c) - Au - nH(x,u) \geqq 0 \quad \text{in } \{u>\psi\} \\ &\text{for every nonnegative constant } c, \end{aligned} \tag{9}$$

and of the following lemma.

Lemma 1 (comparison lemma). *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary, bounded, open set, and $A = -D^i(a_i(p))$ an elliptic differential operator,* i.e.

$$\{a_i(p) - a_i(q)\}\{p^i - q^i\} \geqq 0.$$

If for two functions $u_1, u_2 \in C^{0,1}(\Omega) \cap C^0(\bar{\Omega})$ the inequality

$$Au_1 + nH(x,u_1) - Au_2 - nH(x,u_2) \geqq 0 \tag{10}$$

holds in the distributional sense, then

$$u_1 - u_2 \geqq \min(\gamma, 0) \tag{11}$$

where $\gamma := \inf_{\partial\Omega}(u_1 - u_2)$ provided that A or $H(x,\cdot)$ are strictly monotone.

Proof of the lemma. Let $\gamma^* := \min(\gamma, 0)$, then we have for every $\varepsilon > 0$

$$0 \geqq \min(u_1 - u_2, \gamma^* - \varepsilon) - (\gamma^* - \varepsilon) \in H_c^{1,\infty}(\Omega)$$

so that we obtain from inequality (10)

$$\begin{aligned} 0 \geqq &\int_{u_1 - u_2 < \gamma^* - \varepsilon} \{a_i(Du_1) - a_i(Du_2)\} D^i(u_1 - u_2)\, dx \\ &+ \int_{u_1 - u_2 < \gamma^* - \varepsilon} n\{H(x,u_1) - H(x,u_2)\}\{u_1 - u_2 - (\gamma^* - \varepsilon)\}\, dx. \end{aligned}$$

Both integrands are nonnegative since $(\gamma^* - \varepsilon) < 0$. Hence we deduce from the continuity of u_1, u_2, and the strict monotonicity of A or $H(x,\cdot)$, that $u_1 - u_2 \geqq \gamma^* - \varepsilon$ which proves our assertion, since ε has been chosen arbitrarily.

In order to prove the gradient estimates let us assume that $u \in C^1(\bar{\Omega})$ which is in fact true. A proof is given below.

From Serrin [25], Thm. 10.1 and Thm. 14.3 we conclude that there is a boundary strip

$$\Omega_r := \{x \in \Omega : d(x) < r\},$$
$$d(x) := \operatorname{dist}(x, \partial\Omega),$$

and a function $\delta \in H^{2,\infty}(\Omega_r)$ such that

$$A\delta + nH(x,\delta) \geqq 0 \quad \text{in } \Omega_r, \tag{12}$$

$$\begin{aligned} &\delta \geqq \psi \quad \text{in } \Omega_r, \\ (13) \qquad &\delta(x) = f(x) \quad \text{on } \partial\Omega, \\ &\delta(x) \geqq \sup_\Omega u^* + 2 \cdot \max\left(\sup_\Omega |u^*|, \sup_\Omega \psi\right) \quad \text{for } d(x) = r, \end{aligned}$$

and

$$(14) \qquad |D\delta|_{\partial\Omega} \leqq K_1 \cdot \max(|D\psi|_\Omega, 1)$$

with the same constant as in inequality (4) (compare Appendix I).

Lemma 1 implies therefore

$$(15) \qquad u \leqq \delta \quad \text{in } \Omega_r$$

from which the boundary gradient estimate immediately follows in view of (3).

On the coincidence set $E := \{x \in \Omega\colon\ u(x) = \psi(x)\}$ Du equals $D\psi$ because the difference $u - \psi$ reaches its minimum on E. On $\Omega - E$ we know $u \in C^3(\Omega - E)$ and

$$A u + n H(x, u) = 0$$

so that Theorem 2 of [26] is applicable in order to show that

$$(16) \qquad |Du|_\Omega \leqq C \equiv \exp\{\kappa(\sup_\Omega u - \inf_\Omega u)\} \cdot \max(|D\psi|_\Omega, L, K_3)$$

where κ and K_3 depend only on $\sup\left|\frac{\partial}{\partial x} H\right|$ and on the structure of the coefficients of A (compare [26], p. 570, 585, and 595).

2.2. Existence by a Continuity Method

We shall now prove the existence of a solution u of (**) by a continuity method. Let $\tau \in [0, 1]$ and consider the variational inequalities

$$(**, \tau) \qquad \begin{aligned} &\langle A u_\tau + n\tau H(x, u_\tau),\ v - u_\tau \rangle \geqq 0, \quad \forall v \in \mathbf{K}, \\ &\mathbf{K} := \{v \in H^{1,\infty}(\Omega)\colon v \geqq \psi,\ v_{|\partial\Omega} = f\}. \end{aligned}$$

Define the set T by

$$T := \{\tau \in [0, 1]\colon \text{ there exists a solution } u_\tau \in \mathbf{K} \text{ of } (**, \tau)\}.$$

If one assumes that $0 \in T$, a short proof of the existence will be given below, then it suffices to show that T is both open and closed in $[0, 1]$.

In order to do this let us construct an operator $\tilde{A}$

$$\tilde{A} := -D^i(\tilde{a}_i(p))$$

such that

$$\tilde{a}_i(p) = a_i(p) \quad \text{for } |p| \leqq 2 \cdot C$$

and

$$\nu|\xi|^2 \leqq \frac{\partial \tilde{a}_i}{\partial p^j} \xi^i \xi^j \leqq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,$$

$$0 < \nu \leqq \mu < \infty \quad \text{(see Appendix II)}.$$

Then the inequality

$$\text{(17)} \qquad \begin{aligned} &\langle \tilde{A}\tilde{u}_\tau + n\tau H(x, \tilde{u}_\tau),\, v - \tilde{u}_\tau \rangle \geqq 0, \quad \forall v \in \tilde{\mathbf{K}}, \\ &\tilde{\mathbf{K}} := \{v \in H^{1,2}(\Omega) \colon v \geqq \psi,\, v_{|\partial\Omega} = f\} \end{aligned}$$

has a solution $\tilde{u}_\tau$ for all $\tau \in [0, 1]$ such that $\|\tilde{A}\tilde{u}_\tau\|_\infty$ is uniformly bounded. This follows at once from [4], Théorème II.1, and the fact that $\psi \in C^2(\bar{\Omega})$. Although Brezis and Stampacchia only considered the case $f = 0$ in [4], the proof carries over without any change if one looks at the operator

$$A_0 v := \tilde{A}(v + f) \quad \text{for} \quad v \in H_0^{1,2}(\Omega)$$

and the obstacle $\psi_0 := \psi - f$. The solution u_τ^0 of the variational inequality

$$\text{(17')} \qquad \begin{aligned} &\langle A_0 u_\tau^0 + n\tau H(x, \tilde{u}_\tau),\, v - u_\tau^0 \rangle \geqq 0, \quad \forall v \in \mathbf{K}_0, \\ &\mathbf{K}_0 := \{v \in H_0^{1,2}(\Omega) \colon v \geqq \psi_0\} \end{aligned}$$

then corresponds to the solution of (17) by the equation

$$\text{(18)} \qquad \tilde{u}_\tau = u_\tau^0 + f.$$

It follows from the uniform boundedness of $\|\tilde{A}\tilde{u}_\tau\|_\infty$ that the $\tilde{u}_\tau$'s are uniformly bounded in $C^{1,\alpha}(\bar{\Omega})$ for any α, $0 < \alpha < 1$.

Let us assume now that $\tau \in T$. Then u_τ is a Lipschitz solution of $(**, \tau)$ with Lipschitz constant C_0. If we choose the operator $\tilde{A}$ such that $\tilde{a}_i(p) = a_i(p)$ for $|p| \leqq 2 \cdot C_0$ then $\tilde{A}u_\tau = Au_\tau$, hence

$$\tilde{u}_\tau = u_\tau$$

and $|Du_\tau|_\Omega \leqq C$ in view of the known a priori estimates for C^1 solutions (Replace simply H with τH in Section 2.1).

The closedness of T is now easily derived with the help of a lemma due to Minty, compare [18] or [2].

Lemma 2. *A function $u \in \mathbf{K}$ satisfies the variational inequality* $(**)$ *if and only if*

$$\text{(19)} \qquad \langle Av + nH(x, v),\, v - u \rangle \geqq 0, \quad \forall v \in \mathbf{K}.$$

This follows from the fact that $A + nH(x, \cdot)$ is a monotone, hemicontinuous operator.

Let $\tau_n \in T$ be a sequence which converges to some element $\tau_0 \in [0, 1]$. For the corresponding solutions u_{τ_n} of $(**, \tau_n)$ the estimate

$$|D u_{\tau_n}| \leqq C$$

holds, so that a subsequence of the u_{τ_n}'s converges weakly in $H^{1,\infty}(\Omega)$ and uniformly in $C^0(\bar{\Omega})$ to some function u which will be the solution of $(**, \tau_0)$ by Lemma 2 and the uniqueness of the solution. Hence T is closed.

In order to show that T is open let $\tau_0 \in T$. Then u_{τ_0} is also a solution of (17). Since (17) has a unique solution for each τ (see Lemma 3 below), we deduce from the uniform boundedness of $|\tilde{u}_\tau|_{1,\alpha,\Omega}$ that $|D\tilde{u}_\tau|_\Omega$ depends continuously on τ so that $|D\tilde{u}_\tau| \leqq \frac{3}{2} \cdot C$ in some neighbourhood of τ_0. But for these values of τ

$$\tilde{A} \tilde{u}_\tau = A \tilde{u}_\tau .$$

Hence T is open.

We have to prove that $0 \in T$.

Proposition 1. *The variational inequa' y*

$$(20) \qquad \begin{gathered} \langle A u, v-u \rangle \geqq 0, \quad \forall v \in \mathbf{K}, \\ \mathbf{K} := \{ v \in H^{1,\infty}(\Omega) \colon v \geqq \psi, \; v_{|\partial\Omega} = f \} \end{gathered}$$

has a unique solution $u \in \mathbf{K}$.

Proof. The proposition is due to Miranda, compare [20] and [21], but for completeness we shall prove it once more in a different manner.

We choose $\tau \in [0, 1]$ and consider the inequalities

$$(20, \tau) \qquad \begin{gathered} \langle A u_\tau, v-u_\tau \rangle \geqq 0 \quad \forall v \in \mathbf{K}_\tau, \\ \mathbf{K}_\tau := \{ v \in H^{1,\infty}(\Omega) \colon v \geqq \tau \cdot \psi, \; v_{|\partial\Omega} = \tau \cdot f \} . \end{gathered}$$

As in Section 2.1 we can state a priori estimates for each solution u_τ of $(20, \tau)$.

If we now define the set T by

$$T := \{ \tau \in [0, 1] \colon \text{there exists a solution of } (20, \tau) \} ,$$

then T is evidently not empty for $0 \in T$, and by the same procedure as above we find that T is both open and closed so that $1 \in T$.

Thus Theorem 1 is proved under the additional assumptions made at the beginning. Using the a priori estimates which only depend on the conditions stated in the theorem we shall remove the additional assumptions.

2.3. $H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R})$

We define a nondecreasing Lipschitz function θ by

$$\theta(t)=\begin{cases} m, \; t \leqq m := \inf\limits_{\Omega,\tau} u_\tau^* \\ M, \; t \geqq M := \sup\limits_{\Omega,\tau} u_\tau^* + 2\cdot \max\left(\sup\limits_{\Omega,\tau} |u_\tau^*|, \sup\limits_{\Omega} \psi\right) \\ t, \; m \leqq t \leqq M. \end{cases}$$

The function

$$H_0(x,t) := H(x, \theta(t))$$

is then uniformly Lipschitz continuous and nondecreasing in t, and agrees with H on $\mathbb{R}^n \times [m, M]$.

By smoothing we can find an approximating sequence $H_\varepsilon \in C_c^2(\mathbb{R}^n \times \mathbb{R})$ which tends uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}$ towards H_0 such that $|DH_\varepsilon|_\Omega$ is uniformly bounded and $\frac{\partial H_\varepsilon}{\partial t} \geqq 0$. The H_ε's satisfy

$$n\,|H_\varepsilon(x, f(x))| \leqq (n-1)\,H_{n-1}(x) + \rho(\varepsilon), \qquad \forall x \in \partial\Omega \tag{21}$$

with $\rho(\varepsilon) \to 0$.

Since the proof of Serrin's boundary gradient estimate is still true in this case if ε is sufficiently small, we have only to show that the Dirichlet problem

$$\begin{aligned} A u_{\tau,\varepsilon}^* + n\tau H_\varepsilon(x, u_{\tau,\varepsilon}^*) &= 0 \\ u_{\tau,\varepsilon|\partial\Omega}^* &= f \end{aligned} \tag{22, τ, ε}$$

has a solution $u_{\tau,\varepsilon}^* \in C^1(\bar{\Omega})$ for every $\tau \in [0,1]$ in order to get a solution u_ε of

$$\langle A u_\varepsilon + n H_\varepsilon(x, u_\varepsilon), v - u_\varepsilon \rangle \geqq 0, \qquad \forall v \in \mathbf{K}. \tag{23, ε}$$

To prove (22, τ, ε) we remember that by assumption, ($*$, τ) has a solution u_τ^* which agrees with the solution of (22, τ, 0) since $H_0(x, u_\tau^*) = H(x, u_\tau^*)$. By similar arguments as we used above in order to prove that T is both open and closed, we can show that (22, τ, ε) has a solution $u_{\tau,\varepsilon}^* \in C^{1,\alpha}(\bar{\Omega})$ with

$$|Du_{\tau,\varepsilon}^*| \leqq \tfrac{3}{2} \cdot \sup_\tau |Du_\tau^*|_\Omega \tag{24}$$

for sufficiently small ε's independent of τ.

Applying Minty's lemma to (23, ε) we get the solution of ($**$).

2.4. $\psi \in H^{1,\infty}(\Omega)$

Finally let us assume that $\psi \in H^{1,\infty}(\Omega)$. Then there is a sequence $\psi_\varepsilon \in C_c^2(\mathbb{R}^n)$ with

$$\psi_{\varepsilon|\partial\Omega} < f, \quad \psi_\varepsilon \rightrightarrows \psi \text{ in } \Omega, \tag{25}$$

and

$$|D\psi_\varepsilon|_\Omega \leqq |D\psi|_\Omega \tag{26}$$

for which the inequality (**) has solutions u_ε. As the a priori estimates depend only on $|\psi|_\Omega$, $|D\psi|_\Omega$, and known constants we obtain in a similar manner as above the final existence result.

3. Hölder Continuous Obstacles

In the preceding section we have shown that the variational inequality (**) has a Lipschitz solution u provided that the obstacle ψ is Lipschitz. But what happens when the obstacle is only Hölder continuous, say $\psi \in C^{0,\alpha}(\bar{\Omega})$?

The best that we can expect is that the "solution" u is also of class $C^{0,\alpha}(\bar{\Omega})$. In order to avoid the explanation what "solution" now means (the reader is referred to [18] where this has been done for continuous obstacles) we shall show

Theorem 2. *Under the assumptions of Theorem* 1 *it follows that the Hölder constant* $|u|_{0,\alpha}$ *of the solution* u *of*

$$\langle Au + nH(x,u), v-u\rangle \geqq 0, \quad \forall v \in \mathbf{K} \tag{**}$$

is bounded by

$$|u|_{0,\alpha} \leqq K_4(1 + |\psi|_{0,\alpha}) \tag{27}$$

where K_4 *depends only on the constants in the estimate* (5) *and on the diameter of* Ω.

Proof. $|\psi|_{0,\alpha}$ is defined by

$$|\psi|_{0,\alpha} := \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x-y|^\alpha}. \tag{28}$$

It is well known that we can extend ψ as a Hölder continuous function to the whole space $\mathbb{R}^n$ such that the Hölder constant remains unchanged (compare e.g. [34]).

Let η be a Friedrich's mollifier, that is $0 \leqq \eta \in C_c^\infty(B)$, $B := \{x \in \mathbb{R}^n : |x| < 1\}$, such that $\int \eta(x)\,dx = 1$. For $t > 0$ let

$$\psi_t(x) := \int \eta(z)\,\psi(x+tz)\,dz - |\psi|_{0,\alpha} \cdot t^\alpha. \tag{29}$$

Then

$$\psi_t(x) \leqq \psi(x) \tag{30}$$

and

$$|D\psi_t| \leqq K_5 \cdot |\psi|_{0,\alpha} \cdot t^{\alpha-1} \tag{31}$$

where K_5 depends only on η (compare [32], Lemma 1).

Furthermore we need the following lemma:

Lemma 3. *If u and u' are two solutions of the variational inequality* (**) *with the corresponding obstacles ψ, ψ' and with boundary values f and f', then*

$$|u-u'| \leqq \max\big(\sup_{\Omega} |\psi-\psi'|, \sup_{\partial\Omega} |f-f'|\big). \tag{32}$$

Proof. With the help of Lemma 1 it follows from the inequality

$$Au+nH(x,u)-Au'-nH(x,u') \geqq 0 \quad \text{in } \{u'>\psi'\} \tag{33}$$

that

$$u-u' \geqq \min\big(\inf_{\partial\{u'>\psi'\}} (u-u'), 0\big), \tag{34}$$

i.e.

$$u-u' \geqq \min\big(\inf_{\Omega} (u-\psi'), \inf_{\partial\Omega} (f-f'), 0\big)$$

or finally

$$u-u' \geqq \min\big(\inf_{\Omega} (\psi-\psi'), \inf_{\partial\Omega} (f-f'), 0\big) \tag{35}$$

from which (32) immediately follows.

To prove the theorem we choose now $0 \neq h \in \mathbb{R}^n$. From (30), (31), (4), and (5) we conclude that there is a solution $u_{|h|}$ of Theorem 1 with the obstacle $\psi_{|h|}$ such that

$$|Du_{|h|}|_{\Omega} \leqq K_6 \cdot \max\left(|\psi|_{0,\alpha} |h|^{\alpha-1}, 1\right) \tag{36}$$

where K_6 does not depend on ψ.

Now let x and h be such, that $x, x+h \in \Omega$, then

$$\begin{aligned} |u(x+h)-u(x)| \leqq & |u(x+h)-u_{|h|}(x+h)| \\ & + |u_{|h|}(x+h)-u_{|h|}(x)| \\ & + |u_{|h|}(x)-u(x)|. \end{aligned} \tag{37}$$

Lemma 3 implies that the first and the last summand are bounded by

$$2 \cdot |\psi|_{0,\alpha} \cdot |h|^{\alpha} \tag{38}$$

while the second term is estimated by

$$K_6 \cdot \max(|\psi|_{0,\alpha}, d^{1-\alpha}) \cdot |h|^\alpha \tag{39}$$

because of (36), where $d := \operatorname{diam}(\Omega)$.

The reader may refer to the paper of Giusti [32] for more general considerations in the case $H=0$. In my opinion his results could be transferred if $H=H(u)$ and $\dfrac{\partial H}{\partial t} \geqq 0$.

Remark 1. The method of proving Theorem 2 could also be used to get similar results for the variational inequalities considered in [18].

4. Variational Approach

In this paragraph we shall show that the solution of the variational inequality (**) agrees with the solution of the variational problem

$$F(v) := \int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\, dx + \int_\Omega \int_0^{v(x)} nH(x,t)\, dt\, dx \to \min \tag{***}$$

in the class of functions $v \in H^{1,2}(\Omega) \cap \{v|_{\partial\Omega} = f\} \cap \{v \geqq \psi\} \cap C^0(\bar{\Omega})$.

It comes out that a solution u of (***) also minimizes the functional

$$\Phi(v) := F(v) + \int_{\partial\Omega} |f-v|\, d\mathcal{H}_{n-1} \tag{****}$$

in the class $v \in H^{1,1}(\Omega) \cap C^0(\bar{\Omega}) \cap \{v \geqq \psi\}$ by which we are able to find solutions of (**) or (***) even for continous boundary values f provided that $\psi \in C^{0,1}(\Omega) \cap C^0(\bar{\Omega})$ and $\psi|_{\partial\Omega} < f$.

Under the assumptions of Theorem 1 we shall now prove that the solution u of (**) also solves the variational problem (***). We use a trick similar to the one before (compare also [28], Lemma 5.2) and replace the integrand $g(p) := (1+|p|^2)^{\frac{1}{2}}$ in F by a twice continuously differentiable function $\tilde{g}$, such that $\tilde{g}(p) = g(p)$ for $|p| \leqq R$, where R is an arbitrary number greater than $2 \cdot |Du|_\Omega$, and $D^2\tilde{g}$ is a uniformly elliptic matrix.

We shall furthermore assume that the mean curvature H is bounded so that the corresponding functional $\tilde{F}$ has a minimum $\tilde{u}$ in the function class

$$\tilde{\mathbf{K}} := \{v \in H^{1,2}(\Omega) \colon v \geqq \psi,\ v|_{\partial\Omega} = f\}$$

and $\tilde{u}$ also solves the corresponding variational inequality

$$\langle \tilde{A}\tilde{u} + nH(x,\tilde{u}), v - \tilde{u} \rangle \geqq 0, \quad \forall v \in \tilde{\mathbf{K}}. \tag{40}$$

Since $\tilde{A}u = Au$, it follows by the comparison lemma that $u = \tilde{u}$. Hence u minimizes F in the class $\tilde{\mathbf{K}} \cap \{|Dv| \leqq R\}$. Since R is arbitrary and F is continuous in the topology of $H^{1,2}(\Omega)$, we find a solution if H is bounded.

To get rid of "the boundedness of H" consider for every positive integer k the truncated functions $H_k := \min(H, k) + \max(H, -k) - H$. They are bounded and fulfill the assumptions of Theorem 1 if k is large enough. Hence there are solutions u_k of (***) converging to the desired solution u for $|Du_k|_\Omega$ is uniformly bounded and

$$\int_\Omega \int_0^{v(x)} nH_k(x,t)\,dt\,dx \to \int_\Omega \int_0^{v(x)} nH(x,t)\,dt\,dx$$

for every $v \in H^{1,2}(\Omega) \cap C^0(\bar{\Omega})$.

In order to prove that u also minimizes the functional Φ we use the same trick as Giusti does in [7], Lemma 3.1.

For any positive integer k let

$$\eta_k := \max(0, 1 - kd(x))$$

and for $v \in H^{1,\infty}(\Omega) \cap \{v \geqq \psi\}$ let

(41) $$v_k := v + (u - v) \cdot \eta_k.$$

Then $v_k \in \tilde{\mathbf{K}} \cap C^0(\bar{\Omega})$ and v_k converges in $L^\infty(\Omega)$ to v. In Appendix III we show that

(42) $$\int_\Omega (1 + |Dv_k|^2)^{\frac{1}{2}}\,dx \to \int_\Omega (1 + |Dv|^2)^{\frac{1}{2}}\,dx + \int_{\partial\Omega} |u - v|\,d\mathscr{H}_{n-1}$$

which proves the assertion since Φ is continuous in $H^{1,1}(\Omega) \cap C^0(\bar{\Omega})$.

5. Continuous Boundary Data

Let us assume now that $f \in C^0(\bar{\Omega})$,

$$n\,|H(x, f(x))| \leqq (n-1)\,H_{n-1}(x), \qquad \forall x \in \partial\Omega,$$

and that there is a sequence $f_\varepsilon \in C^2(\bar{\Omega})$ which tends uniformly in Ω to f such that the Dirichlet problem

$$A u^*_{\tau,\varepsilon} + n\tau H(x, u^*_{\tau,\varepsilon}) = 0$$

$$u^*_{\tau,\varepsilon|\partial\Omega} = f_\varepsilon$$

has C^1 solutions $u^*_{\tau,\varepsilon}$ for every $\tau \in [0,1]$. Sufficient conditions (on H) guaranteeing this can be found in [25].

Furthermore let $\psi \in C^{0,1}(\Omega) \cap C^0(\bar{\Omega})$ be such, that $\psi_{|\partial\Omega} < f_{|\partial\Omega}$. Then ψ can be uniformly approximated by a sequence $\psi_\varepsilon \in C^2(\bar{\Omega})$ with $\psi_{\varepsilon|\partial\Omega} < f_{\varepsilon|\partial\Omega}$. Let u_ε be the corresponding solution of (**) which exists for sufficiently small ε since

$$n\,|H(x, f_\varepsilon(x))| \leqq (n-1)\,H_{n-1}(x) + \rho(\varepsilon), \qquad \forall x \in \partial\Omega$$

with $\rho(\varepsilon) \to 0$ (compare the considerations in Section 2.3).

Then we shall prove

Theorem 3. *The solutions u_ε converge uniformly to a function*

$$u \in H^{1,1}(\Omega) \cap C^{0,1}(\Omega) \cap C^0(\bar{\Omega}) \cap \{v_{|\partial\Omega} = f\} \cap \{v \geqq \psi\}$$

which solves the variational problem (∗∗∗∗).

Proof. From Lemma 3 we conclude that the u_ε's are a Cauchy sequence in $C^0(\bar{\Omega})$ which converges to some continuous function u. The proof will be finished if we can derive interior gradient estimates for u_ε

$$(43) \qquad |D u_\varepsilon|_{\Omega'} \leqq \text{const}(\Omega', |D\psi|_{\Omega'}), \qquad \forall \Omega' \subset\subset \Omega.$$

In order to do this observe that

$$(44) \qquad u_\varepsilon^* \leqq u_\varepsilon.$$

Hence there is a boundary strip Ω_r where r depends only on the modulus of continuity of ψ_ε and u_ε^*(and is in fact independent of ε) such that

$$(45) \qquad \psi_\varepsilon < u_\varepsilon \quad \text{in } \Omega_r$$

if ε is sufficiently small, hence

$$(46) \qquad A u_\varepsilon + nH(x, u_\varepsilon) = 0 \quad \text{in } \Omega_r.$$

If we now choose $\Omega' \subset\subset \Omega$ such that $\partial\Omega' \subset \Omega_r$, then

$$(47) \qquad |D u_\varepsilon|_{\partial\Omega'} \leqq L = L(\Omega', r)$$

where the constant depends only on dist$(\partial\Omega, \Omega')$ and r which follows at once from the *a priori estimates* of Ladyzhenskaya and Ural'tseva stated in [16]. Although they considered only the case $H = H(x)$, the proof carries over word for word to our case since $\frac{\partial H}{\partial t} \geqq 0$ (compare [16], inequality (2.22), p. 691).

As soon as the boundary estimates have been established, we know from our former considerations (see Section 2.1) that

$$(48) \qquad |D u_\varepsilon|_{\Omega'} \leqq C = C(L, |D\psi_\varepsilon|_{\Omega'}) = C(L, |D\psi|_{\Omega'}).$$

The rest of the proof is now standard and will be omitted.

Appendix I

Theorem A 1. *Let u be a C^1 solution of the variational inequality* (∗∗). *Then*

$$|Du|_{\partial\Omega} \leqq K_1 \cdot \max(|D\psi|_\Omega, |Du^*|_\Omega, 1).$$

Compare inequality (4) *in Section 2.*

Proof. Let us assume that $\partial\Omega$ is of class C^3. Then there is a boundary strip Ω_{d_0}, $d_0<\frac{1}{K}$, such that $d\in C^2(\bar{\Omega}_{d_0})$.

In view of [25], Theorem 10.1, the operator $A+nH(x,\cdot)$ is boundedly nonlinear with

$$\Phi(\rho)=\frac{\text{const}}{\rho} \tag{A 1}$$

(see [25] for the notations) where the constant depends only on $\sup|H|$.

According to Serrin we can find to every given pair of sufficiently large positive constants α, M a real number r, $0<r<d_0$, and a real function $h\in C^2([0,r])$ with

$$h(0)=0, \quad h(r)=M, \quad h'\geqq\alpha \tag{A 2}$$

such that $\delta(x):=f(x)+h(d(x))$ satisfies in Ω_r the inequality

$$A\delta+nH(x,\delta)\geqq 0. \tag{A 3}$$

From the definition of h it follows moreover that

$$|h''|\leqq\text{const} \quad \text{and} \quad h'\leqq\text{const}\cdot(\alpha+1). \tag{A 4}$$

If we choose $\alpha>|D\psi|_\Omega+|Df|_\Omega$ and $M\geqq\sup_\Omega u-\inf_\Omega f$, then

$$\delta\geqq\psi \quad \text{in } \Omega_r \tag{A 5}$$

and

$$\delta\geqq u \quad \text{on } \partial\Omega_r. \tag{A 6}$$

If $\partial\Omega$ is only of class C^2, we find by approximation (compare [25], Theorem 14.3) a sequence δ_ε the elements of which satisfy (A 3), (A 4), (A 5), and (A 6). A subsequence then converges to some function $\delta\in H^{2,\infty}(\Omega_r)$ that still satisfies these inequalities.

Appendix II

Let $a_i\in C^2(\mathbb{R}^n)$, $i=1,\dots,n$, be a strictly monotone vector field, i.e.

$$\frac{\partial a_i}{\partial p^j}\xi^i\xi^j>0, \quad \forall 0\neq\xi\in\mathbb{R}^n. \tag{A 7}$$

Then for every constant $C>0$ there is a *uniformly* monotone vector field $\tilde{a}_i$ which agrees with a_i on $|p|\leqq C$ and there are constants $0<\nu\leqq\mu<\infty$ such that

$$\nu|\xi|^2\leqq\frac{\partial\tilde{a}_i}{\partial p^j}\xi^i\xi^j\leqq\mu|\xi|^2, \quad \forall\xi\in\mathbb{R}^n. \tag{A 8}$$

Proof (Compare [4], Appendice). We choose positive functions $\psi, g \in C^3(\mathbb{R}_+)$ such that

$$\psi(t)=1 \quad \text{for } 0 \leqq t \leqq 2\cdot C,$$

$$\psi(t)=0 \quad \text{for } t \geqq 3\cdot C,$$

and g convex

$$g(t)=0 \quad \text{for } 0 \leqq t \leqq C,$$

$$g(t)=\alpha\cdot t \quad \text{for } t \geqq 2\cdot C, \alpha>0.$$

Let

$$\tilde{a}_i(p) := \psi(|p|^2)\cdot a_i(p) + k\cdot g'(|p|^2)\cdot p^i,$$

i.e., with $b_i(p) := g'(|p|^2)\cdot p^i$,

$$\tilde{a}_i(p) = \psi(|p|^2)\cdot a_i(p) + k\cdot b_i(p),$$

where b_i is a monotone vector field. Then $\dfrac{\partial \tilde{a}_i}{\partial p^j}$ equals

1. $\dfrac{\partial a_i}{\partial p^j} + k\dfrac{\partial b_i}{\partial p^j}$ if $|p| \leqq 2\cdot C$,

2. $2\cdot\psi'\cdot a_i\cdot p^j + \psi\cdot\dfrac{\partial a_i}{\partial p^j} + k\cdot\alpha\cdot\delta^{ij}$ if $2\cdot C \leqq |p| \leqq 3\cdot C$,

and

3. $k\cdot\alpha\cdot\delta^{ij}$ if $|p| \geqq 3\cdot C$.

Choosing k sufficiently large one verifies immediately the uniform monotonicity of $\tilde{a}_i$.

Appendix III

Proposition A 1. *For* $k\in\mathbb{N}$ *let*

$$\eta_k := \max(0, 1-kd(x))$$

and for $u, v \in H^{1,\infty}(\Omega)$ *put* $v_k := v + (u-v)\cdot\eta_k$. *Then*

$$\int_\Omega (1+|Dv_k|^2)^{\frac{1}{2}}\,dx \to \int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\,dx + \int_{\partial\Omega} |u-v|\,d\mathcal{H}_{n-1}.$$

Proof. Since

$$\int_\Omega (1+|Du-Dv|^2)^{\frac{1}{2}}\,dx \leqq \int_\Omega (1+|Du|^2)^{\frac{1}{2}}\,dx + \int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\,dx$$

it is enough to prove

Lemma A 1. *Let* $0 \leqq w \in H^{1,\infty}(\Omega)$. *Then*

$$\text{(A 9)} \qquad \lim \int_{1-kd(x)>0} kw\,dx = \int_{\partial\Omega} w\,d\mathcal{H}_{n-1}.$$

Proof. Without loss of generality we may assume that $\partial\Omega\in C^3$ and that $w\geqq\varepsilon>0$. Let $\Omega_k:=\{x\in\Omega\colon 1-kd(x)>0\}$. Then

$$\begin{aligned} k\cdot\int_{\Omega_k} w\,dx &= k\cdot\int_{\Omega_k} w\cdot D^i d\cdot D^i d\,dx \\ &= -k\cdot\int_{\Omega_k} D^i(wD^i d)\cdot d\,dx + k\cdot\int_{kd(x)=1} wD^i d\cdot d\cdot \nu_i\,d\mathscr{H}_{n-1} \end{aligned}$$

where ν_i are the components of the outward normal vector at $\partial\Omega_k$.

The first integral converges to zero while the second tends to

$$\int_{\partial\Omega} w\,d\mathscr{H}_{n-1}.$$

References

1. Bakel'man, I.Y.: Geometric problems in quasilinear elliptic equations. Russ. math. Surveys **25**, 45–109 (1971)
2. Brezis, H.: Equations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier **18**, 115–175 (1968)
3. Brezis, H.: Problèmes unilatereaux. J. Math. pur. appl. IX Ser. **51**, 1–168 (1972)
4. Brezis, H., Stampacchia, G.: Sur la regularité de la solution d'inéquations elliptiques. Bull. Soc. math. France **96**, 153–180 (1968)
5. Browder, F.E.: Problèmes non linéaires. Montréal: Les Presses de l'Université 1966
6. Giaquinta, M., Pepe, L.: Esistenza e regolarità per il problema dell' area minima con ostacoli in n variabili. Ann. Scuola norm. sup. Pisa, Sci. fis. mat., III. Ser. **25**, 481–507 (1971)
7. Giusti, E.: Superfici minime cartesiane con ostacoli discontinui. Arch. rat. Mech. Analysis **35**, 47–82 (1969)
8. Hartmann, P., Stampacchia, G.: On some non linear elliptic differential functional equations. Acta math. **115**, 271–310 (1966)
9. Hildebrandt, S.: On the regularity of solutions of two-dimensional variational problems with obstructions. Commun. pure appl. Math. **25**, 479–496 (1972)
10. Hildebrandt, S., Kaul, H.: Two-dimensional variational problems with obstructions, and Plateau's problem for H-surfaces in a Riemannian manifold. Commun. pure appl. Math. **25**, 187–223 (1972)
11. Jenkins, H., Serrin, J.: The Dirichlet problem for the minimal surface equation in higher dimensions. J. reine angew. Math. **229**, 170–187 (1968)
12. Kinderlehrer, D.S.: The coincidence set of solutions of certain variational inequalities. Arch. rat. Mech. Analysis **40**, 231–250 (1971)
13. Kinderlehrer, D.S.: Variational inequalities with lower dimensional obstacles. Israel J. Math. **10**, 339–348 (1971)
14. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and quasilinear elliptic equations. New York-London: Academic Press 1968
15. Ladyzhenskaya, O.A., Ural'tseva, N.N.: On some classes of nonuniformly elliptic equations. Sem. Math., V.A. Steklov math. Inst., Leningrad **11**, 47–53 (1970)
16. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. Commun. pure appl. Math. **23**, 677–703 (1969)
17. Lewy, H., Stampacchia, G.: On the regularity of the solution of a variational inequality. Commun. pure appl. Math. **22**, 153–188 (1969)

18. Lewy, H., Stampacchia, G.: On existence and smoothness of solutions of some non-coercive variational inequalities. Arch. rat. Mech. Analysis **41**, 241–253 (1971)
19. Lions, J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Paris: Dunod 1969
20. Miranda, M.: Frontiere minimali con ostacoli. Ann. Univ. Ferrara, n. Ser., Sez. VII, vol. **XVI** (1971)
21. Miranda, M.: Un principo di massimo forte per le frontiere minimale e una sua applicazione alla risoluzione del problema al contorno per l'equazione delle superfici di area minima. Rend. Sem. mat. Univ. Padova **XLV**, 355–366 (1971)
22. Morrey, C. B.: Multiple integrals in the calculus of variations. Berlin-Heidelberg-New York: Springer 1966
23. Necas, J.: Les méthodes directes en théorie des équations elliptiques. Paris: Masson et C[ie] 1967
24. Nitsche, J. C. C.: Variational problems with inequalities as boundary conditions or how to fashion a cheap hat for Giacometti's brother. Arch. rat. Mech. Analysis **35**, 83–113 (1969)
25. Serrin, J.: The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. Philos. Trans. roy. Soc. London, Ser. A **264**, 413–496 (1969)
26. Serrin, J.: Gradient estimates for solutions of nonlinear elliptic and parabolic equations. In: Contributions to nonlinear functional analysis, pp. 565–602 (E. H. Zarantonello editor). New York-London: Academic Press 1971
27. Stampacchia, G.: Équations elliptiques du second ordre à coefficients discontinus. Montréal: Les Presses de l'Université 1966
28. Stampacchia, G.: On some regular multiple integral problems in the calculus of variations. Commun. pure appl. Math. **16**, 383–421 (1963)
29. Stampacchia, G.: Variational inequalities. In: Theory and applications of monotone operators, pp. 101–192. A. Ghizetti (editor). Gubbio: Oderisi 1969
30. Stampacchia, G.: Regularity of solutions of some variational inequalities. In: Nonlinear Functional Analysis Proc. Symp. Pure Math. Vol. XVIII, part 1, Chicago 1968, pp. 271–281 Providence: Amer. Math. Soc. 1970
31. Tomi, F.: Minimal surfaces and surfaces of prescribed mean curvature spanned over obstacles. Math. Ann. **190**, 248–264 (1971)

Supplement

32. Giusti, E.: Boundary behavior of non-parametric minimal surfaces. Preprint
33. Giusti, E.: Minimal surfaces with obstacles. Preprint
34. Minty, G. J.: On the extension of Lipschitz, Lipschitz-Hölder continuous, and monotone function. Bull. Amer. math. Soc. **76**, 334–339 (1970)

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