0. Introduction

In a complete \((n+1)\)-dimensional manifold \(N\) we want to find closed hypersurfaces \(M\) of \textit{prescribed curvature}, so-called Weingarten hypersurfaces. To be more precise, let \(\Omega\) be a connected open subset of \(N\), \(f \in C^{\omega}(\Omega)\), \(F\) a smooth, symmetric function defined in the positive cone \(\Gamma_+ \subset \mathbb{R}^n\), then we look for a convex hypersurface \(M \subset \Omega\) such that

\[
F|_M = f(x) \quad \forall x \in M,
\]

where \(F|_M\) means that \(F\) is evaluated at the vector \((\kappa_i(x))\) the components of which are the principal curvatures of \(M\).

This is in general a fully nonlinear partial differential equation problem, which is elliptic if we assume \(F\) to satisfy

\[
\frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in} \quad \Gamma_+.
\]

Classical examples of curvature functions \(F\) are the elementary symmetric polynomials of order \(k\), \(H_k\), defined by

\[
H_k = \sum_{i_1 < \ldots < i_k} \kappa_{i_1} \cdots \kappa_{i_k} \quad , \quad 1 \leq k \leq n.
\]

\(H_1\) is the mean curvature \(H\), \(H_2\) is the scalar curvature—for hypersurfaces in Euclidean space—, and \(H_n\) is the Gaussian curvature \(K\).

For technical reasons it is convenient to consider the homogeneous polynomials of degree 1

\[
\sigma_k = H_k^{1/k}
\]

instead of \(H_k\). Then, the \(\sigma_k\)'s are not only monotone increasing but also \textit{concave}. Their \textit{inverses} \(\tilde{\sigma}_k\), defined through

\[
\tilde{\sigma}_k(\kappa_j) = \frac{1}{\sigma_k(\kappa^{-1}_j)}
\]
share these properties; a proof of this non-trivial result can be found in [11]. \( \tilde{\sigma}_1 \) is the so-called harmonic curvature \( G \), and, evidently, we have \( \tilde{\sigma}_n = \sigma_n \).

The general curvature functions we have in mind are defined in Section 1, we shall call those functions to be of class (K); special functions belonging to that class are the \( n \)-th root of the Gaussian curvature, the harmonic curvature, the inverse of the length of the second fundamental form, i.e.

\[
F(\kappa) = \frac{1}{\left( \sum_{i=1}^{n} \kappa_i^{-2} \right)^{1/2}}
\]

and, more generally, the inverses of the complete symmetric functions \( \gamma_k \), \( 1 \leq k \leq n \), homogeneous of degree 1 which are defined through

\[
\gamma_k(\kappa) = \left( \sum_{|\alpha| = k} \kappa^\alpha \right)^{1/k}
\]

Our main assumption in the existence proof is a barrier assumption.

**Definition 0.1.** Let \( M_1, M_2 \) be strictly convex, closed hypersurfaces in \( N \), homeomorphic to \( S^n \) and of class \( C^{4, \alpha} \) which bound a connected open subset \( \Omega \), such that the mean curvature vector of \( M_1 \) points outside of \( \Omega \) and the mean curvature vector of \( M_2 \) points inside of \( \Omega \). \( M_1, M_2 \) are barriers for \((F, f)\) if

\[
F|_{M_1} \leq f
\]

and

\[
F|_{M_2} \geq f
\]

**Remark 0.2.** In view of the Harnack inequality we deduce from the properties of the barriers that they do not touch, unless both coincide and are solutions of our problem. In this case \( \Omega \) would be empty.

Then we can prove

**Theorem 0.3.** Let the sectional curvature of \( N \) be non-positive, let \( F \) be of class (K), \( 0 < f \in C^{2, \alpha}(\overline{\Omega}) \) and assume that \( M_1, M_2 \) are barriers for \((F, f)\), then the problem

\[
F|_M = f
\]

has a strictly convex solution \( M \subset \overline{\Omega} \) of class \( C^{4, \alpha} \).

In a separate paper we shall consider closed Weingarten hypersurfaces in space forms for a class of curvature functions that includes the \( \sigma_k \)'s, cf. [8].
The existence of closed Weingarten hypersurfaces in $\mathbb{R}^{n+1}$ has been studied extensively in previous papers: the case $F = H$ by Bakelman and Kantor [1], Treibergs and Wei [13], the case $F = K$ by Oliker [12], Delanoë [4], and for general curvature functions by Caffarelli, Nirenberg and Spruck [3]. In all papers—except in [4]—the authors imposed a sign condition for the radial derivative of the right-hand side to prove the existence. This condition was necessary for two reasons, first to derive the a priori estimates for the $C^1$-norm and secondly to apply the inverse function theorem, i.e. the kernel of the linearized operator had to be trivial.

Without this condition the kernel is no longer trivial and the inverse function theorem or Leray-Schauder type arguments fail.

We therefore use the evolution method to approximate stationary solutions. But there is still the difficulty of obtaining the $C^1$-estimates: either one has to impose some artificial condition on the right-hand side, i.e. the condition depends on the choice of a special coordinate system, or one has to stay in the class of convex hypersurfaces where the $C^1$-estimates are a trivial consequence of the convexity, but then the preservation of the convexity has to be proved and this can only be achieved for special curvature functions like the Gaussian curvature, or by assuming $f$ to be concave, for details see [8].

The paper is organized as follows: In Section 1 we define the curvature functions of class (K) and give sufficient conditions for a curvature function to belong to that class, cf. Lemma 1.4.

In Section 2 we formulate the evolution problem and prove short-time existence.

In Section 3 we derive the evolution equation for some geometric quantities like the metric and the second fundamental form.

In Section 4 we demonstrate that the geometric setting can be lifted isometrically to the universal cover, so that without loss of generality we may assume that $\mathcal{N}$ is simply connected.

In Section 5 we prove that the flow stays in $\overline{\mathcal{O}}$.

In Section 6 we derive a priori estimates in the $C^1$-norm.

In Section 7 we obtain the parabolic equations satisfied by $h_{ij}$ resp. $v = \sqrt{1 + |Du|^2}$.

In Section 8 the $C^2$-estimates are derived, while in Section 9 the convergence to a smooth stationary solution is proved.

1. Curvature Functions

Let $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\overline{\Gamma}_+)$ be a symmetric function satisfying the condition

\begin{equation}
F_i = \frac{\partial F}{\partial \mathcal{K}_i} > 0;
\end{equation}

then, $F$ can also be viewed as a function defined on the space of symmetric, positive definite matrices $\mathcal{G}_+$, or to be more precise, at least in this section, let $(h_{ij}) \in \mathcal{G}_+$.
with eigenvalues $\kappa_i$, $1 \leq i \leq n$, then define $\hat{F}$ on $\mathcal{S}_+$ by

$$
\hat{F}(h_{ij}) = F(\kappa_i).
$$

(1.2)

It is well known, see e.g. [2], that $\hat{F}$ is as smooth as $F$ and that $\hat{F}^{ij} = \frac{\partial F}{\partial h_{ij}}$ satisfies

$$
\hat{F}^{ij} \xi_i \xi_j = \frac{\partial F}{\partial \kappa_i} |\xi_i|^2,
$$

(1.3)

where we use the summation convention throughout this paper unless otherwise stated.

Moreover, if $F$ is concave or convex then $\hat{F}$ is also concave or convex, i.e.

$$
\hat{F}^{ij, kl} \eta_{ij} \eta_{kl} \leq 0 \quad \text{or} \quad \hat{F}^{ij, kl} \eta_{ij} \eta_{kl} \geq 0
$$

(1.4)

for any symmetric $(\eta_{ij})$, where

$$
\hat{F}^{ij, kl} = \frac{\partial^2 \hat{F}}{\partial h_{ij} \partial h_{kl}}.
$$

(1.5)

An even sharper estimate is valid, namely,

**Lemma 1.1.** Let $F$, $\hat{F}$ be defined as above, then

$$
\hat{F}^{jj, kl} \eta_{ij} \eta_{kl} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \eta_{ij} \eta_{jj} + \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j} (\eta_{ij})^2,
$$

(1.6)

for any $(\eta_{ij}) \in \mathcal{S}$, where $\mathcal{S}$ is the space of all symmetric matrices and where $F_i = \frac{\partial F}{\partial \kappa_i}$. The second term on the right-hand side of (1.6) is non-positive if $F$ is concave and non-negative if it is convex and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

In [6, Lemma 2] it is shown that

$$
\left(\frac{\partial F}{\partial \kappa_i} - \frac{\partial F}{\partial \kappa_j}\right)(\kappa_i - \kappa_j) \leq 0
$$

(1.7)

if $F$ is concave and that the reverse inequality holds in case it is convex, hence the second term of the right-hand side in (1.6) is non-positive resp. non-negative.

The proof of (1.6) is very elementary but rather lengthy, so we shall only indicate the main steps.

We also want to mention that $F$ need not to be defined on the positive cone, any open, convex cone will do.

**Proof of Lemma 1.1.** First, let us remark that by continuity we may assume the eigen-
values of the matrix \( (h_{ij}) \), where \( F \) is evaluated, to be simple, if not, we can approximate \( (h_{ij}) \) by matrices with this property. Let \( \kappa_i \) be the eigenvalues of \( (h_{ij}) \) and \( \xi = (\xi_k) \) be the corresponding eigenvectors.

Let \( r \xi, s \xi \) be two eigenvectors, then we define the matrix \([r,s] \) through

\[
[r, s]_{ij} = \frac{1}{2} \{ r \xi_i s \xi_j + r \xi_j s \xi_i \}.
\]

We want to evaluate terms like

\[
(1.8)
\]

\[
\hat{F}^{ij, kl} [r_1, r_2]_{ij} [r_3, r_4]_{kl}.
\]

For simplicity we restrict the ranges of \( r_1, \ldots, r_4 \) to \{1, \ldots, 4\}, i.e. \([1, 1]\) represents a generic pair \([r_1, r_1]\) and \([1, 2]\) a generic pair \([r_1, r_2]\) with \( r_1 \neq r_2 \).

We shall consider several cases.

1. Case. Let us first consider a perturbation

\[
(1.10)
\]

\[
\tilde{h}_{ij} = h_{ij} + \varepsilon [1, 2]_{ij}.
\]

The new non-trivial eigenvalues are

\[
\tilde{\kappa}_1 = \frac{\kappa_1 + \kappa_2}{2} + \sqrt{\frac{(\kappa_1 - \kappa_2)^2}{4} + \frac{\varepsilon^2}{4}}
\]

\[
\tilde{\kappa}_2 = \frac{\kappa_1 + \kappa_2}{2} - \sqrt{\frac{(\kappa_1 - \kappa_2)^2}{4} + \frac{\varepsilon^2}{4}}
\]

Let \( R \) be the square root on the right-hand side, then

\[
(1.12)
\]

\[
F^{ij} \xi_i \xi_j = \hat{F}^{ij} [1, 2]_{ij} = \frac{\partial F}{\partial \kappa_1 R^4} \frac{1}{\varepsilon} \frac{\partial F}{\partial \kappa_2 R^4}
\]

and

\[
(1.13)
\]

\[
\hat{F}^{ij, kl} [1, 2]_{ij} [1, 2]_{kl} \big|_{\varepsilon = 0} = \frac{1}{2} \frac{F_1 - F_2}{\kappa_1 - \kappa_2} = 2 \frac{F_1 - F_2}{\kappa_1 - \kappa_2} (1, 2)_{12}^2
\]

\[
= \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j} (1, 2)_{ij}^2.
\]
2. Case. Choose

\[ \tilde{h}_{ij} = h_{ij} + \varepsilon[1, 2]_{ij} + \delta[3, 4]_{ij} \]

and conclude from (1.12)

\[ \hat{F}^{ij, kl}[1, 2]_{ij}[3, 4]_{kl} = 0. \]

3. Case. Choose

\[ \tilde{h}_{ij} = h_{ij} + \varepsilon[1, 2]_{ij} + \delta[3, 3]_{ij} \]

and use the same arguments to obtain

\[ \hat{F}^{ij, kl}[1, 2]_{ij}[3, 3]_{kl} = 0. \]

4. Case. Choose

\[ \tilde{h}_{ij} = h_{ij} + \varepsilon[1, 1]_{ij} \]

and deduce

\[ \hat{F}^{ij, kl}[1, 1]_{ij}[1, 1]_{kl} = \frac{\partial^2 F}{\partial \kappa_1 \partial \kappa_1} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}[1, 1]_{ij}[1, 1]_{jj}. \]

5. Case. Choose

\[ \tilde{h}_{ij} = h_{ij} + \varepsilon[1, 1]_{ij} + \delta[2, 2]_{ij} \]

and deduce

\[ \hat{F}^{ij, kl}[1, 1]_{ij}[2, 2]_{kl} = \frac{\partial^2 F}{\partial \kappa_1 \partial \kappa_2} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}[1, 1]_{ij}[2, 2]_{jj}. \]

6. Case. Choose

\[ \tilde{h}_{ij} = h_{ij} + \varepsilon[1, 1]_{ij} + \delta[1, 2]_{ij} \]

and deduce from (1.12)

\[ \hat{F}^{ij, kl}[1, 1]_{ij}[1, 2]_{kl} = 0. \]
7. Case. Choose

\[ \tilde{h}_{ij} = h_{ij} + \varepsilon[1, 2]_{ij} + \delta[1, 3]_{ij} . \]

The three non-trivial eigenvalues are the solutions of the cubic equation

\[ \frac{\delta^2}{4}(\kappa - \kappa_2) + \frac{\varepsilon^2}{4}(\kappa - \kappa_3) - (\kappa - \kappa_1)(\kappa - \kappa_2)(\kappa - \kappa_3) = 0 . \]

They depend smoothly on the parameters \( \varepsilon, \delta \) and we deduce from (1.25)

\[ \frac{\partial \kappa}{\partial \varepsilon} = \frac{\partial \kappa}{\partial \delta} = \frac{\partial^2 \kappa}{\partial \varepsilon \partial \delta} = 0 \]

at \( \varepsilon = \delta = 0 \), where \( \kappa \) represents any of the three eigenvalues. Hence, we obtain

\[ \hat{F}^{ij, kl}[1, 2]_{ij}[1, 3]_{kl} = 0 . \]

Now, let \( (\eta_{ij}) \in F \), then

\[ (\eta_{ij}) = \sum_{r, s} \eta_{rs}[r, s] \equiv \eta_{rs}[r, s] \]

and we conclude from the previous particular results

\[ \hat{F}^{ij, kl}_{ij} \eta_{ij} \eta_{kl} = \hat{F}^{ij, kl}_{ij}[r, s]_{ij}[p, q]_{kl} \eta_{rs} \eta_{pq} \]

\[ = \sum_{p, r} \hat{F}^{ij, kl}_{ij}[r, r]_{ij}[p, p]_{kl} \eta_{rr} \eta_{pp} + 2 \sum_{r \neq s} \hat{F}^{ij, kl}_{ij}[r, s]_{ij}[r, s]_{kl}(\eta_{rs})^2 \]

\[ = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \eta_{ii} \eta_{jj} + 2 \sum_{i \neq j} \left\{ \frac{F_i - F_j}{\kappa_i - \kappa_j} \sum_{r \neq s} (\eta_{rs}[r, s])^2 \right\} \]

\[ = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2 \]

**Definition 1.2.** A symmetric function \( F \in C^0(\Gamma_+) \cap C^{2, \alpha}(\Gamma_+) \) homogeneous of degree 1 is said to be of class (K) if

\[ F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+, \]

\[ F \text{ is concave}, \]

\[ F|_{\partial \Gamma_+} = 0 , \]
and

\[ F^{ij,kl} \eta_{ij} \eta_{kl} \leq 2F^{-1}(F^{ij} \eta_{ij})^2 - F^{ik}h^{jl} \eta_{ij} \eta_{kl} \quad \forall \eta \in \mathcal{F} \]

where \( F \) is evaluated at \((h_{ij}) \in \mathcal{F}_+ \) and \((\hat{h}^{ij}) = (h_{ij})^{-1} \).

We immediately deduce from (1.33)

**Lemma 1.3.** Let \( F \) be of class \((K)\), let \( \kappa_m \) be the largest eigenvalue of \((h_{ij}) \in \mathcal{F}_+ \), then for any \((\eta_{ij}) \in \mathcal{F} \) we have

\[ F^{ij,kl} \eta_{ij} \eta_{kl} \leq 2F^{-1}(F^{ij} \eta_{ij})^2 - \kappa_m^{-1} F^{ij} \eta_{im} \eta_{jm} , \]

where \( F \) is evaluated at \((h_{ij}) \).

For the rest of the paper we shall no longer distinguish between \( F \) and \( \tilde{F} \); instead we shall consider \( F \) to be defined both on \( \mathcal{F}_+ \) and \( \Gamma_+ \).

**Lemma 1.4.** Let \( F \in C^0(\bar{\mathcal{F}}_+) \cap C^{2,\alpha}(\Gamma_+) \) be symmetric, homogeneous of degree 1, monotone increasing and convex. Then, its inverse \( \tilde{F} \) is of class \((K)\).

**Proof.** We first show that \( \tilde{F} \) is concave.

We have \( \tilde{F}(\kappa_i) = \frac{1}{F(\kappa_i^{-1})} \) and hence

\[ \tilde{F}_i = F^{-2}F_i \kappa_i^{-2} , \]

\[ \tilde{F}_{ij} = 2F^{-3}F_iF_j \kappa_i^{-2} \kappa_j^{-2} - F^{-2}F_{ij} \kappa_i^{-2} \kappa_j^{-2} - 2F^{-2}F_i \kappa_i^{-3} \delta_{ij} , \]

and therefore, we obtain

\[ \tilde{F}_{ij} \xi^i \xi^j \leq 2F^{-3}(F_i \kappa_i^{-2} \xi^i)^2 - 2F^{-2}F_i \kappa_i^{-3} |\xi|^2 . \]

We estimate further

\[ F_i \kappa_i^{-2} \xi = F_i^{1/2} \kappa_i^{-1/2} F_i^{1/2} \kappa_i^{-3/2} \xi \leq (F_i \kappa_i^{-1})^{1/2} (F_i \kappa_i^{-3} |\xi|^2)^{1/2} \]

and conclude that the right-hand side of (1.37) is non-positive, where we used in addition the homogeneity of \( F \).

Next, we prove that \( \tilde{F} \) satisfies the condition (1.33). Thus, let \((h_{ij}) \in \mathcal{F}_+ \), \((\hat{h}^{ij}) = (h_{ij})^{-1} \) and
Then,

\[
\tilde{\tilde{F}}_{ij} = F^{-2} F_{rs} \tilde{h}^r i \tilde{h}^s j \frac{\partial h_{ab}}{\partial h_{ij}} = \tilde{\tilde{F}}_{rs} \frac{1}{2} \{ \tilde{\tilde{h}}^r i \tilde{\tilde{h}}^s j + \tilde{\tilde{h}}^r j \tilde{\tilde{h}}^s i \},
\]

(1.40)

The last term in this expression is equal to

\[
-2 F_{rs} \frac{1}{4} \{ \tilde{\tilde{h}}^r i \tilde{\tilde{h}}^s j + \tilde{\tilde{h}}^r j \tilde{\tilde{h}}^s i \} \tilde{\tilde{h}}^r i \tilde{\tilde{h}}^s j + \tilde{\tilde{h}}^r j \tilde{\tilde{h}}^s i \}
\]

(1.41)

The last term in this expression is equal to

\[
-2 F_{rs} \frac{1}{4} \{ \tilde{\tilde{h}}^r i \tilde{\tilde{h}}^s j + \tilde{\tilde{h}}^r j \tilde{\tilde{h}}^s i \} \tilde{\tilde{h}}^r i \tilde{\tilde{h}}^s j + \tilde{\tilde{h}}^r j \tilde{\tilde{h}}^s i \}
\]

(1.42)

and we deduce

\[
\tilde{\tilde{F}}_{ij} h_{ij} \eta_{ij} \eta_{kl} \leq 2 F^{-1} (\tilde{\tilde{F}}_{ij} \eta_{ij})^2 - 2 F \tilde{h}^i \eta_{ij} \eta_{kl} \quad \forall \eta \in \mathcal{F}.
\]

(1.43)

The remaining conditions which functions of class (K) have to satisfy are easily verified.

Remark 1.5.

(i) The mean curvature, the length of the second fundamental form and the \( \gamma_k \) satisfy the assumptions of the lemma, hence their inverses are of class (K).

For the mean curvature and the length of the second fundamental form the required properties are obvious, while the non-trivial result for the \( \gamma_k \) can be found in [11, p.105].

(ii) Straightforward computation shows that the n-th root of the Gaussian curvature is of class (K).

The preceding considerations are also applicable if the \( \kappa_i \) are the principal curvatures of a hypersurface \( M \) with metric \( (g_{ij}) \). \( F \) can then be looked at as being defined on the space of all symmetric tensors \( (h_{ij}) \) with eigenvalues \( \kappa_i \) with respect to the metric.
is then a contravariant tensor of second order. Sometimes, it will be convenient to circumvent the dependence on the metric by considering $F$ to depend on the mixed tensor

$$h^i_j = g^{ik}h_{kj}.$$  

Then

$$F^j_i = \frac{\partial F}{\partial h^j_i}$$

is also a mixed tensor with contravariant index $j$ and covariant index $i$.

## 2. The evolution problem

Let $N$ be a complete $(n+1)$-dimensional Riemannian manifold and $M$ a closed hypersurface. Geometric quantities in $N$ will be denoted by $(\tilde{g}_{\alpha\beta})$, $(\tilde{R}_{\alpha\beta\gamma\delta})$, etc., and those in $M$ by $(g_{ij})$, $(R_{ijkl})$, etc.. Greek indices range from 0 to $n$ and Latin from 1 to $n$; the summation convention is always used. Generic coordinate systems in $N$ resp. $M$ will be denoted by $(\tilde{x}_\alpha)$ resp. $(x^i)$. Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function $u$ on $N$, $(u_\alpha)$ will be the gradient and $(u_{\alpha\beta})$ the Hessian, but, e.g. the covariant derivative of the curvature tensor will be abbreviated by $\tilde{R}_{\alpha\beta\gamma\delta;\varepsilon}$. We also point out that

$$\tilde{R}_{\alpha\beta\gamma\delta;i} = \tilde{R}_{\alpha\beta\gamma\delta;x}^i x_i^\varepsilon$$

with obvious generalizations to other quantities.

In local coordinates $x^\alpha$ and $\xi^i$ the geometric quantities of the hypersurface $M$ are connected through the following equations

$$x^\alpha_{ij} = -h_{ij}^\alpha$$

the so-called Gauß formula. Here, and also in the sequel, a covariant derivative is always a full tensor, i.e.

$$x^\alpha_{ij} = x^\alpha_{ij} - \Gamma^k_{ij} x^\alpha_k + \tilde{\Gamma}^\alpha_{\beta\gamma} x^\beta_i x^\gamma_j.$$

The comma indicates ordinary partial derivatives.
In this implicit definition (2.2) the second fundamental form \( (h_{ij}) \) is taken with respect to \(-\nu\).

The second equation is the Weingarten equation

\[
(2.4) \quad \nu^\alpha_i = h^k_i x^\alpha_k,
\]

where we remember that \( \nu^\alpha_i \) is full tensor.

Finally, we have the Codazzi equation

\[
(2.5) \quad h_{ij,k} - h_{ik,j} = \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha_i x^\beta_j x^\gamma_k x^\delta_l,
\]

and the Gauß equation

\[
(2.6) \quad R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk} + \overline{R}_{\alpha\beta\gamma\delta} x^\alpha_i x^\beta_j x^\gamma_k x^\delta_l.
\]

We want to prove that the equation

\[
(2.7) \quad F = f
\]

has a solution. For technical reasons it is convenient to solve instead of (2.7) the equivalent equation

\[
(2.8) \quad \Phi(F) = \Phi(f)
\]

where \( \Phi \) is real function defined on \( R_+ \) such that

\[
(2.9) \quad \Phi > 0 \quad \text{and} \quad \dot{\Phi} \leq 0.
\]

For notational reasons let us abbreviate

\[
(2.10) \quad \tilde{f} = \Phi(f)
\]

To solve (2.8), we look at the evolution problem

\[
(2.11) \quad \dot{x} = - (\Phi - \tilde{f}) \nu, \quad x(0) = x_0
\]

where \( x_0 \) is an embedding of an initial strictly convex hypersurface \( M_0 \) diffeomorphic to \( S^n \), \( \Phi = \Phi(F) \), and \( F \) is evaluated for the principal curvatures of the flow hypersurfaces \( M(t) \), or, equivalently, we may assume that \( F \) depends on the second fundamental form \( (h_{ij}) \) and the metric \( (g_{ij}) \) of \( M(t) \); \( x(t) \) is the embedding for \( M(t) \).

This is a parabolic problem, so short-time existence is guaranteed—an exact proof is given below—and under suitable assumptions we shall be able to prove that the
solution exists for all time and that the velocity tends to zero if \( t \) goes to infinity.

Consider now a tubular neighbourhood \( \mathcal{U} \) of the initial hypersurface \( M_0 \), then we can introduce so-called normal Gaussian coordinates \( x^\alpha \), such that the metric in \( \mathcal{U} \) has the form

\[
(2.12) \quad ds^2 = dr^2 + \bar{g}_{ij}dx^i dx^j
\]

where \( r = x^0 \), \( \bar{g}_{ij} = \bar{g}_{ij}(r, x) \); here we use slightly ambiguous notation.

A point \( p \in \mathcal{U} \) can be represented by its signed distance from \( M_0 \) and its base point \( x \in M_0 \), thus \( p = (r, x) \).

Let \( M \subset \mathcal{U} \) be a hypersurface which is a graph over \( M_0 \), i.e.

\[
(2.13) \quad M = \{(r, x): r = u(x), x \in M_0\}.
\]

The induced metric of \( M \), \( g_{ij} \), can then be expressed as

\[
(2.14) \quad g_{ij} = \bar{g}_{ij} + u_i u_j
\]

with inverse

\[
(2.15) \quad g^{ij} = \bar{g}^{ij} - \frac{u^i u^j}{v^2} \frac{1}{v}
\]

where \( (\bar{g}^{ij}) = (\bar{g}_{ij})^{-1} \) and

\[
(2.16) \quad u^i = \bar{g}^{ij} u_j \quad v^2 = 1 + \bar{g}^{ij} u_i u_j
\]

The normal vector \( v \) of \( M \) then takes the form

\[
(2.17) \quad (v^\alpha) = v^{-1}(1, -u^i)
\]

if \( x^0 \) is chosen appropriately.

From the Gauß formula we immediately deduce that the second fundamental form of \( M \) is given by

\[
(2.18) \quad v^{-1} h_{ij} = -u_{ij} + \bar{h}_{ij},
\]

where
is the second fundamental form of the level surfaces \( \{ r = \text{const} \} \), and where the second covariant derivatives of \( u \) are defined with respect to the induced metric.

At least for small \( t \) the hypersurfaces \( M(t) \) are graphs over \( M_0 \) and the embedding vector looks like

\[
\begin{align*}
    x^0(t) &= u(t, x^i(t)) \\
    x^i(t) &= x^i(t, \xi^i)
\end{align*}
\]

where the \( \xi^i \) are local coordinates for \( M(t) \) independent of \( t \).

Furthermore,

\[
\begin{align*}
    \dot{x}^0 &= \frac{\partial u}{\partial t} + x^i u_i \\
    \dot{x}^i &= \Phi - \tilde{\chi} \end{align*}
\]

and from (2.11) we conclude

\[
\begin{align*}
    \dot{x}^0 &= - (\Phi - \tilde{\chi}) v^{-1} \\
    \dot{x}^i &= v^{-1} u^i (\Phi - \tilde{\chi})
\end{align*}
\]

hence, we obtain

\[
\frac{\partial u}{\partial t} = - (\Phi - \tilde{\chi}) v
\]

This is a scalar equation, which can be solved on a cylinder \([0, \varepsilon] \times M_0 \) for small \( \varepsilon \), if the principal curvatures of the initial hypersurface \( M_0 \) are strictly positive. The equation (2.22) for the embedding vector is then a classical ordinary differential equation of the form

\[
\dot{x} = \varphi(t, x)
\]

We have therefore proved

**Theorem 2.1.** The evolution problem (2.11) has a solution on a small time interval \([0, \varepsilon]\).

### 3. The evolution equations of some geometric quantities

In this section we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces \( M(t) \) evolve. All time derivatives are total derivatives.

**Lemma 3.1** (Evolution of the metric). The metric \( g_{ij} \) of \( M(t) \) satisfies the evolution
equation

\begin{equation}
\dot{g}_{ij} = -2(\Phi - \tilde{f})h_{ij} .
\end{equation}

\textbf{Proof.} Let $\xi^i$ be local coordinates for $M(t)$, then

\begin{equation}
g_{ij} = \bar{g}_{\alpha\beta} x^\alpha_i x^\beta_j
\end{equation}

and thus

\begin{equation}
\dot{g}_{ij} = 2\bar{g}_{\alpha\beta} x^\alpha_i x^\beta_j .
\end{equation}

On the other hand, differentiating

\begin{equation}
\dot{x}^\alpha = - (\Phi - \tilde{f})\nu^\alpha
\end{equation}

with respect to $\xi^i$ yields

\begin{equation}
\dot{x}_i^\alpha = - (\Phi - \tilde{f})_i \nu^\alpha - (\Phi - \tilde{f})\nu_i^\alpha
\end{equation}

and the desired result follows from the Weingarten equation.

\textbf{Lemma 3.2} (Evolution of the normal). The normal vector $\nu$ evolves according to

\begin{equation}
\dot{\nu} = \nabla_M (\Phi - \tilde{f}) = g^{ij} (\Phi - \tilde{f})_i x_j .
\end{equation}

\textbf{Proof.} Since $\nu$ is a unit normal vector we have $\dot{\nu} \in T(M)$. Furthermore, differentiating

\begin{equation}
0 = \langle \nu, x_i \rangle
\end{equation}

with respect to $t$, we deduce

\begin{equation}
\langle \nu, x_i \rangle = - \langle \nu, \dot{x}_i \rangle = (\Phi - \tilde{f})_i .
\end{equation}

\textbf{Lemma 3.3} (Evolution of the second fundamental form). The second fundamental form evolves according to

\begin{equation}
\dot{h}^i_j = (\Phi - \tilde{f})^i_j + (\Phi - \tilde{f}) h^k_i h^j_k + (\Phi - \tilde{f}) \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha_i \nu^\beta_j \nu^\gamma_k h^\delta^{ij}
\end{equation}

and

\begin{equation}
\dot{h}_{ij} = (\Phi - \tilde{f})_{ij} - (\Phi - \tilde{f}) h^k_i h_{kj} + (\Phi - \tilde{f}) \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha_i \nu^\beta_j \nu^\gamma_k h^\delta^{ij} .
\end{equation}
Proof. We use the Ricci identities to interchange the covariant derivatives of \( \xi^i \) with respect to \( t \)

\[
\frac{d}{dt}(\xi^i) = (\ddot{\xi}^i)_j - \dddot{\xi}^i \sigma_{j}\xi^\gamma x^\delta
\]

(3.11) \[ = g^{k\ell}(\Phi - \ddot{\Phi})_{\ell}x^i_{j} + g^{k\ell}(\Phi - \ddot{\Phi})_{\ell}x^i_{j} - \dddot{\xi}^i \sigma_{j}\xi^\gamma x^\delta \]

For the second equality we used (3.6).

On the other hand, in view of the Weingarten equation

(3.12) \[
\frac{d}{dt}(\xi^i) = \frac{d}{dt}(h^k_{ij}x^i) = h^k_{ij}x^i + h^k_{ij}x^i.
\]

Multiplying the resulting equation with \( g_{\alpha\beta}x^j \) we conclude

(3.13) \[
\dot{h}^i_{kj} - (\Phi - \ddot{\Phi})h^i_{kj} &= (\Phi - \ddot{\Phi})h^i_{ij} + (\Phi - \ddot{\Phi})\dddot{\xi}^i \sigma_{j}\xi^\gamma x^\delta
\]

or equivalently (3.9).

To derive (3.10), we differentiate

(3.14) \[
h_{ij} = h^k_{ij}g_{kj}
\]

with respect to \( t \) and use (3.3).

Lemma 3.4 (Evolution of \( \Phi - \ddot{\Phi} \)). The term \( \Phi - \ddot{\Phi} \) evolves according to the equation

(3.15) \[
(\Phi - \ddot{\Phi})' - \Phi F^{ij}(\Phi - \ddot{\Phi})_{ij} = \Phi F^{ij}_{\ell}h_{\ell k}h^k_{ij}(\Phi - \ddot{\Phi}) + \ddot{\Phi}v^\alpha(\Phi - \ddot{\Phi})
\]

+ \Phi F^{ij}R_{\alpha\beta\gamma\delta}\xi^\alpha x^\beta v^\gamma x^\delta(\Phi - \ddot{\Phi})

where

(3.16) \[
(\Phi - \ddot{\Phi})' = \frac{d}{dt}(\Phi - \ddot{\Phi})
\]

and

(3.17) \[
\Phi = \frac{d}{dr}\Phi(r).
\]

Proof. When we differentiate \( F \) with respect to \( t \) it is advisable to consider \( F \) as a function of the mixed tensor \( h^i_{j} \); then we obtain
The result now follows from (3.9) and (3.4).

4. Lifting of the problem to the universal cover

Let us first recall the definition of a strictly convex hypersurface: strictly convex means that the second fundamental form has a sign.

Then we define

**Definition 4.1.** Let $M$ be a strictly convex, closed hypersurface homeomorphic to $S^n$, then $v$ is the *outward* unit normal if

$$\langle \Delta_M x, v \rangle < 0. \tag{4.1}$$

This definition is consistent with the usual definition of the interior of a convex body bounded by $M$ if the sectional curvature of the ambient space $N$ is non-positive, cf. the considerations below.

In the sequel, we shall always assume that the second fundamental form of a strictly convex hypersurface is *positive* definite, i.e. the normal $v$ in the Gauss formula (2.2) is the outward normal.

In this section we want to show that the open set $\Omega$ bounded by the barriers $M_1$, $M_2$ is a *distinguished* open set, i.e. it can be isometrically lifted to the universal cover $\tilde{N}$.

By assumption, we have $k_N \leq 0$, thus the universal cover is diffeomorphic to $\mathbb{R}^{n+1}$, any geodesic in $\tilde{N}$ is minimizing, and the geodesic spheres around a point are strictly convex with respect to the inner normal, cf. [9, pp. 143–163].

Let $M \subset \tilde{N}$ be a strictly convex, closed hypersurface homeomorphic to $S^n$, then $\tilde{N} \setminus M$ has two components $\Omega_-$ and $\Omega_+$, one of which is bounded and simply connected. Let $\Omega_-$ be the bounded component; we call it the interior of $M$. Then we can prove

**Proposition 4.2.** $M$ is star-shaped with respect to any interior point, i.e. let $x_0 \in \Omega_-$, then any geodesic $\gamma$ emanating from $x_0$ intersects $M$ exactly once; let $\dot{\gamma}$ be the tangent vector at that point then

$$\langle \dot{\gamma}, v \rangle > 0, \tag{4.2}$$

where $v$ is the outward normal according to Definition 4.1.

**Proof.** First, we shall show that $-v$ points into $\Omega_-$.

Fix $x_0 \in \Omega_-$ and introduce geodesic polar coordinates $x^\alpha$ around $x_0$, i.e.
Let $\vec{x} \in M$ be such that 

$$
(4.4) \quad r(\vec{x}) = \sup_M r.
$$

Let $\xi^i$ be local coordinates for $M$ near $\vec{x}$, then we have in $\vec{x}$ 

$$
(4.5) \quad 0 = r_i = r_\alpha x_i^\alpha
$$

and 

$$
(4.6) \quad 0 \geq r_{ij} = r_{\alpha\beta} x_i^\alpha x_j^\beta + r_{\alpha} x_{ij}^\alpha.
$$

Here, $r_\alpha = v_\alpha$, and the first term on the right-hand side is the second fundamental form of the geodesic sphere through that point and hence positive definite, i.e. we have in view of the Gauß formula 

$$
(4.7) \quad h_{ij} \geq r_{\alpha\beta} x_i^\alpha x_j^\beta > 0,
$$

which proves that Definition 4.1 is consistent with the geometric notion of interior in this case.

Next, let $\vec{x} \in M$ be such that 

$$
(4.8) \quad d(x_0, \vec{x}) = \inf \{ d(x_0, x) : x \in M \}.
$$

Let $\gamma_{\vec{x}}$ be the geodesic connecting $x_0$ and $\vec{x}$, and $[x_0, \vec{x})$ be its half-open segment, then 

$$
(4.9) \quad [x_0, \vec{x}) \subset \Omega_-
$$

and 

$$
(4.10) \quad \langle \gamma_{\vec{x}}, v \rangle > 0;
$$

it is obvious, where the last expression has to be evaluated.

Now, let $x \in M$ be arbitrary and let $\Gamma \subset M$ be any curve connecting $\vec{x}$ and $x$ 

$$
(4.11) \quad \Gamma = \{ x(t) : 0 \leq t \leq 1 \}, \quad x(0) = \vec{x}.
$$

Define 

$$
(4.12) \quad \Lambda = \{ t : \langle \gamma_{x(t)}, v \rangle > 0 \text{ and } [x_0, x(t)) \subset \Omega_- \}.
$$
Then, $\Lambda \neq \emptyset$, since $0 \in \Lambda$, and we shall show that $\Lambda$ is both open and closed and hence that it coincides with the interval $[0, 1]$.

(iii) $\Lambda$ is open. If not, then, in view of the uniqueness of the geodesics, we would deduce the existence of a sequence $t_k$ converging to $t_0 \in \Lambda$ such that there are $x_k \in [x_0, x(t_k)] \cap M$ satisfying

$$x_k \to x(t_0) \quad \text{and} \quad \langle \dot{\gamma}_{x_k}, \nu \rangle \leq 0$$

(4.13) clearly a contradiction.

(iv) $\Lambda$ is closed. Let $t_k \in \Lambda$, $t_k \to t_0$ and $t_0 \notin \Lambda$. Then, there are two possibilities: first, suppose

$$[x_0, x(t_0)) \cap \Omega_+ \neq \emptyset.$$  

(4.14) This would imply that

$$[x_0, x(t_k)) \cap \Omega_+ \neq \emptyset$$

(4.15) for all but a finite number of $k$’s, a contradiction.

Thus, we have

$$[x_0, x(t_0)) \subset \Omega_-$$

(4.16) but

$$\langle \dot{\gamma}_{x(t_0)}, \nu \rangle = 0.$$  

(4.17) Now, choose Riemannian normal coordinates $x^\alpha$ in $x(t_0)$, then $\gamma_{x(t_0)}$ is contained in $T_{x(t_0)}M$. In a neighbourhood of $x(t_0)$ we can write $M$ as a graph over $T_{x(t_0)}M$

$$M = \{ x^0 = u(x^i) \}$$

(4.18) If we choose the coordinates such that in $x(t_0)$

$$\frac{\partial}{\partial x_0} = -\nu,$$

(4.19) we have in $x(t_0)$

$$h_{ij} = u_{ij}$$

(4.20)
where the derivatives of \( u \) are ordinary partial derivatives, i.e. the Euclidean Hessian of \( u \) is in a neighbourhood of \( x(t_0) \) positive definite, or equivalently, \( M \) is (locally) strictly convex in \( \mathbb{R}^{n+1} \). Thus, \( \Omega_- \) is (locally) completely contained in the half-space defined by \( T_{x_0}M \) contradicting (4.16) and the fact that \( \gamma_{x(t_0)} \) is contained in \( T_{x_0}M \).

**Corollary 4.3.** The interior of a strictly convex hypersurface \( M \subset N \) homeomorphic to \( S^n \) is convex.

Let us consider the domain \( \Omega \subset N \) bounded by the barriers \( M_1, M_2 \). Each barrier is homeomorphic to \( S^n, n \geq 2 \), hence each \( M_i \) has a tubular neighbourhood \( \mathcal{U}_i \) which is simply connected, i.e. there is a well defined lift to \( \tilde{N} \). More precisely, let

\[
\pi: \tilde{N} \to N
\]

be the covering map, then each \( \pi^{-1}(\mathcal{U}_i) \) consists of several disjoint copies such that the restriction of \( \pi \) to each copy is an isometry on \( \mathcal{U}_i \). Let \( \tilde{M}_i, \tilde{M}_i' \) be two generic elements of \( \pi^{-1}(M_i) \) and let \( \langle \tilde{M}_i \rangle, \langle \tilde{M}_i' \rangle \) be the corresponding open convex bodies, then we have

**Lemma 4.4.** Let \( \tilde{M}_i \neq \tilde{M}_i' \), then

\[
\langle \tilde{M}_i \rangle \cap \langle \tilde{M}_i' \rangle = \emptyset.
\]

**Proof.** \( \tilde{M}_i' \) is the image of \( \tilde{M}_i \) under a deck transformation which is an isometry, hence \( \langle \tilde{M}_i' \rangle \) is the image of \( \langle \tilde{M}_i \rangle \) under the same deck transformation and so the diameters of the convex bodies are the same.

Thus, if \( \tilde{M}_i \neq \tilde{M}_i' \) and

\[
\langle \tilde{M}_i \rangle \cap \langle \tilde{M}_i' \rangle \neq \emptyset
\]

we would have \( \langle \tilde{M}_i \rangle \) is strictly contained in \( \langle \tilde{M}_i' \rangle \) or vice versa, but this is impossible since the diameters are the same.

**Corollary 4.5.** For each \( \langle \tilde{M}_i \rangle \), \( \pi|_{\langle \tilde{M}_i \rangle} \) is an isometry. Let \( \langle M_i \rangle \) be the images, then

\[
\Omega = \langle M_2 \rangle \setminus \langle M_1 \rangle.
\]

**Proof.** The first claim is evident. To prove (4.24) we only have to show

\[
\Omega \supset \langle M_2 \rangle.
\]
Let
\[(4.26) \quad A = \Omega \cap \langle M_2 \rangle.\]

(i) \(A\) is non-empty, since the tubular neighbourhood \(\mathcal{U}_2\), previously defined, corresponds to a tubular neighbourhood \(\tilde{\mathcal{U}}_2\) of \(\tilde{M}_2\) and the notions interior and exterior relative to \(M_2\) resp. \(\tilde{M}_2\) are the same.

(ii) \(A\) is evidently open.

(iii) \(A\) is closed in \(\Omega\), for let
\[(4.27) \quad x_k \in A, \quad x_k \to x \in \Omega,\]
then we also know \(x \in \langle M_2 \rangle\) but \(x \not\in M_2\).

Thus, we have proved that \(A = \Omega\) since \(\Omega\) is connected.

Having laid so much groundwork in this context, let us also consider the case when the ambient space \(N\) is a space form with positive curvature, and let us show that the problem can still be lifted to the universal cover; without loss of generality we shall assume that \(\tilde{N} = S^{n+1}\). The basic definitions are the same as in the preceding considerations.

First, let us quote a result due to Do Carmo and Warner [5]

**Theorem 4.6.** Let \(M \subset S^{n+1}\) be a strictly convex hypersurface diffeomorphic to \(S^n\), then \(M\) is contained in an open hemisphere and is the boundary of a convex body.

Actually, Do Carmo and Warner’s result is slightly more general, but that is irrelevant in our context.

Since the shortest geodesic between two points in an open hemisphere is unique, Proposition 4.2 remains valid with the obvious restriction that only geodesics contained in the hemisphere are considered; the other former considerations also apply in this situation and we derive the following theorem

**Theorem 4.7.** Suppose that the universal cover of \(N\) either has non-positive sectional curvature or that it is \(S^{n+1}\), then the data of our problem \(\Omega, M_1, M_2\) and \(f\) can be lifted to the universal cover \(\tilde{N}\), and \(\Omega\) is the difference of two convex bodies, one of which is contained in the other.

In the following we shall therefore assume that \(N\) is simply connected.

5. **Barriers and a priori estimates in the \(C^0\)-norm**

The ambient space \(N\) has by assumption non-positive curvature, and in the preceding section we have shown that we may assume that \(N\) is simply connected. Therefore, we can introduce geodesic polar coordinates \((x^\alpha) = (r, x^i) = (r, x)\) around a point in \(\langle M_1 \rangle\) such that
and the second fundamental form $\bar{h}_{ij}$ of a geodesic sphere $\{r = \text{const}\}$ is uniformly positive definite in $\bar{\Omega}$.

Let $M(t)$ be a solution of the evolution problem (2.11) in a maximal time interval $I = [0, T^*)$ such that the hypersurfaces are strictly convex. Then, in view of Proposition 4.2 each $M(t)$ can be represented as a graph

\begin{equation}
M(t) = \{(r, x): r = u(t, x), \ x \in S_0\},
\end{equation}

where $S_0$ is a fixed geodesic sphere. The barriers $M_i$ are also graphs of positive functions $u_i$. We can then prove

**Lemma 5.1.** Choose as initial hypersurface $M_0$ either $M_1$ or $M_2$, then we have for the embedding vector $x = x(t)$

\begin{equation}
x(t) \in \bar{\Omega} \quad \forall t \in I.
\end{equation}

**Proof.** We shall only consider the case when $M_0 = M_1$. From Lemma 5.2 below we then obtain

\begin{equation}
\Phi - \tilde{f} \leq 0 \quad \forall t.
\end{equation}

For all $t$ the flow hypersurfaces are the graphs of functions $u(t)$. The equations (2.23) and (5.4) then yield

\begin{equation}
\frac{\partial u}{\partial t} \geq 0,
\end{equation}

i.e. the flow moves into $\Omega$ and

\begin{equation}
\inf_{S_0} u_1 \leq u \quad \forall t.
\end{equation}

Thus, let us assume that for $t = t_0 > 0$ it is the first time that the flow $M(t)$ touches $M_2$. Let $\bar{x} = x(t_0) = (u(t_0, \xi_0), \xi_0)$ be that point. In a neighbourhood $B_R = B_R(\xi_0)$ of $\xi_0$ define

\begin{equation}
\varphi = u_2 - u \geq 0 \quad u = u(t_0, \cdot).
\end{equation}

Now, because of (5.4) $u$ satisfies the inequality
and \( u_2 \) the reverse inequality

\[
\Phi - \tilde{f} \geq 0 \quad \text{in } B_R,
\]

since \( M_2 \) is an upper barrier. Here, we note, that the elliptic operator in the above inequalities is evaluated at \( u \) resp. \( u_2 \).

We then conclude—if we choose \( B_R \) small—, that \( \Phi \) satisfies a linearized elliptic inequality of the form

\[
- a^{ij} \varphi_{ij} + b^i \varphi_i + c \varphi \geq 0 \quad \text{in } B_R.
\]

Since \( \varphi \) is nonnegative, the Harnack inequality tells us that \( \varphi \) has to vanish identically in \( B_R \), i.e. if the flow touches \( M_2 \) at \( t = t_0 \), then \( M(t_0) = M_2 \) and \( M_2 \) is a solution of the problem (2.8). The flow is then stationary for \( t > t_0 \).

**Lemma 5.2.** Let \( M(t) \) be a solution of the evolution problem (2.11) defined on a maximal interval \( [0, T^*) \). As initial hypersurface \( M_0 \) we choose either \( M_1 \) or \( M_2 \); then we obtain

\[
\Phi - \tilde{f} \leq 0 \quad \forall t
\]

if \( M_0 = M_1 \), and

\[
\Phi - \tilde{f} \geq 0 \quad \forall t
\]

if \( M_0 = M_2 \).

**Proof.** In Lemma 3.4 we have shown that \( \Phi - \tilde{f} \) satisfies a linear parabolic equation; therefore, the proclaimed estimates follow from the maximum principle, since the inequalities are satisfied initially at \( t = 0 \).

**6. A priori estimates in the \( C^1 \)-norm**

The result of Lemma 5.1 implies

\[
\inf_{S_0} u_1 \leq u \leq \sup_{S_0} u_2.
\]

We shall show that \( Du \) and hence the induced metric of \( M(t) \) is uniformly bounded, cf. (2.14), as long as the \( M(t) \) remain convex.
Lemma 6.1. Let \( M = \text{graph } u|_{S_0} \) be a closed convex hypersurface represented in normal Gaussian coordinates then the quantity \( v = \sqrt{1 + |Du|^2} \) can be estimated by

\[
(6.2) \quad v \leq c(|u|, S_0, \bar{g}_{ij}).
\]

Proof. We have

\[
(6.3) \quad g_{ij} = \bar{g}_{ij} + u_i u_j, \quad \bar{g}_{ij} = \bar{g}_{ij}(u, x).
\]

Define

\[
(6.4) \quad \|Du\|^2 = g^{ij} u_i u_j, \quad |Du|^2 = \bar{g}^{ij} u_i u_j,
\]

then

\[
(6.5) \quad \|Du\|^2 = \frac{|Du|^2}{v^2}
\]

and

\[
(6.6) \quad v^{-2} = 1 - \|Du\|^2.
\]

Let \( \varphi \) be defined through

\[
(6.7) \quad \varphi = \log v + \lambda u,
\]

where the parameter \( \lambda \) will be chosen later, and let \( x_0 \in S_0 \) be such that

\[
(6.8) \quad \varphi(x_0) = \sup_{S_0} \varphi.
\]

Then, we have at \( x_0 \)

\[
(6.9) \quad 0 = \varphi_i = v^{-1} v_i + \lambda u_i
\]

or

\[
(6.10) \quad 0 = v^{-1} v_i u^i + \lambda \|Du\|^2.
\]

Differentiating \( v \), we obtain

\[
(6.11) \quad v_i = u_{ij} u^j v^3,
\]

i.e.
We then conclude from (2.18)

$$v_i u^i = u_{ij} u^i u^j v^3.$$  

(6.13) 

$$0 = -h_{ij} u^i u^j v^2 + \lambda \|Du\|^2 + \overline{h}_{ij} u^i u^j v^2.$$  

We now observe that

(6.14) 

$$u^i = g^{ij} u_j = \overline{g}^{ij} u_j v^{-2}.$$  

Thus, let $\overline{\kappa}$ be an upper bound for the eigenvalues of $\overline{h}_{ij}$, then

(6.15) 

$$\overline{h}_{ij} u^i u^j v^2 \leq \overline{\kappa} v^{-2} |Du|^2$$  

and we deduce in view of (6.13)

(6.16) 

$$0 \leq (\overline{\kappa} + \lambda) |Du|^2 v^{-2}$$  

at $x_0$.

Let us now choose $\lambda = -\overline{\kappa} - \varepsilon$, then we conclude $Du = 0$ and

(6.17) 

$$\varphi \leq \varphi(x_0) = \lambda u(x_0),$$  

or equivalently

(6.18) 

$$v \leq e^{\lambda \{u(x_0) - u\}} \leq e^{\lambda \{\sup u - \inf u\}}.$$  

By letting $\varepsilon$ tend to zero we finally conclude

(6.19) 

$$v \leq e^{\overline{\kappa} \{\sup u - \inf u\}}.$$
7. The evolution equations for $h_{ij}$ and $v$

Let us first derive the parabolic equation for the second fundamental form.

**Lemma 7.1.** Let $M(t)$ be a solution of the problem (2.11), then the second fundamental form satisfies

$$h_{ij} - \Phi F^{kl} h_{ij;kl} = \Phi F^{kl} h_{kr} h_{ij} + (\Phi - \tilde{f}) h_{ij} - \Phi F h_{ij} - f_{\alpha \beta} x_i x_j + f_{a} v^\alpha h_{ij}$$

$$+ \Phi F_{i} F_{j} + \Phi F^{kl, rs} h_{kri} h_{rjs} + (\Phi - \tilde{f}) R_{\alpha \beta \gamma \delta} v^\alpha x_i v^\gamma x_j$$

(7.1)

$$+ 2 \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_r x_i x_j x_k h_{ij} - \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_r x_i x_j x_k h_{ij}$$

$$- \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_r x_i x_j x_k h_{ij} + \Phi F^{kl, rs} R_{\alpha \beta \gamma \delta} x_r x_i x_j x_k h_{ij}$$

$$- \Phi F R_{\alpha \beta \gamma \delta} v^\alpha x_i v^\gamma x_j + \Phi F^{kl} R_{\alpha \beta \gamma \delta} v^\alpha x_i v^\gamma x_j$$

$$+ \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_i x_k x_j v^\alpha x_i v^\gamma x_j + v^\alpha x_i x_k x_j x_i$$

(7.2)

**Proof.** We start with equation (3.10) and shall evaluate the term

$$\Phi_{i} = \Phi F_{i} = \Phi F^{kl} h_{kl,i}$$

and

$$\Phi_{ij} = \Phi F^{kl} h_{kl,ij} + \Phi F^{kl} h_{kl,i} F_{rjs} + \Phi F^{kl, rs} h_{kljs} h_{rs,j}.$$  

(7.3)

(7.4)

Next, we replace $h_{kl;ij}$ by $h_{ij;kl}$. Differentiating the Codazzi equation

$$h_{kl;ij} = h_{ik;lj} + R_{\alpha \beta \gamma \delta} x_{k} x_{l} x_{i} x_{j}$$

we obtain

$$h_{kl;ij} = h_{ik;lj} + R_{\alpha \beta \gamma \delta} x_{k} x_{l} x_{i} x_{j}$$

$$+ R_{\alpha \beta \gamma \delta} \{ v_{j} x_{k} x_{i} x_{l} x_{j} + v_{k} x_{l} x_{i} x_{j} x_{l} + v_{l} x_{k} x_{i} x_{j} x_{k} + v_{k} x_{l} x_{i} x_{j} x_{l} \}.$$  

(7.5)

(7.6)

We now use the Ricci identities

$$h_{ik;lj} = h_{ik;jl} + h_{ak} R_{ilj}^a + h_{al} R_{klj}^a$$

and differentiate once again the Codazzi equation

$$h_{ik;lj} = h_{ik;jl} + h_{ak} R_{ilj}^a + h_{al} R_{klj}^a.$$  

(7.7)
to replace $h_{kl:ij}$ by $h_{ij:kl}$.

To replace $\tilde{f}_{ij}$ we use the chain rule

\begin{align}
\tilde{f}_i &= \tilde{f}_\alpha x_i^\alpha \\
\tilde{f}_{ij} &= \tilde{f}_\alpha \beta x_i^\alpha x_j^\beta + \tilde{f}_\alpha x_{ij}^\alpha
\end{align}

Then, using the Gauß equation and Gauß formula, the symmetry properties of the Riemann curvature tensor and the homogeneity of $F$, i.e.

\begin{align}
F &= F^{kl} h_{kl}
\end{align}

we deduce from (3.10) the equation (7.1).

Since the mixed tensor $h_i^j$ is a more natural geometric object, let us look at the evolution equation for $h_i^j$ that can be derived from (3.9).

**Lemma 7.2.** The evolution equation for $h_i^j$ (no summation over $i$) has the form

\begin{align}
\dot{h}_i^j - \Phi F^{kl} h_{kl}^j &= \Phi F^{kl} h_k^r h_l^i + (\Phi - \tilde{f}) h_k^l h_i^j - \Phi F^{kl} h_k^j h_i^l - \tilde{f}_\alpha \beta x_i^\alpha x_j^\beta g_{ki} + \tilde{f}_\alpha \beta x_i^\alpha x_j^\beta g_{ki} \\
&+ \Phi F_l F^i + \Phi F^{kl} h_{kl}^j h_{rs}^i g_{mi} + (\Phi - \tilde{f}) R_{\alpha \beta \gamma \delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta g_{mi} \\
&+ 2 \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_k^\alpha x_i^\beta x_l^\delta x_m^\gamma g_{mi} h_r^l - 2 \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_k^\alpha x_i^\beta x_l^\delta x_m^\gamma g_{mi} h_r^l \\
&+ \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_k^\alpha x_i^\beta x_l^\delta x_m^\gamma g_{mi} h_r^l - \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_i^\alpha x_k^\beta x_j^\gamma x_l^\delta g_{mi} \\
&+ \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_i^\alpha x_k^\beta x_j^\gamma x_l^\delta g_{mi} h_r^l + \Phi F^{kl} R_{\alpha \beta \gamma \delta} x_i^\alpha x_k^\beta x_l^\gamma x_m^\delta g_{mi} h_r^l
\end{align}

Let $M$ be a hypersurface that can be written as a graph in a normal Gaussian coordinate system $(x^\alpha) = (r, x^i)$. From the relation (2.17) we conclude

\begin{align}
v = \sqrt{1 + |Du|^2} = (r^\alpha v^\alpha)^{-1}.
\end{align}

For the hypersurfaces $M(t)$ defined by the flow (2.11) we obtain

**Lemma 7.3.** Consider the flow in a normal Gaussian coordinate system where the $M(t)$ can be written as graphs of a function $u(t)$. Then $v$ satisfies the evolution equation
Proof. Differentiate (7.12) to obtain

\[ \dot{v} - \Phi F^{ij}\dot{v}_{ij} = -\Phi F^{ij}h_{ik}h_{j}^{k}v - 2v^{-1}\Phi F^{ij}v_{ij} + r_{\alpha\beta}v^{\alpha}v^{\beta}[(\Phi - \tilde{f}) - \Phi F]v^2 \]

(7.13)

\[ + \Phi F^{ij}R_{\alpha\beta\gamma\delta}v^{\alpha}\beta x_{i}^{\gamma}x_{j}^{\delta} + e_{e}^{em}e_{mk}v^2 + 2\Phi F^{ij}r_{\alpha\beta}h_{i}^{k}h_{j}^{\alpha}v^2 \]

\[ + \Phi F^{ij}r_{\alpha\beta}\gamma v^{\alpha}\beta x_{i}^{j}v^2 + \tilde{f}_{\alpha}^{\gamma}x_{m}^{\gamma}r_{\beta}x_{k}^{\alpha}v^2 \]

We also have to calculate the time derivative of \( v \)

\[ \dot{v} = -v^2\{r_{\alpha\beta}v^{\alpha}x_{i}^{\alpha} + r_{\alpha}v^{\alpha}\} \]

(7.14)

\[ v_{ij} = 2v^{-1}v_{i}v_{j} - v^2\{r_{\alpha\beta}v^{\alpha}\beta x_{i}^{\alpha} + r_{\alpha}v^{\alpha}\} \]

(7.15)

where we have used (3.6).

Now, by inserting (7.15) and (7.16) in the left-hand side of (7.13) and simplifying the resulting expression with the help of the Weingarten and Codazzi equations we arrive at the desired conclusion.

Lemma 7.4. For convex hypersurfaces which stay in a compact domain we have

\[ \left| F^{ij}r_{\alpha\beta}h_{i}^{k}h_{j}^{\alpha} \right| \leq cF \]

(7.17)

Proof. Choose a coordinate system \( \xi^{i} \) such that in a fixed but arbitrary point in \( M \)

\[ g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_{ij} \delta_{ij} \]

(7.18)

Then,

\[ \left| F^{ij}r_{\alpha\beta}h_{i}^{k}h_{j}^{\alpha} \right| \leq \sum_{i} \left| F^{ij}h_{i}^{k} \right| \sup |D^2 r| = F^{ij}h_{ij} \sup |D^2 r| = F \sup |D^2 r| . \]

(7.19)

8. A priori estimates in the \( C^2 \)-norm

Let \( M(t) \) be a solution of the evolution problem (2.11) with initial hypersurface \( M_0 = M_1 \) defined in a maximal time interval \( I = [0, T^*) \). We also assume that \( F \) is of class (K) as in Definition 1.2, and we choose \( \Phi(t) = -t^{-1} \). Let \( M(t) \) be represented as the graph of a function \( u \) in geodesic polar coordinates, then, from (2.11) we deduce
and taking the relation (2.18) into account we conclude

\[ \dot{u} - \Phi F^{ij} u_{ij} = -(\Phi - \tilde{f}) v^{-1} + \Phi F v^{-1} - \Phi \tilde{F}^{ij} \tilde{h}_{ij}. \]

Here, the $\tilde{h}_{ij}$ are uniformly positive definite in $\tilde{\Omega}$, i.e. we can estimate

\[ F^{ij} \tilde{h}_{ij} \geq c F^{ij} g_{ij} \geq c F(1, \ldots, 1) \]

with a positive constant $c$. The second estimate in (8.3) follows from

**Lemma 8.1.** Let $F \in C^2(\Gamma_+)$ be homogeneous of degree 1, monotone increasing and concave, then

\[ F^{ij} g_{ij} \geq F(1, \ldots, 1). \]

A proof can be found in [14, Lemma 3.2].

We first note that in view of Lemma 5.2 we know that

\[ \Phi \leq \tilde{f} \quad \text{or} \quad F \leq f \]

and that in view of the results in Section 5 the flow stays in the compact set $\tilde{\Omega}$. Furthermore, due to the choice of $\Phi$ and the condition (1.32) the $M(t)$ are strictly convex during the evolution and, hence, $Du$ uniformly bounded.

An estimate for the second derivatives of $u$ is given in

**Lemma 8.2.** Let $F$ be of class (K), then the principal curvatures of the evolution hypersurfaces $M(t)$ are uniformly bounded.

Proof. Let $\phi$ be defined by

\[ \phi = \sup \{ h_{ij} \eta^i \eta^j : \|\eta\| = 1 \} \]

and $w$ by

\[ w = \log \phi + \log v + \lambda u \]

where $\lambda$ is a large positive parameter. We claim that $w$ is bounded.

Let $0 < T < T^*$, and $x_0 = x(t_0), 0 < t_0 \leq T$, be a point in $M(t_0)$ such that

\[ \sup_{M_0} w < \sup \{ \sup_{M(t)} w : 0 < t \leq T \} = w(x_0). \]
We then can introduce a Riemannian normal coordinate system $\xi^i$ at $x_0 = M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h_n^n. \quad (8.9)$$

Let $\eta = (\eta^i)$ be the contravariant vector defined by

$$\eta = (0, \ldots, 0, 1) \quad (8.10)$$

and set

$$\tilde{\varphi} = \frac{h_{ij} \eta^i \eta^j}{g_{ij} \eta^i \eta^j}. \quad (8.11)$$

$\tilde{\varphi}$ is well defined in a neighbourhood of $(t_0, \xi_0)$.

Now, define $\tilde{w}$ by replacing $\varphi$ by $\tilde{\varphi}$ in (8.7); then $\tilde{w}$ assumes its maximum at $(t_0, \xi_0)$. Moreover, at $(t_0, \xi_0)$ we have

$$\dot{\tilde{\varphi}} = h_n^n \quad (8.12)$$

and the spacial derivatives do also coincide; in short, $\tilde{\varphi}$ satisfies at $(t_0, \xi_0)$ the same differential equation (7.11) as $h_n^n$. For the sake of greater clarity, let us therefore treat $h_n^n$ like a scalar and pretend that $w$ is defined by

$$w = \log h_n^n + \log v + \lambda u. \quad (8.13)$$

At $(t_0, \xi_0)$ we have $\tilde{w} \geq 0$, and, in view of the maximum principle, we deduce from (7.11), (7.13) and (8.2)

$$0 \leq -F^{-1} h_n^n - c(\Phi - \tilde{f}) + c + \Phi F^{ij} g_{ij} c - \lambda(\Phi - \tilde{f}) v^{-1} + \lambda F^{-1} v^{-1} - \lambda \Phi F^{ij} \tilde{h}_{ij} - \Phi F^{ij} (\log v)_i (\log v)_j + \Phi F^{ij} (\log h_n^n)_i (\log h_n^n)_j + \{\Phi F_n F_n + \Phi F^{kl} h_{kl,n} h_{rs,m} g^{mn}\} h_n^n \quad (8.14)$$

where we have estimated bounded terms by a positive constant $c$, assumed that $h_n^n \geq 1$, and also observed (8.5).

Now, the last term in the preceding inequality is estimated from above by
\[ -(h^n_n)^{-2} \Phi F^{ij} h_{in;n} h_{jn;m} g^{mn}, \]

cf. Lemma 1.3, in view of the choice of $\Phi$. Moreover, because of the Codazzi equation we have
\[ h_{in;n} = h_{nn;i} + \bar{R}_{\alpha\beta\gamma\delta} \gamma_{\alpha\beta} \gamma_{\gamma\delta}, \]

and hence, when we abbreviate the curvature term by $\bar{R}_i$, we conclude that (8.15) is equal to
\[ -(h^n_n)^{-2} \Phi F^{ij} (h^n_n + \bar{R}_i)(h^n_j + \bar{R}_j). \]

Thus, the terms in (8.14) containing the derivatives are estimated from above by
\[ -\Phi F^{ij} (\log v)_i (\log v)_j - 2(h^n_n)^{-1} \Phi F^{ij} (\log h^n_n) \bar{R}_j. \]

Moreover, at $\xi_0$ $Dw$ vanishes, i.e.
\[ D\log h^n_n = -D\log v - \lambda Du \]

and (8.18) is further estimated from above by
\[ (h^n_n)^{-1} c\lambda \Phi F^{ij} g_{ij}, \]

where we assumed $\lambda \geq 1$.

Summarizing, we deduce from (8.14)
\[ 0 \leq \{-F^{-1} h^n_n + c + \lambda F^{-1} v^{-1} - \lambda (\Phi - \bar{\Phi}) v^{-1} - c(\Phi - \bar{\Phi}) \} \]
\[ + \left\{ c\Phi F^{ij} g_{ij} + (h^n_n)^{-1} c\lambda \Phi F^{ij} g_{ij} - \lambda \Phi F^{ij} \bar{h}_{ij} \right\}. \]

We now choose $\lambda$ very large and assume that
\[ h^n_n > \mu, \]

where $\mu$ is also large, and we deduce that the terms involving $\Phi$ sum up to something negative if we choose $\mu$ large. Thus, we conclude that we are left with
\[ 0 \leq -F^{-1} h^n_n + c + \lambda F^{-1} v^{-1} - \lambda (\Phi - \bar{\Phi}) v^{-1} - c(\Phi - \bar{\Phi}), \]
i.e. \( h^n_n \) and hence \( w \) is a priori bounded at \( (t_0, \xi_0) \).

To complete the a priori estimates we have to show that the principal curvatures can be bounded from below by a positive constant, or equivalently, since \( F \) vanishes on \( \partial \Gamma^+ \), that \( F \) is bounded from below by a positive constant.

**Lemma 8.3.** Let \( F \) be of class \( (K) \), then there is a positive constant \( \varepsilon_0 \) such that

\[
(8.24) \quad \varepsilon_0 \leq F
\]
during the evolution.

**Proof.** Consider the function

\[
(8.25) \quad w = -(\Phi - \tilde{f}) + \lambda u,
\]
where \( \lambda \) is large. Let \( 0 < T < T^* \) and suppose

\[
(8.26) \quad \sup_{M_0} w < \sup \{ \sup_{M(t)} w : 0 \leq t \leq T \}.
\]

Then, there is \( x_0 = x(t_0), 0 < t_0 \leq T \), such that

\[
(8.27) \quad w(x_0) = \sup \{ \sup_{M(t)} w : 0 \leq t \leq T \}.
\]

From (3.15), (8.2) and the maximum principle we then infer

\[
(8.28) \quad 0 \leq -\Phi F^{ij} h_{ik} k_j^k (\Phi - \tilde{f}) - \Phi F^{ij} \tilde{R}_{\alpha \beta \gamma \delta} v^\alpha x_i^\beta v^\gamma x_j^\delta (\Phi - \tilde{f})
- \tilde{f} \alpha v^\alpha (\Phi - \tilde{f}) - \lambda (\Phi - \tilde{f}) v^{-1} + \lambda \Phi F v^{-1} - \lambda \Phi F^{ij} \tilde{h}_{ij}
\]

Let \( \kappa \) be an upper bound for the principle curvatures, then the first term on the right-hand side can be estimated by

\[
(8.29) \quad -\Phi F^{ij} h_{ij} \kappa (\Phi - \tilde{f}) = -\Phi F \kappa (\Phi - \tilde{f}) = -\kappa F^{-1} (\Phi - \tilde{f})
\]
the second term is non-positive because \( K_N \leq 0 \); from the remaining terms the last one is negative and has as dominating factor \( \lambda \Phi \), hence \( F \) cannot be too small at \( x_0 \) and the lemma is proved.

**9. Convergence to a stationary solution**

We are now ready to prove Theorem 0.3. Let \( M(t) \) be the flow with initial hypersurface \( M_0 = M_1 \). Let us look at the scalar version of the flow (2.23)
This is a scalar parabolic differential equation defined on the cylinder

\[ \frac{\partial u}{\partial t} = -(\Phi - \tilde{f}) v. \]

with initial value \( u_0 = u_1 \in C^{4, \alpha}(S_0) \). \( S_0 \) is a geodesic sphere equipped with the induced metric. In view of the a priori estimates we have proved in the preceding sections, we know that

\[ |u|_{2,0,S_0} \leq c \]

and

\[ F \text{ is uniformly elliptic in } u \]

independent of \( t \). Furthermore, \( F \) is concave and thus, we can apply the regularity results in Krylov [10, Chapter 5.5] to conclude that uniform \( C^{2, \alpha} \)-estimates are valid, leading further to uniform \( C^{4, \alpha} \)-estimates in view of the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e. \( T^* = \infty \).

Now, integrate (9.1) and observe that the right-hand side is nonnegative to obtain

\[ u(t, x) - u(0, x) = \int_0^t (\Phi - \tilde{f}) v \geq -\int_0^t (\Phi - \tilde{f}) , \]

i.e.

\[ \int_0^\infty |\Phi - \tilde{f}| < \infty \quad \forall x \in S_0. \]

Thus, for any \( x \in S_0 \) there is a sequence \( t_k \to \infty \) such that \( (\Phi - \tilde{f}) \to 0 \).

On the other hand, \( u(\cdot, x) \) is monotone increasing and therefore

\[ \lim_{t \to \infty} u(t, x) = \tilde{u}(x) \]

exists and is of class \( C^{4, \alpha}(S_0) \) in view of the a priori estimates. We finally deduce that \( \tilde{u} \) is a stationary solution of our problem and that

\[ \lim_{t \to \infty} (\Phi - \tilde{f}) = 0. \]
References


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