Randomized local search

Local search guided by unsatisfied clauses

Consider a Boolean formula $\varphi$ in $n$ variables in 3-CNF.

Suppose that $\varphi$ is satisfiable and fix any satisfying assignment $\sigma_0$.

Then given any assignment $\sigma$ and any clause of $\varphi$ that is not made true by $\sigma$, the assignments $\sigma$ and $\sigma_0$ must differ on at least one of the variables in this clause.

Consequently, when choosing a variable uniformly at random from any fixed unsatisfied clause and flipping the value of the assignment $\sigma$ for this variable, with probability at least $\frac{1}{3}$ one obtains an assignment that is strictly closer to $\sigma_0$ than $\sigma$.

In particular, when starting at an assignment $\sigma$ where $d(\sigma, \sigma_0) \leq j$ and iterating such random flips $j$ times or more, one reaches $\sigma_0$ or another satisfying assignment with probability of at least $3^{-j}$.

Randomized local search

Local search and random walks

Consider an invocation LocalSearch($\varphi$, $\sigma$, $r$) such that $\varphi$ has a satisfying assignment $\sigma_0$ that has distance of $j \leq r$ from $\sigma$.

The probability of finding some satisfying assignment during this invocation are at least as good as reaching the origin in a one-dimensional random walk of length $r$ that starts at position $j$ and moves left and right with probability 1/3 and 2/3, respectively.

In the exursus on random walks below, it is demonstrated that the origin is reached in such a random walk with probability of at least $2^{-j}$ in case $r \geq 27j$.

By invoking LocalSearch with $r = 27n$, the probability of finding a satisfying assignment is at least $2^{-j}$, provided that the initial assignment has distance at most $j$ to some satisfying assignment.

Randomized local search

Algorithm LocalSearch($\varphi$, $\sigma$, $r$) (Randomized Local Search)

Input: A Boolean formula $\varphi$ in $n$ variables in 3-CNF.

An assignment $\sigma \in \{0, 1\}^n$ and a natural number $r$.

While $r \geq 1$ and $\varphi(\sigma) = \text{false}$

Pick the least clause of $\varphi$ that is not made true by $\sigma$.

Pick a variable in this clause uniformly at random.

Flip the value of the assignment $\sigma$ at this variable.

Let $r = r - 1$.

Output: The modified assignment $\sigma$.

Suppose $\varphi$ is satisfiable and $\sigma_0$ is any fixed satisfying assignment. The hamming distance $d(\sigma, \sigma_0)$ either increases or decreases by 1 during each iteration of the while loop in algorithm LocalSearch. Furthermore, a decrease occurs with probability of at least $1/3$.

Randomized local search

The probability of starting close to a satisfying assignment

In case the an assignment $\sigma$ is chosen uniformly at random from all assignments in $\{0, 1\}^n$, what is the probability for choosing a assignment that has distance $j$ to a satisfying assignment?

Assuming that $\varphi$ is satisfiable, fix any satisfying assignment $\sigma_0$.

Choosing the assignment $\sigma$ uniformly at random amounts to tossing a fair coin for each bit of $\sigma$.

Equivalently, one could toss a fair coin for each position in order to decide whether $\sigma$ should agree with the corresponding bit of $\sigma_0$.

Accordingly, the number of places where $\sigma$ and the fixed assignment $\sigma_0$ differ will be distributed according to a Binomial distribution with parameter 1/2, i.e.,

$$\text{Prob}[d(\sigma, \sigma_0) = j] = \binom{n}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{n-j} = \binom{n}{j} \left(\frac{1}{2}\right)^n.$$
Randomized local search

Success probability of randomized local search

What is the probability of finding a satisfying assignment on an invocation LocalSearch(\(\varphi, \sigma, 27n\)), where \(\varphi\) is satisfiable and \(\sigma\) is chosen uniformly at random?

By putting together the probabilities for an initial assignment to be at distance \(j\) from some fixed satisfying assignment and the lower bound of \(p_j/2 = 2^{-j+1}\) on the success probability in this case, the probability of finding a satisfying assignment is at least

\[
\sum_{j=0}^{n} \Pr[d(\sigma, \sigma_0) = j] \frac{p_j}{2} = \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2^n} \frac{1}{2^{2j}} = \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2^{j+1}} = \frac{1}{2^{n+1}} \left(1 + \frac{1}{2}\right)^n = \frac{1}{2} \left(\frac{3}{4}\right)^n.
\]

Randomized local search

By the preceding discussion, the following algorithm, when applied to a satisfiable Boolean formula, will return a satisfying assignment with probability at least 0.6.

The running time of the algorithm is a polynomial in \(n\) times the number of invocations of LocalSearch, hence is in \(O(1.34^n)\).

Algorithm LocalSearch(\(\varphi\))  (Iterated Local Search)

| Input: A Boolean formula \(\varphi\) in \(n\) variables in 3-CNF. |
| Let \(i = 1\) and \(\sigma = 0^n\). |
| While \(i \leq 2 \cdot \left(\frac{3}{4}\right)^n\) and \(\varphi(\sigma) = \text{false}\) |
| Pick an assignment \(\sigma\) uniformly at random from \(\{0, 1\}^n\). |
| Let \(\sigma := \text{LocalSearch}(\varphi, \sigma, 27n)\). |
| Let \(i := i + 1\). |
| Output: The current assignment \(\sigma\). |

Randomized local search

Independent trials

If a chance experiment with probability \(1/n\) for success is repeated independently \(n\) times, the probability of obtaining at least one success is at least 0.6.

The probability of obtaining no success is equal to \((1 - 1/n)^n\), which goes increasingly to \(1/e < 0.4\) when \(n\) goes to infinity.

Iterated local searches

For a satisfiable formula \(\varphi\) and an assignment \(\sigma\) chosen uniformly at random, an invocation LocalSearch(\(\varphi, \sigma, 27n\)) will return a satisfying assignment with probability of at least \((3/4)^n/2\).

When invoking the algorithm \(2(3/4)^n\) times independently, the probability of finding a satisfying assignment is at least 0.6 in case the input formula is indeed satisfiable and, otherwise, is 0.

Excursus on random walks

Random walks

Consider the chance experiment where a token moves at random on the \(x\)-axis such that at times \(t = 0, 1, \ldots\) the token is at position \(X_t\) in \(\{0, 1, \ldots\}\), if \(X_t = 0\), then \(X_{t+1} = X_t + 1\) = \(X_t + 2\) = \(\ldots\), if \(X_t \neq 0\), then \(X_{t+1}\) is equal to \(X_t - 1\) or \(X_t + 1\) with probability \(\alpha\) and \(1 - \alpha\), respectively.

Such a sequence of random variable \(X_0, X_1, \ldots\) is called a \textit{one-dimensional random walk} with the origin as \textit{absorbing point} and uniform \textit{transition probabilities} \(\alpha\) and \(1 - \alpha\).

The sequence \(X_0, X_1, \ldots\) forms a \textit{Markov chain} because the \(X_i\) satisfy the \textit{Markov property}, i.e., for all \(t\) the probability distribution of \(X_{t+1}\) is determined by the value of \(X_t\) alone, in the sense that the probability distribution of \(X_{t+1}\) conditioned on \(X_t\) and on \(X_1, \ldots, X_t\) is the same.
Excursus on random walks

The probability of reaching the origin

For a random walk $X_0, X_1, \ldots$ as above and with the value of the parameter $\alpha$ understood, let $p_j = p_j(\alpha)$ denote the probability that the token starting at position $j$ eventually reaches the origin, i.e., the probability that $X_i = 0$ for some $i \geq 0$, given that $X_0 = j$.

Trivially, $p_0 = 1 = p_0^0$ and $p_1 = p_1^1$. Indeed we have for all $j \geq 0$,

$$p_j = p_j^j$$

because reaching the origin from position $j$ amounts to

- first reaching position $j - 1$, starting at position $j$,
- then reaching position $j - 2$, starting at position $j - 1$,
- then reaching position $j - 3$, starting at position $j - 2$, and so on,

where the corresponding events all have probability $p_1$ and are mutually independent.

Excursus on random walks

Calculating the probabilities $p_j$

The random walk where $\alpha = 1/2$ is called the symmetric random walk. The fact that a symmetric random walk starting at any position $j$ with probability 1 will eventually reach the origin is sometimes referred to as gambler’s ruin.

**Case $\alpha < 1/2$.**

We will argue in a minute that in this case the probability $p_1$ must differ from 1, hence the only admissible solution is

$$p_0 = 1, \quad p_1 = \frac{\alpha}{1 - \alpha}, \quad \text{and} \quad p_j = p_j^j = \left(\frac{\alpha}{1 - \alpha}\right)^j \quad \text{for all} \quad j \geq 0. $$

For the case $\alpha = 1/3$ most relevant to us, we get $p_j = 1/2^j$ and, more general, in case $\alpha = 1/k$, we have

$$\frac{\alpha}{1 - \alpha} = \frac{1/k}{(k - 1)/k} = \frac{1}{k - 1}, \quad \text{hence} \quad p_j = \frac{1}{(k - 1)^j}. $$

Excursus on random walks

Calculating the probabilities $p_j$

Furthermore, we have $p_j = \alpha p_{j-1} + (1 - \alpha)p_{j+1}$ for all $j > 0$.

For $j = 1$ we obtain an equation that determines $p_1$, i.e.,

$$p_1 = \alpha p_0 + (1 - \alpha)p_2 = \alpha + (1 - \alpha)p_1^2, \quad \text{which is equivalent to}$$

$$0 = p_1^2 - \frac{1}{1 - \alpha}p_1 + \frac{\alpha}{1 - \alpha} = (p_1 - 1)(p_1 - \frac{\alpha}{1 - \alpha}).$$

**Case $\alpha \geq 1/2$.**

For such $\alpha$, we have $\alpha/(1 - \alpha) \geq 1$, hence the second solution for $p_1$ is at least 1, i.e., either agrees with the first solution or cannot be a probability.

As a consequence, the only admissible solution for the value of the probability $p_1$ is $p_1 = 1$.

By $p_j = p_j^j$, for $\alpha \geq 1/2$ the $p_j$ are all equal to 1.

Excursus on random walks

Calculating the probabilities $p_j$

**Case $\alpha < 1/2$ (continued).** It remains to show that for $\alpha < 1/2$ the probability $p_1$, and hence $p_2, p_3, \ldots$ are strictly smaller than 1.

If we let $R_i$ be the indicator variable of step $i$ being to the right, then $\text{Prob}[R_i = 1] = 1 - \alpha = 1/2 + \epsilon$ for some $\epsilon > 0$.

Reaching the origin eventually implies that for some $t \geq j$ the origin is reached first after exactly $t$ steps, which in turn implies that at most half of the these $t$ steps have been to the right.

Similar to the discussion on probability amplification, the Chernov-Hoeffding bound then yields for all $j > 0$,

$$p_j \leq \sum_{t=j}^{\infty} \text{Prob} \left[ \sum_{i=1}^{t} R_i \leq t/2 \right] < \sum_{t=j}^{\infty} (1 - \epsilon/2)^t < 1,$$

So the $p_j$ are smaller than the tail sums of a converging series, thus become arbitrarily small for large $j$, hence $p_1 < 1$ due to $p_j = p_j^j$. 
Excursus on random walks

Restricting the length of the random walk

The probabilities \( p_j \) we have just been calculated give the probability of eventually reaching the origin when starting at position \( j \).

We will see next that for all sufficiently large \( j \), the probability of reaching the origin from position \( j \) will become only marginally smaller if the total number of steps allowed is restricted to \( dj \) for some appropriate constant \( d \) that depends on \( \alpha \).

We will restrict attention to fixed values \( \alpha = 1/3 \) and \( d = 27 \). The argument extends naturally to the more general case \( \alpha = 1/k \).

In the case \( \alpha = 1/3 \), after not having reached the origin within \( 27j \) steps the expected position of the token is \( j + 9j \), hence with very high probability the position of the token is, say, \( 4j \) or larger and the probability of still reaching the origin from there is much smaller than for the initial position \( j \).

Accordingly, we have

\[
L = \text{Prob}[E] = \text{Prob}[F] + \text{Prob}[S] + \text{Prob}[L].
\]

We will argue next that \( \text{Prob}[S] \) and \( \text{Prob}[L] \) are rather small compared to \( \text{Prob}[E] \), and that hence \( \text{Prob}[F] \geq \text{Prob}[E]/2 \).

Excursus on random walks

Restricting the length of the random walk

The event \( L \) occurs if and only if the token reaches the origin eventually.

For a given initial position \( X_0 = j \), we define the following events

\( E \) the token reaches the origin eventually,

\( F \) the token reaches the origin within the first \( 27j \) steps,

\( S \) the token reaches the origin after strictly more than \( 27j \) steps and \( X_{27j} \leq 4j \)

\( L \) the token reaches the origin after after strictly more \( 27j \) steps and \( X_{27j} > 4j \) (the latter condition implies the former one).

The last three events partition the first one, hence we have

\[
p_j = \text{Prob}[E] = \text{Prob}[F] + \text{Prob}[S] + \text{Prob}[L].
\]

In order to apply the Chernov-Hoeffding bound, let \( R_i \) be the indicator variable of step \( i \) being to the right, i.e., \( R_i = 1 \) in case \( X_i - X_{i-1} = 1 \) and \( R_i = 0 \), otherwise.

Let \( R = R_1 + \cdots + R_{27j} \) and observe that \( E[R] = 27j \delta = 18j \).

For \( \delta = 1/6 \), the Chernov-Hoeffding bound yields

\[
\text{Prob}[S] \leq \text{Prob}[X_{27j} \leq 4j] = \text{Prob}[R \leq 15j]
\]

\[
= \text{Prob}[R \leq (1 - \delta)E[R]] \leq e^{-\frac{\delta^2}{2}E[R]} = e^{-\frac{1}{36}27j < 2^{-2j} = \frac{p_j}{2}},
\]

where the last inequality holds because of \( \log e > 1.44 > 4/3 \).