Computing from projections of random points: $K$-triviality and geometry

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Recall: \( k/n \)-bases

Definition
Let \( k < n \). A sequence \( A \in 2^\omega \) is a \( k/n \)-base if there is a Martin-Löf random sequence \( Z = Z_1 \oplus \cdots \oplus Z_n \) such that \( A \) is computable from the join of any \( k \) of the \( n \) parts of \( Z \).

Theorem (Greenberg, M., Nies)
Let \( A \in 2^\omega \). The following are equivalent:

1. \( A \) is a \( k/n \)-base,
2. \( A \) is a \( k/n \)-base witnessed by \( \Omega \),
3. \( A \) obeys \( c_{\Omega,k/n} \), where \( c_{\Omega,k/n}(x,s) = (\Omega_s - \Omega_x)^{k/n} \)
4. [with Turetsky] \( A \) is \( K \)-trivial and computable from some \( k/n \) part of \( \Omega \).

The \( k/n \)-bases form an ideal in the Turing degrees—A proper subideal of the \( K \)-trivials—that is generated by its c.e. elements. Depends only on the rational \( k/n \).
There are two obstacles to generalizing from $1/2$-bases to $k/n$-bases:

1. We used the fact that if $U \subseteq 2^\omega \times 2^\omega$ is open, then one of $\pi_1(U)$ and $\pi_2(U)$ has measure at least $\sqrt{\lambda(U)}$.

For $k/n$-bases, we need to relate the measure of $U \subseteq (2^\omega)^n$ to the measures of its $k$-dimensional projections (along the coordinate axis).

2. We used the fact that if $A$ is a noncomputable $1/2$-base witnessed by $Z_1 \oplus Z_2$, then both $Z_1$ and $Z_2$ must be density-one points.

For $k/n$-bases, we might hope that if $A$ is a noncomputable $k/n$-base witnessed by $Z = Z_1 \oplus \cdots \oplus Z_n$, then the join of any $k$ parts of $Z$ is a density-one point.

Not true in the “degenerate” case, i.e., where some join of $k - 1$ parts of $Z$ computes $A$. In general, any minimal join of coordinates computing $A$ is a density-one point.
The Loomis–Whitney inequality

For $k/n$-bases, we need to relate the measure of $U \subseteq (2^\omega)^n$ to the measures of its $k$-dimensional projections.

Consider the $2/3$ case.

What do the areas of the projections say about the volume of the wood block?

By the Loomis–Whitney inequality:

$$\lambda(\text{Block}) \leq \sqrt{\lambda(G)\lambda(M)\lambda(N)}.$$
The Loomis–Whitney inequality

As a “sanity check”, let us consider how the Loomis–Whitney inequality scales.

If we transform the block by compressing one dimension by a factor of two...

We get:

$$\frac{1}{2} \lambda(\text{Block}) \leq \sqrt{\frac{1}{2} \lambda(G) \frac{1}{2} \lambda(M) \lambda(N)},$$

which is equivalent to the original inequality.
The Loomis–Whitney inequality

Fix \( n \geq 1 \).

**Definition.** For \( F \subseteq \{1, \ldots, n\} \), let \( \pi_F : (2^\omega)^n \to (2^\omega)^{|F|} \) be defined by \( \pi_F(X_1 \oplus \cdots \oplus X_n) = \bigoplus_{i \in F} X_i \).

**Theorem (Loomis and Whitney 1949)**

Let \( 0 < k < n \) and let \( U \subseteq (2^\omega)^n \) be Borel. Then:

\[
\lambda(U)^\binom{n}{k} \leq \left( \prod_{\substack{F \subseteq \{1, \ldots, n\} \\ |F|=k}} \lambda(\pi_F[U]) \right)^{n/k}.
\]

**Corollary ("At least one projection is big")**

Let \( 0 < k < n \) and let \( U \subseteq (2^\omega)^n \) be Borel. Then there is some \( F \subseteq \{1, \ldots, n\} \) with \( |F| = k \) such that

\[
\lambda(\pi_F[U]) \geq \lambda(U)^{k/n}.
\]
Generalizing Loomis–Whitney

Fix \( n \geq 1 \) and let \( \mathcal{F} \subseteq \mathcal{P}(\{1, 2, \ldots, n\}) \). We want to relate the measure of \( U \subseteq (2^\omega)^n \) to the measures of its projections from \( \mathcal{F} \).

A normalized weighting of \( \mathcal{F} \) is a sequence \( \langle x_F \rangle_{F \in \mathcal{F}} \) of nonnegative real numbers such that \( \sum_{\{F \in \mathcal{F} : i \in F\}} x_F \leq 1 \) for all \( i \leq n \).

**Theorem (Greenberg, M., Nies)**

If \( \langle x_F \rangle \) is a normalized weighting of \( \mathcal{F} \) and \( U \subseteq (2^\omega)^n \) is Borel, then

\[
\lambda(U) \leq \prod_{F \in \mathcal{F}} \lambda(\pi_F[U])^{x_F}.
\]

The choice of a normalized weighting guarantees that this inequality scales properly. If we transform \( U \) to \( V \) by compressing the \( i \)th dimension by a factor of two, then:

\[
\lambda(V) = \frac{1}{2} \lambda(U) \leq \frac{1}{2} \prod_{F \in \mathcal{F}} \lambda(\pi_F[U])^{x_F}
\]

\[
\leq \prod_{F \in \mathcal{F} : i \in F} \left(\frac{1}{2}\right)^{x_F} \prod_{F \in \mathcal{F}} \lambda(\pi_F[U])^{x_F} = \prod_{F \in \mathcal{F}} \lambda(\pi_F[V])^{x_F}.
\]
Generalizing Loomis–Whitney

We want the generalized form of “At least one projection is big”, but our generalized Loomis–Whitney inequality depends on the normalized weighting of $\mathcal{F}$.

We can optimize the weighting:

**Definition.** The norm of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \sup \left\{ \sum_{F \in \mathcal{F}} x_F : \langle x_F \rangle \text{ is a normalized weighting of } \mathcal{F} \right\}.$$  

- $\|\mathcal{F}\|$ is defined by a *linear programming problem*, i.e., the optimization of a linear objective function subject to linear inequality constraints.
- If $\emptyset \notin \mathcal{F}$, then $\|\mathcal{F}\|$ exists (and is finite).
- If $\mathcal{F} \neq \emptyset$, then $\|\mathcal{F}\| \geq 1$.

From now on, assume that $\emptyset \notin \mathcal{F}$ and $\mathcal{F} \neq \emptyset$.

- $\|\mathcal{F}\|$ measures the “amount of disjointness” of $\mathcal{F}$.  

Generalizing Loomis–Whitney

Theorem (Greenberg, M., Nies)

If $\langle x_F \rangle$ is a normalized weighting of $\mathcal{F}$ and $U \subseteq (2^\omega)^n$ is Borel, then

$$\lambda(U) \leq \prod_{F \in \mathcal{F}} \lambda(\pi_F[U])^{x_F}.$$ 

Corollary ("At least one projection is big")

Let $U \subseteq (2^\omega)^n$ be Borel. There is some $F \in \mathcal{F}$ such that

$$\lambda(\pi_F[U]) \geq \lambda(U)^{1/\|\mathcal{F}\|}.$$ 

Moreover, this cannot be improved.

Proof. Assume not and let $\langle x_F \rangle$ be a normalized weighting realizing $\|\mathcal{F}\|$. Then

$$\lambda(U) \leq \prod_{F \in \mathcal{F}} \lambda(\pi_F[U])^{x_F} < \prod_{F \in \mathcal{F}} \lambda(U)^{x_F/\|\mathcal{F}\|} = \lambda(U).$$

The proof that this inequality is tight (at least for some $U$) relies on the dual to the linear program defining $\|\mathcal{F}\|$. 

We are ready to generalize $k/n$-bases to arbitrary families.

**Definition**
Let $\mathcal{F} \subseteq \mathcal{P} \{1, 2, \ldots, n\}$. A sequence $A \in 2^\omega$ is an $\mathcal{F}$-base if there is a Martin-Löf random sequence $Z = Z_1 \oplus \cdots \oplus Z_n$ such that $A$ is computable from $\pi_F(Z)$ for every $F \in \mathcal{F}$.

**Theorem (Greenberg, M., Nies)**
The following are equivalent:
1. $A$ is an $\mathcal{F}$-base,
2. $A$ is an $\mathcal{F}$-base “witnessed by $\Omega$”,
3. $A$ is a $1/\|\mathcal{F}\|$-base. I.e., $A$ obeys $c_{\Omega, 1/\|\mathcal{F}\|}(x, s) = (\Omega_s - \Omega_x)^{1/\|\mathcal{F}\|}$.

Note that arbitrary families do not give us new ideals!
Example: cyclic \( k/n \)-bases

The definition of a \( k/n \)-base looks quite demanding; for example, a 3/7-base is assumed to be computable from \( \binom{7}{3} = 35 \) different reals.

We can get away with a weaker hypothesis.

**Definition.** Fix \( 0 < k < n \). For each \( i \leq n \), let

\[
F_i = \{i, i + 1 \pmod{n}, \ldots, i + k - 1 \pmod{n}\}.
\]

Let \( \mathcal{F} = \{F_i\}_{i<n} \). A set \( A \) is a cyclic \( k/n \)-base if it an \( \mathcal{F} \)-base.

Note that this definition only requires \( A \) to be computable from \( n \) different reals. And yet:

**Proposition.** A set is a cyclic \( k/n \)-base if and only if it is a \( k/n \)-base.

**Proof.** Consider the constant weighting \( x_F = 1/k \) for all \( F \in \mathcal{F} \). This is normalized since every \( i \leq n \) is an element of precisely \( k \) many sets in \( \mathcal{F} \). Hence \( \|\mathcal{F}\| \geq n/k \) and so \( 1/\|\mathcal{F}\| \leq k/n \). Therefore, every cyclic \( k/n \)-base is a \( k/n \)-base. 

\[\square\]
Example: degenerate \( k/n \)-bases

Definition
Assume that \( 1 < k < n \). We call \( A \) a degenerate \( k/n \)-base if there is a Martin-Löf random \( Z \) that witnesses that \( A \) is a \( k/n \)-base and there is also a \( G \subseteq \{1, \ldots, n\} \) such that \( |G| < k \) and \( A \leq_T \pi_G(Z) \).

Degenerate \( k/n \)-bases are always \( p \)-bases for a smaller \( p \):

Proposition
Let \( p = \max \left\{ \frac{k}{n+1}, \frac{k-1}{n-1} \right\} \). A set is a degenerate \( k/n \)-base if and only if it is a \( p \)-base.

Example
\( A \) is a degenerate \( 3/6 \)-base if and only if it is a \( 3/7 \)-base.

\[
\begin{array}{|c|c|c|c|}
\hline
k & 2 & 3 & 4 & 5 \\
\hline
p & 2/7 & 3/7 & 3/5 & 4/5 \\
\hline
\end{array}
\]
Thank You!