# Computing from projections of random points: A dense hierarchy of subideals of the $K$-trivial degrees

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22nd June 2015

Joint work with Joe Miller and Andre Nies

# Background:

# $K$-triviality, covering, cost

# $K$-triviality

**Theorem (Nies;Nies,Hirschfeldt;Nies,Hirschfeldt,Stephan)**

*The following are equivalent for $A \in 2^\omega$:*

1. *$A$ is K-trivial: $K(A\upharpoonright_n) =^+ K(n)$;*
2. *$A$ is low for K: $K^A =^+ K$;*
3. *$A$ is low for ML randomness: $\mathrm{MLR}^A = \mathrm{MLR}^\varnothing$;*
4. *$A$ is a base for ML randomness: $A \leqslant_T Z$ for some $Z \in \mathrm{MLR}^A$.*

Solovay proved that there are noncomputable $K$-trivial sets; Zambella constructed a c.e. $K$-trivial set; Muchnik constructed a set which is low for $K$; Kučera and Terwijn constructed a set which is low for ML randomness.

# Structure of $K$-triviality

**Theorem (Chaitin;Nies;Downey,Hirschfeldt,Nies,Stephan)**

1. *There are only countably many $K$-trivial sets.*
2. *The $K$-trivial sets induce a $\Sigma^0_3$ ideal in the Turing degrees.*
3. *This ideal is c.e.-generated.*
4. *Every $K$-trivial set is superlow.*

# Covering

**Theorem (Kučera;Hirschfeldt,Miller)**

*Every $\Delta^0_2$ random sequence computes a noncomputable c.e. set; indeed a random computes a noncomputable c.e. set if and only if it is not weakly 2 random.*

**Theorem (Hirschfeldt,Nies,Stephan)**

*If A is c.e. and computable from an incomplete random sequence then A is K-trivial.*

**Theorem (Bienvenu,Day,Greenberg,Kučera,Miller,Nies,Turetsky)**

*If A is K-trivial then A is computable from some incomplete random sequence.*

# Stronger variants of covering

**Theorem (Kučera)**

*If $Z$ is a $\Delta^0_2$ random sequence then there is a noncomputable c.e. set, computable from both halves of $Z$.*

Note that both halves are low.

**Question (Stephan)**

1. Is every $K$-trivial set computable from a low random sequence?
2. Is every $K$-trivial set computable from both halves of a random sequence?

**Theorem (Bienvenu,Greenberg,Kučera,Nies,Turetsky)**

*No and no.*

**Question**

What $K$-trivial sets are computable from both halves of a random?

# Cost functions

How would you answer this question? What are ways to characterise subclasses of the $K$-trivials? Most characterisations of $K$-triviality are extremal.

**Theorem (Nies)**

*A set $A$ is $K$-trivial if and only if there is a computable approximation $\langle A_s \rangle$ of $A$ such that*

$$\sum_{s<\omega} \left( \Omega_s - \Omega_{|A_{s-1} \wedge A_s|} \right)$$

*is finite.*

We say that $A$ obeys the cost function $\mathbf{c}_\Omega(x) = \Omega - \Omega_x$.

# 1/2-bases

# $1/2$-bases

**Theorem**

*The following are equivalent for a set $A$:*

1. *$A$ is a $1/2$-base: it is computable from both halves of a random sequence.*
2. *$A$ is computable from both halves of Chaitin's $\Omega$.*
3. *$A$ obeys the cost function $\mathbf{c}_{\Omega,1/2}(x) = \sqrt{\Omega - \Omega_x}$.*

**Theorem**

*The collection of $1/2$-bases induces a $\Sigma^0_3$-ideal in the Turing degrees, generated by its c.e. elements; the two halves of Chaitin's $\Omega$ form an exact pair for this ideal.*

**Theorem (with Turetsky)**

*A c.e. set is a $1/2$-base if and only if it is computable from one of the halves of Chaitin's $\Omega$.*

# Where does the square root come from?

First direction:

1. The halves of Chaitin's $\Omega$ are captured by a $1/2$-Oberwolfach test: a test $\langle G_\sigma \rangle$ (where $\sigma \in 2^{<\omega}$), nested, such that $\lambda(G_\sigma) \leqslant 2^{-n/2}$; the null set is $\bigcap_n G_{\Omega\upharpoonright_n}$.

2. A $1/2$-Oberwolfach test can be covered by a $\mathbf{c}_{\Omega,1/2}$-bounded test: a weak 2-test $\langle U_n \rangle$ such that $\lambda(U_n) \leqslant \mathbf{c}_{\Omega,1/2}(n)$.

3. Generalised Kučera: if $A$ obeys $\mathbf{c}$ then $A$ is computable from any random set which is captured by a $\mathbf{c}$-bounded test.

# Second direction: hungry sets

As a warmup, we sketch a direct argument showing that a c.e. $K$-trivial set obeys $\mathbf{c}_\Omega$. Let $A$ be $K$-trivial; let $Z$ be an $A$-random sequence which computes $A$: $\Phi(Z) = A$.

What we want: a process of confirmation of initial segments of $A$: at stage $s$ we believe that $A_s \upharpoonright_k$ is correct. The idea: $\tau$ is believed if many oracles compute it.

We build “hungry sets” $G_\tau$ with the properties:

- $G_\tau \subseteq \Phi^{-1}[\tau]$;
- They are pairwise disjoint;
- The goal for $G_\tau$ is $\Omega_{|\tau|+1} - \Omega_{|\tau|}$.

# Hungry sets: how they are useful

Suppose that every true initial segment of $A$ is eventually confirmed.

Our speedup of the enumeration of $A$ is a seuqnece $s_0 < s_1 < s_2 < \ldots$ sucht that $A_s \upharpoonright_n$ is confirmed at stage $s_n$.

Let $n < \omega$. The cost of the change from $A_{s_n}$ to $A_{s_{n+1}}$ is $\Omega_{n+1} - \Omega_k$, where $k = |A_{s_n} \wedge A_{s_{n+1}}|$. We charge this cost against the measure of

$$G_{A_{s_n}} \upharpoonright_{k+1} \cup G_{A_{s_n}} \upharpoonright_{k+2} \cup \cdots \cup G_{A_{s_n}} \upharpoonright_n .$$

These sets are pairwise disjoint across $n$'s.

# Hungry sets: how to get them

Recursively fill $G_\tau$ from $\Phi^{-1}[\tau]$; when it is satiated, move to the next extension of $\tau$.

Suppose some $\tau \prec A$ is the least which is not confirmed. This means that

$$\Phi^{-1}[\tau] \subseteq G_{\tau\restriction_0} \;\cup\; G_{\tau\restriction_1} \;\cup\; \cdots \cup\; G_\tau.$$

So

$$Z \in \bigcup_{\tau \prec A} G_\tau.$$

The measure of the union is bounded by $\Omega$.

Doing this over with constants $\epsilon > 0$ shows that $Z$ is not $A$-random.

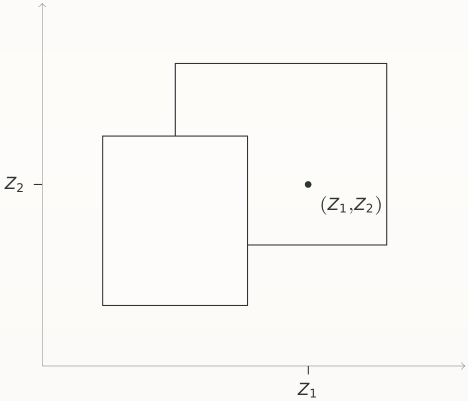
# Hungry sets: two oracles

We adapt this hungry sets argument to $1/2$-bases. Now we have $Z_1$ and $Z_2$, relatively random; and $\Phi_i(Z_i) = A$ for $i = 1, 2$. Our hungry sets $G_\tau$ will be subsets of $\Phi_1^{-1}[\tau] \times \Phi_2^{-1}[\tau]$. We are aiming to capture the random point $(Z_1, Z_2)$.

Main idea: Suppose that $\tau$ is believed: $\lambda(G_\tau) = \Omega_{|\tau|+1} - \Omega_{|\tau|}$. Then either the projection $\pi_1(G_\tau)$ or $\pi_2(G_\tau)$ has measure $\sqrt{\Omega_{|\tau|+1} - \Omega_{|\tau|}}$.

# Hungry sets: problems

Problem: can't keep the projections disjoint.



# Hungry sets: solutions

Say $\tau \prec \tau'$, $\tau$ still appears correct at stage $s$ but $\tau'$ suddenly not. The idea is to extract oracles mapping to $\tau'$ from $G_\tau$ and refill it with new stuff: need to re-certify.

This would give us a difference test capturing $(Z_1, Z_2)$.

# Difference tests

**Definition (Franklin,Ng)**

A difference test is a test of the form $\langle U_n \cap P \rangle$ where $P$ is a $\Pi^0_1$ class (an effectively closed set), $U_n$ are uniformly c.e. and $\lambda(U_n \cap P) \leqslant 2^{-n}$.

**Theorem (Franklin,Ng)**

*The following are equivalent for a random sequence $Z$:*

1. *$Z$ is captured by some difference test;*
2. *$Z$ is complete: $Z \geqslant_T \emptyset'$.*

# Lebesgue density

Recall that the (lower) density of $P$ at $Z$ is

$$\liminf_{n\to\infty} \lambda(P \mid Z\upharpoonright_n).$$

**Theorem (Bienvenu,Hölzl,Miller,Nies)**

*The following are equivalent for a random sequence $Z$ and a $\Pi^0_1$ class $P$:*

1. *$Z$ is captured by a difference test based on $P$;*
2. *$P$ has density 0 at $Z$.*

# Back to the solution

So we got a difference test capturing $(Z_1, Z_2)$. But $(Z_1, Z_2)$ could be complete, so where's the contradiction?

Observe that in this case our effectively closed set is the product class $P_1 \times P_2$: $P_i$ is the class of oracles found to compute $A$ incorrectly via $\Phi_i$.

So the density of $P_1 \times P_2$ at $(Z_1, Z_2)$ is zero.

But then either $P_1$ has zero density at $Z_1$, or $P_2$ has zero density at $Z_2$. So some $Z_i \geqslant_T \varnothing'$. And then $Z_{1-i}$ is 2-random and cannot compute $A$.

- Other problems when $A$ is not c.e.

# $p$-bases

# $k/n$-bases

Let us generalise.

**Definition**

A set $A$ is a $k/n$-base if there is a random tuple $(Z_1, Z_2, \ldots, Z_n)$ such that $A$ is computable from the join of any $k$ of the $Z_i$'s.

# $k/n$-bases

**Theorem**

*The following are equivalent for a set A:*

1. *A is a $k/n$-base.*
2. *A is a $k/n$-base, witnessed by Chaitin's $\Omega$.*
3. *A obeys the cost function $\mathbf{c}_{\Omega,k/n}(x) = (\Omega - \Omega_x)^{k/n}$.*

**Theorem**

*The collection of $k/n$-bases induces a $\Sigma^0_3$-ideal in the Turing degrees, generated by its c.e. elements.*

**Corollary**

*Every $1/2$-base is a $2/4$-base.*

**Theorem (with Turetsky)**

*A c.e. set is a $k/n$-base if and only if it is computable from some $k/n$ part of $\Omega$.*

# $\mathcal{F}$-bases

An even more general notion turns out to be useful, for example in classifying “cyclic” $k/n$-bases.

Joe will discuss.

# Other results

# Other ideals

For $p \in (0,1) \cap \mathbb{Q}$, let $\mathcal{B}_p$ be the collection of $p$-bases.

- if $p < q$ then $\mathcal{B}_p \subsetneq \mathcal{B}_q$.

For $r \in [0,1]$ let

- $\mathcal{B}_{<r} = \bigcup_{p<r} \mathcal{B}_p$; and
- $\mathcal{B}_{>r} = \bigcap_{p>r} \mathcal{B}_p$.

Both are ideals.

**Proposition**

$\mathcal{B}_{<r} \neq \mathcal{B}_{>r}$ *if and only if $r$ is left-$\Pi^0_3$.*

# $1/\omega$-bases

**Theorem**

*The following are equivalent for a set $A$:*

1. *There is an infinite random sequence $(Z_1, Z_2, \ldots)$ such that $A$ is computable from each $Z_i$.*
2. *There is a computable partition of $\Omega$ into infinitely many columns such that $A$ is computable from each column.*
3. *There is an infinite sequence $(Z_1, Z_2, \ldots)$ such that the join of any finitely many $Z_i$ is random, and such that $A$ is computable from each $Z_i$.*
4. *$A \in \mathcal{B}_{>0}$.*

# Robust computability

For $X, Y \subseteq \omega$ write $X \sim Y$ if

$$\lim_{n\to\infty} \frac{|(X \triangle Y) \cap n|}{n} = 0.$$

**Definition (Hirschfeldt,Jockusch,Kuyper,Schupp)**

$A$ is robustly reducible to $Z$ if $A \leqslant_T Y$ for all $Y \sim Z$.

**Theorem**

*The following are equivalent for a set A:*

1. *A is robustly reducible to a random set;*
2. *A is robustly reducible to $\Omega$;*
3. *For some $\epsilon > 0$, A is computable from every Y such that the density of $Y \triangle \Omega$ is at most $\epsilon$;*
4. *$A \in \mathcal{B}_{<1}$.*

Hirschfedlt et al. proved $(2) \Leftrightarrow (3)$ and $(1) \Rightarrow (4)$.

# LR-hardness

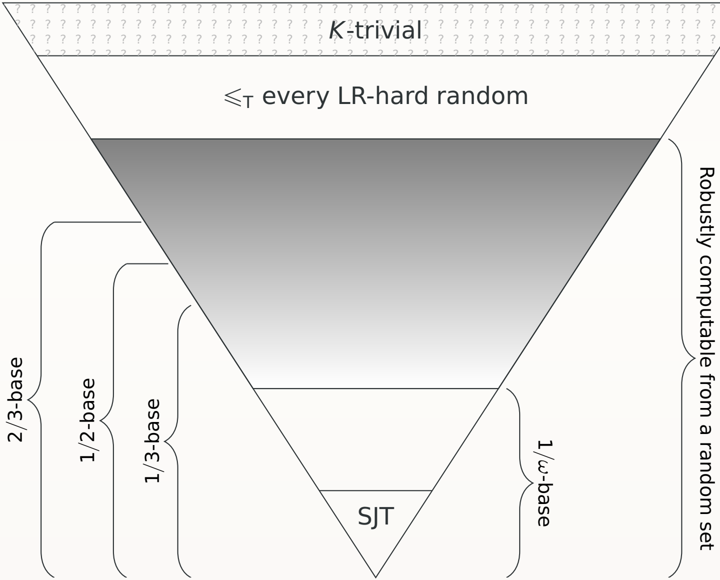
A set $Z$ is LR-hard if $\mathrm{MLR}^Z \subseteq \mathrm{MLR}^{\emptyset'}$. This is equivalent to being almost everywhere dominating (Kjos-Hanssen,Miller,Solomon).

**Theorem**

*Every set in $\mathcal{B}_{<1}$ (and more) is computable from every LR-hard random set.*

**Question**

Is every $K$-trivial set computable from every LR-hard random set?



$K$-trivial
$\leqslant_T$ every LR-hard random
SJT
2/3-base
1/2-base
1/3-base
1/$\omega$-base
Robustly computable from a random set

Thank you