Compressibility of Closed Sets

Douglas Cenzer, Chris Porter, Ferit Toska
CCR 2015, Heidelberg, Germany

June 26, 2015

Table of contents

Background

- Kolmogorov complexity $M(x) = y$ where M is a universal prefix free machine.
- For the optimal compression, with $K(y) = |x|$, the compressed strings may be taken to be random, that is, incompressible.
- The compression of an infinite sequence $Y \in 2^{\mathbb{N}}$ to $X \in 2^{\mathbb{N}}$ via a Turiing machine M has long been an interesting topic
- Kučera-Gács showed that any infinite sequence Y is Turing reducible to a ML-random sequence R.

Background

- Kolmogorov complexity $M(x) = y$ where M is a universal prefix free machine.
- For the optimal compression, with $K(y) = |x|$, the compressed strings may be taken to be random, that is, incompressible.
- The compression of an infinite sequence $Y \in 2^{\mathbb{N}}$ to $X \in 2^{\mathbb{N}}$ via a Turiing machine M has long been an interesting topic
- Kučera-Gács showed that any infinite sequence Y is Turing reducible to a ML-random sequence R.

Background

- Kolmogorov complexity $M(x) = y$ where M is a universal prefix free machine.
- For the optimal compression, with $K(y) = |x|$, the compressed strings may be taken to be random, that is, incompressible.
- The compression of an infinite sequence $Y \in 2^{\mathbb{N}}$ to $X \in 2^{\mathbb{N}}$ via a Turiing machine M has long been an interesting topic
- Kučera-Gács showed that any infinite sequence Y is Turing reducible to a ML-random sequence R.

Background

- Kolmogorov complexity $M(x) = y$ where M is a universal prefix free machine.
- For the optimal compression, with $K(y) = |x|$, the compressed strings may be taken to be random, that is, incompressible.
- The compression of an infinite sequence $Y \in 2^{\mathbb{N}}$ to $X \in 2^{\mathbb{N}}$ via a Turiing machine M has long been an interesting topic
- Kučera-Gács showed that any infinite sequence Y is Turing reducible to a ML-random sequence R.

Use

- $\rho^X(n)$ is the *use* of oracle X to compute output $Y \upharpoonright n$
- Ryabkov showed that the optimal compression of Y will have *relative use* $\rho^-(X,Y) = \liminf_n \rho^X(n)/n$ equal to the constructive Hausdorff dimension $dim(Y)$.
- Moreover,this compression can be done uniformly for all elements of a subset S of $2^{\mathbb{N}}$, with optimal compression $dim(S)$.
- In both cases, compression was obtained arbitrarily close to optimal.

Use

- $\rho^X(n)$ is the *use* of oracle X to compute output $Y \upharpoonright n$
- Ryabkov showed that the optimal compression of Y will have *relative use* $\rho^-(X, Y) = \liminf_n \rho^X(n)/n$ equal to the constructive Hausdorff dimension $dim(Y)$.
- Moreover,this compression can be done uniformly for all elements of a subset S of $2^{\mathbb{N}}$, with optimal compression $dim(S)$.
- In both cases, compression was obtained arbitrarily close to optimal.

Use

- $\rho^X(n)$ is the *use* of oracle X to compute output $Y \upharpoonright n$
- Ryabkov showed that the optimal compression of Y will have *relative use* $\rho^-(X, Y) = \liminf_n \rho^X(n)/n$ equal to the constructive Hausdorff dimension $dim(Y)$.
- Moreover,this compression can be done uniformly for all elements of a subset S of $2^{\mathbb{N}}$, with optimal compression $dim(S)$.
- In both cases, compression was obtained arbitrarily close to optimal.

Use

- $\rho^X(n)$ is the *use* of oracle X to compute output $Y \upharpoonright n$
- Ryabkov showed that the optimal compression of Y will have *relative use* $\rho^-(X, Y) = \liminf_n \rho^X(n)/n$ equal to the constructive Hausdorff dimension $dim(Y)$.
- Moreover,this compression can be done uniformly for all elements of a subset S of $2^{\mathbb{N}}$, with optimal compression $dim(S)$.
- In both cases, compression was obtained arbitrarily close to optimal.

Optimal Compression

- Doty gave a similar result for the constructive packing dimension $Dim(Y) = \rho^+(X, Y) = \limsup_n \rho^X(n)/n$.
- Moreover, optimal compression was achieved.
- Doty also considered computable and sub-computable dimension and compression.

Optimal Compression

- Doty gave a similar result for the constructive packing dimension $Dim(Y) = \rho^+(X, Y) = \limsup_n \rho^X(n)/n$.
- Moreover, optimal compression was achieved.
- Doty also considered computable and sub-computable dimension and compression.

Optimal Compression

- Doty gave a similar result for the constructive packing dimension $Dim(Y) = \rho^+(X, Y) = \limsup_n \rho^X(n)/n$.
- Moreover, optimal compression was achieved.
- Doty also considered computable and sub-computable dimension and compression.

Constructive Dimension

Constructive dimension for $X \in 2^{\mathbb{N}}$ is characterized as follows:

Theorem (Mayordomo)

$$dim(X) = \liminf_n \frac{K(X \upharpoonright n)}{n}$$

Theorem (Athreya)

$$Dim(X) = \limsup_n \frac{K(X \upharpoonright n)}{n}$$

Constructive Dimension

Constructive dimension for $X \in 2^{\mathbb{N}}$ is characterized as follows:

Theorem (Mayordomo)

$$dim(X) = \liminf_n \frac{K(X \upharpoonright n)}{n}$$

Theorem (Athreya)

$$Dim(X) = \limsup_n \frac{K(X \upharpoonright n)}{n}$$

Constructive Dimension

Constructive dimension for $X \in 2^{\mathbb{N}}$ is characterized as follows:

Theorem (Mayordomo)

$$dim(X) = \liminf_n \frac{K(X \upharpoonright n)}{n}$$

Theorem (Athreya)

$$Dim(X) = \limsup_n \frac{K(X \upharpoonright n)}{n}$$

Strict Process Machines

- $M : \{0,1\}^* \to \{0,1\}^*$ such that $dom(M)$ is a tree and if $v \preceq u$, then $M(v) \preceq M(u)$.

  Compare to Levin's monotone machines, where if $v \prec u$ and both $M(u)$ and $M(v)$ are defined, then $M(v) \preceq M(u)$.

- For total M, this defines a computable function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. M total on a tree T defines F on the closed set $[T]$.

- There is a universal strict process machine.

Strict Process Machines

- $M : \{0,1\}^* \to \{0,1\}^*$ such that $dom(M)$ is a tree and if $v \preceq u$, then $M(v) \preceq M(u)$.

  Compare to Levin's monotone machines, where if $v \prec u$ and both $M(u)$ and $M(v)$ are defined, then $M(v) \preceq M(u)$.

- For total M, this defines a computable function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. M total on a tree T defines F on the closed set $[T]$.

- There is a universal strict process machine.

Strict Process Machines

- $M : \{0,1\}^* \to \{0,1\}^*$ such that $dom(M)$ is a tree and if $v \preceq u$, then $M(v) \preceq M(u)$.

  Compare to Levin's monotone machines, where if $v \prec u$ and both $M(u)$ and $M(v)$ are defined, then $M(v) \preceq M(u)$.

- For total M, this defines a computable function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. M total on a tree T defines F on the closed set $[T]$.

- There is a universal strict process machine.

Strict Process Machines

- $M : \{0,1\}^* \to \{0,1\}^*$ such that *dom*(M) is a tree and if $v \preceq u$, then $M(v) \preceq M(u)$.

  Compare to Levin's monotone machines, where if $v \prec u$ and both $M(u)$ and $M(v)$ are defined, then $M(v) \preceq M(u)$.

- For total M, this defines a computable function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. M total on a tree T defines F on the closed set $[T]$.

- There is a universal strict process machine.

Description

- Let M be a strict process machine. Define $F_M : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $X \in dom(F_M) \iff \{|M(X \upharpoonright n)| : n \in \mathbb{N}\}$ is unbounded, and $F_M(X) = Y$ where X is the M-description of Y
- X **describes** $Y = F_M(X)$ **with rate** c if

$$(\exists^\infty n)\,(c \cdot n \leq |M(X \upharpoonright n)|)$$

  We say that Y is c-compressible by M
- M **complexity of** Y**:** $s_M(y) = \inf\{\frac{1}{c} : y \text{ is } c\text{-compressible by } M\}$.

Description

- Let M be a strict process machine. Define $F_M : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $X \in dom(F_M) \iff \{|M(X \restriction n)| : n \in \mathbb{N}\}$ is unbounded, and $F_M(X) = Y$ where X is the M-description of Y
- X **describes** $Y = F_M(X)$ **with rate** c if

$$(\exists^\infty n)\,(c \cdot n \leq |M(X \restriction n)|)$$

  We say that Y is c-compressible by M
- M **complexity of** Y**:** $s_M(y) = \inf\{\frac{1}{c} : y \text{ is } c\text{-compressible by } M\}$.

Description

- Let M be a strict process machine. Define $F_M : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $X \in dom(F_M) \iff \{|M(X \upharpoonright n)| : n \in \mathbb{N}\}$ is unbounded, and $F_M(X) = Y$ where X is the M-description of Y
- X **describes** $Y = F_M(X)$ **with rate** c if

$$(\exists^\infty n)\,(c \cdot n \leq |M(X \upharpoonright n)|)$$

  We say that Y is c-compressible by M
- M **complexity of** Y**:**
  $s_M(y) = \inf\{\frac{1}{c} \ : \ y \text{ is } c\text{-compressible by } M\}$.

Complexity

- P **is an** M**-description of** Q if $Q = \{F_M(X) : X \in P\}$
- P **describes** Q **with a rate** $c > 0$ if each $X \in P$ describes $Y = F_M(X)$ with rate c.
- The **SPM complexity of** Q**:**
- $sp_M(P, Q) = \inf\{\frac{1}{c} : \; P$ describes Q with a rate $c\}$
- $sp_M(Q) = \inf\{s_M(P, Q) : P$ is an M-description of $Q\}$
- $sp(Q) = sp_M(Q)$ for a universal strict process machine M.

Complexity

- P **is an** M**-description of** Q if $Q = \{F_M(X) : X \in P\}$
- P **describes** Q **with a rate** $c > 0$ if each $X \in P$ describes $Y = F_M(X)$ with rate c.
- The **SPM complexity of** Q**:**
- $sp_M(P, Q) = \inf\{\frac{1}{c} : \ P$ describes Q with a rate $c\}$
- $sp_M(Q) = \inf\{s_M(P, Q) : P$ is an M-description of $Q\}$
- $sp(Q) = sp_M(Q)$ for a universal strict process machine M.

Complexity

- P **is an** M**-description of** Q if $Q = \{F_M(X) : X \in P\}$
- P **describes** Q **with a rate** $c > 0$ if each $X \in P$ describes $Y = F_M(X)$ with rate c.
- The **SPM complexity of** Q**:**
- $sp_M(P, Q) = \inf\{\frac{1}{c} : P$ describes Q with a rate $c\}$
- $sp_M(Q) = \inf\{s_M(P, Q) : P$ is an M-description of $Q\}$
- $sp(Q) = sp_M(Q)$ for a universal strict process machine M.

Complexity

- P **is an** M**-description of** Q if $Q = \{F_M(X) : X \in P\}$
- P **describes** Q **with a rate** $c > 0$ if each $X \in P$ describes $Y = F_M(X)$ with rate c.
- The **SPM complexity of** Q**:**
- $sp_M(P, Q) = \inf\{\frac{1}{c} : P \text{ describes } Q \text{ with a rate } c\}$
- $sp_M(Q) = \inf\{s_M(P, Q) : P \text{ is an } M\text{-description of } Q\}$
- $sp(Q) = sp_M(Q)$ for a universal strict process machine M.

Complexity

- P **is an** M**-description of** Q if $Q = \{F_M(X) : X \in P\}$
- P **describes** Q **with a rate** $c > 0$ if each $X \in P$ describes $Y = F_M(X)$ with rate c.
- The **SPM complexity of** Q**:**
- $sp_M(P, Q) = \inf\{\frac{1}{c} : P \text{ describes } Q \text{ with a rate } c\}$
- $sp_M(Q) = \inf\{s_M(P, Q) : P \text{ is an } M\text{-description of } Q\}$
- $sp(Q) = sp_M(Q)$ for a universal strict process machine M.

Complexity

- P **is an** M**-description of** Q if $Q = \{F_M(X) : X \in P\}$
- P **describes** Q **with a rate** $c > 0$ if each $X \in P$ describes $Y = F_M(X)$ with rate c.
- The **SPM complexity of** Q**:**
- $sp_M(P, Q) = \inf\{\frac{1}{c} : \ P$ describes Q with a rate $c\}$
- $sp_M(Q) = \inf\{s_M(P, Q) : P$ is an M-description of $Q\}$
- $sp(Q) = sp_M(Q)$ for a universal strict process machine M.

Results of Toska

Theorem

For any $X \in 2^{\mathbb{N}}$, $dim(X) = sp(X)$.

Theorem

For any $Q \subset 2^{\mathbb{N}}$, $sp(Q) = \sup\{sp(Y) : Y \in Q\}$

Corollary

For any $Q \subset 2^{\mathbb{N}}$, $dim(Q) = sp(Q)$.

Results of Toska

Theorem

For any $X \in 2^{\mathbb{N}}$, $dim(X) = sp(X)$.

Theorem

For any $Q \subset 2^{\mathbb{N}}$, $sp(Q) = \sup\{sp(Y) : Y \in Q\}$

Corollary

For any $Q \subset 2^{\mathbb{N}}$, $dim(Q) = sp(Q)$.

Results of Toska

Theorem

For any $X \in 2^{\mathbb{N}}$, $dim(X) = sp(X)$.

Theorem

For any $Q \subset 2^{\mathbb{N}}$, $sp(Q) = \sup\{sp(Y) : Y \in Q\}$

Corollary

For any $Q \subset 2^{\mathbb{N}}$, $dim(Q) = sp(Q)$.

For Closed Sets

We can now improve the corollary as follows:

- If Q is closed, then Q can be optimally compressed by a closed set P with $dim(P) = 1$.

Question

Can we make P a random closed set

- If Q is effectively closed, then Q may be compressed arbitrarily close to optimal by a Π^0_1 class P.
- If $dim(Q)$ is computable, then there is optimal compression.

For Closed Sets

We can now improve the corollary as follows:

- If Q is closed, then Q can be optimally compressed by a closed set P with $dim(P) = 1$.

Question

Can we make P a random closed set

- If Q is effectively closed, then Q may be compressed arbitrarily close to optimal by a Π^0_1 class P.
- If $dim(Q)$ is computable, then there is optimal compression.

For Closed Sets

We can now improve the corollary as follows:

- If Q is closed, then Q can be optimally compressed by a closed set P with $dim(P) = 1$.

Question

Can we make P a random closed set

- If Q is effectively closed, then Q may be compressed arbitrarily close to optimal by a Π^0_1 class P.
- If $dim(Q)$ is computable, then there is optimal compression.

For Closed Sets

We can now improve the corollary as follows:

- If Q is closed, then Q can be optimally compressed by a closed set P with $dim(P) = 1$.

Question

Can we make P a random closed set

- If Q is effectively closed, then Q may be compressed arbitrarily close to optimal by a Π^0_1 class P.
- If $dim(Q)$ is computable, then there is optimal compression.

Decidability

Definition

- For a closed set P, let $T_P = \{w \in \{0,1\}^* : (\exists Z \in P) w \prec Z\}$
- P is **effectively closed** (a Π^0_1 class) if T_P is a co-c.e. set
- P is **decidable** if T_P is a computable set
- P is 1-**decidable** if there is a computable function $f : \{0,1\}^* \to (\mathbb{N} \cup \{\infty\})$ such that $f(w) = \text{card}\{Z \in P : w \prec Z\}$.

This can be generalized to $n+1$-decidability by having $f(w) = card\{Z \in D^n(P) : w \prec Z\}$

Decidability

Definition

- For a closed set P, let $T_P = \{w \in \{0,1\}^* : (\exists Z \in P) w \prec Z\}$
- P is **effectively closed** (a Π^0_1 class) if T_P is a co-c.e. set
- P is **decidable** if T_P is a computable set
- P is 1-**decidable** if there is a computable function $f : \{0,1\}^* \to (\mathbb{N} \cup \{\infty\})$ such that $f(w) = \text{card}\{Z \in P : w \prec Z\}$.

This can be generalized to $n+1$-decidability by having $f(w) = card\{Z \in D^n(P) : w \prec Z\}$

Rank One Classes

Definition

Let P be a rank one Π^0_1 class with unique limit path A.

- $R_n(P) = \{Z \in P \,:\, n = \mathit{least}\, k(A \upharpoonright (k+1) \neq Z \upharpoonright (k+1))\}$
- Let $\ell(P) = \lim_n \mathit{card}(R_n)$ (if this exists)
- Let $av(P) = \lim \frac{\sum_{k \leq n} \text{card } R_n}{n+1}$ if it exists

$X \in R_n$ is an isolated path which branches off from A at $A(n)$.
$av(P)$ is the average *population* of these branches.

Rank One Classes

Definition

Let P be a rank one Π^0_1 class with unique limit path A.

- $R_n(P) = \{Z \in P \,:\, n = \mathit{least}\, k(A \upharpoonright (k+1) \neq Z \upharpoonright (k+1))\}$
- Let $\ell(P) = \lim_n \mathit{card}(R_n)$ (if this exists)
- Let $av(P) = \lim \frac{\sum_{k \leq n} \text{card } R_n}{n+1}$ if it exists

$X \in R_n$ is an isolated path which branches off from A at $A(n)$.
$av(P)$ is the average *population* of these branches.

Subshifts

Definition

- The shift operator $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by $\sigma(X) = (X(1), X(2), \dots)$.
- A closed set P is a subshift if it is closed under σ.

- Suppose that P is a subshift of rank one with unique limit path A. Then A is periodic.
- Furthermore, P is 1-decidable.

Subshifts

Definition

- The shift operator $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by $\sigma(X) = (X(1), X(2), \dots)$.
- A closed set P is a subshift if it is closed under σ.

- Suppose that P is a subshift of rank one with unique limit path A. Then A is periodic.
- Furthermore, P is 1-decidable.

Subshifts

Definition

- The shift operator $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by $\sigma(X) = (X(1), X(2), \dots)$.
- A closed set P is a subshift if it is closed under σ.

- Suppose that P is a subshift of rank one with unique limit path A. Then A is periodic.
- Furthermore, P is 1-decidable.

Population of Subshifts

- Let P be a subshift with unique limit path $A = 0^\omega$.
- Then for each n, $R_{n+1} \subseteq R_n$.
- Since each R_n is finite, $\lim_n R_n = R$ exists, and $\ell(P) = card(R) = av(P)$ is an integer.

Population of Subshifts

- Let P be a subshift with unique limit path $A = 0^\omega$.
- Then for each n, $R_{n+1} \subseteq R_n$.
- Since each R_n is finite, $\lim_n R_n = R$ exists, and $\ell(P) = card(R) = av(P)$ is an integer.

Population of Subshifts

- Let P be a subshift with unique limit path $A = 0^\omega$.
- Then for each n, $R_{n+1} \subseteq R_n$.
- Since each R_n is finite, $\lim_n R_n = R$ exists, and $\ell(P) = card(R) = av(P)$ is an integer.

Examples

- $P = \{0^\omega\} \cup \{0^n 1^\omega : n \in \omega\}$
  $\ell(P) = av(P) = 1.$

- $Q = \{0^\omega\} \cup \{0^n 10^\omega : n \in \omega\} \cup \{0^n 110^\omega : n \in \omega\}$
  $\ell(Q) = av(Q) = 2.$

Examples

- $P = \{0^\omega\} \cup \{0^n 1^\omega : n \in \omega\}$
  $\ell(P) = av(P) = 1.$
- $Q = \{0^\omega\} \cup \{0^n 10^\omega : n \in \omega\} \cup \{0^n 110^\omega : n \in \omega\}$
  $\ell(Q) = av(Q) = 2.$

Homeomorphisms of Rank One Classes

- The problem of determining when two Π^0_1 classes of rank one are computably homeomorphic, and hence automorphic, or not, has been studied by Cenzer, Cholak+Downey, Montalbán, and others.
- It follows from this previous work that any two 1-decidable closed sets of rank one are computably homeomorphic.
- We will consider a stronger notion of homeomorphism, where the population of P serves as a characteristic.

Homeomorphisms of Rank One Classes

- The problem of determining when two Π^0_1 classes of rank one are computably homeomorphic, and hence automorphic, or not, has been studied by Cenzer, Cholak+Downey, Montalbán, and others.
- It follows from this previous work that any two 1-decidable closed sets of rank one are computably homeomorphic.
- We will consider a stronger notion of homeomorphism, where the population of P serves as a characteristic.

Homeomorphisms of Rank One Classes

- The problem of determining when two Π^0_1 classes of rank one are computably homeomorphic, and hence automorphic, or not, has been studied by Cenzer, Cholak+Downey, Montalbán, and others.
- It follows from this previous work that any two 1-decidable closed sets of rank one are computably homeomorphic.
- We will consider a stronger notion of homeomorphism, where the population of P serves as a characteristic.

Compression of Rank One Classes

Theorem

Suppose that P and Q are rank one classes such that

- $av(P) = p$, $av(Q) = q$ *exist.*
- P *c-compresses* Q.

Then, $q \cdot c \leq p$.

Main Result

Theorem

Let P and Q be 1-decidable closed sets with unique limit paths, such that $av(P) = p$ and $av(Q) = q$ exist.

Then there exists a bijection $F : P \to Q$ represented by a computable strict process machine M such that P is an M-description of Q and $\lim \frac{|M(X \upharpoonright n)|}{n} = \frac{p}{q}$.

Main Result

Theorem

Let P and Q be 1-decidable closed sets with unique limit paths, such that $av(P) = p$ and $av(Q) = q$ exist.

Then there exists a bijection $F : P \to Q$ represented by a computable strict process machine M such that P is an M-description of Q and $\lim \frac{|M(X \upharpoonright n)|}{n} = \frac{p}{q}$.

Corollaries

Definition

$F : P \to Q$ is a *strong homeomorphism* if there are computable representations M, M' of F, F^{-1} such that, for $X \in P$, $Y \in Q$,

$$\lim \frac{|M(X) \upharpoonright n)|}{n} = \lim \frac{|M'(Y \upharpoonright n)|}{n} = 1$$

Corollary

Let P and Q be 1-decidable closed sets with unique limit paths such that $av(P) = p$ and $av(Q) = q$ exist. Then

- *P strongly c-compresses Q if and only if $c \cdot q \leq p$.*
- *P is strongly homeomorphic to Q if and only if $p = q$.*

Corollaries

Definition

$F : P \to Q$ is a *strong homeomorphism* if there are computable representations M, M' of F, F^{-1} such that, for $X \in P$, $Y \in Q$,

$$\lim \frac{|M(X) \upharpoonright n)|}{n} = \lim \frac{|M'(Y \upharpoonright n)|}{n} = 1$$

Corollary

Let P and Q be 1-decidable closed sets with unique limit paths such that $av(P) = p$ and $av(Q) = q$ exist. Then

- *P strongly c-compresses Q if and only if $c \cdot q \leq p$.*
- *P is strongly homeomorphic to Q if and only if $p = q$.*

Subshifts and strong homeomorphisms

Corollary

*Let P be a subshift with a unique limit point and let $p = \ell(P) = av(P)$.
Then P is strongly homeomorphic to $\{0^\omega\} \cup \{0^n 1^k 0^\omega \ : \ k = 1, \dots p\}$.*

Current and Future Work

- $n+1$ -decidable closed sets
- Compressibility and homeomorphisms of rank $n+1$ closed sets
- Strong homeomorphisms of arbitrary closed sets

Current and Future Work

- $n+1$ -decidable closed sets
- Compressibility and homeomorphisms of rank $n+1$ closed sets
- Strong homeomorphisms of arbitrary closed sets

Current and Future Work

- $n+1$ -decidable closed sets
- Compressibility and homeomorphisms of rank $n+1$ closed sets
- Strong homeomorphisms of arbitrary closed sets

Thank You