On Friedberg Splits

Peter Cholak

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Germany
Computably Enumerable Sets

- $W_e$ is the $e$th c.e. set under some nice acceptable uniform standard enumeration of all c.e. sets.
- $W_e,s \subseteq \{0, 1, \ldots s\}$.
- A c.e. set $R$ is *computable* iff $\overline{R}$ is also a c.e. set.
- $A_0, A_1$ is a *split* of $A$ iff $A_0 \sqcup A_1 = A$ iff $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = A$.
- Focus on splits of noncomputable c.e. sets into c.e. sets.
- If $F \subseteq A$ is finite than $F \sqcup (A - F) = A$.
- A split $A_0, A_1$ is *trivial* if $A_0$ or $A_1$ is computable.
Lemma
Every noncomputable c.e. set $A$ has an infinite computable subset $R$.
Then $A = R \sqcup (A \cap \overline{R})$.

Proof.
$A = a_0, a_1, a_2 \ldots$, in the order of enumeration with no repeats. Let $R = \{a_i | (\forall j \leq i)[a_i > a_j]\}$. $n \in R$ iff, for some $i \leq n$, $n = a_i$, and, for all $j < i$, $a_j < a_i$. \qed
Myhill’s Question

Question

Does every noncomputable c.e. set have a nontrivial split?

Theorem (Friedberg)

Yes!

Myhill’s question appeared in the Journal of Symbolic Logic in June 1956, Volume 21, Number 2 on page 215 in the “Problems” section of the JSL. This question was the eighth problem appearing in this section. The question about the existence of maximal sets, also answered by Friedberg, was ninth.
**Friedberg Splits**

**Definition**

$A_0 \sqcup A_1 = A$ is a *Friedberg Split* of $A$ iff, for all $e$, if $W_e - A$ is not c.e. then $W_e - A_i$ are also not c.e.

**Lemma**

*A Friedberg split of a noncomputable set is a nontrivial split.*

**Proof.**

Assume $A_0$ is computable. So $\overline{A_0}$ is a c.e. set.

$\overline{A_0} - A = \overline{A_0} - A_1 = \overline{A}$. So this set is not a c.e. set. But then $\overline{A_0} - A_0 = \overline{A_0}$ must not be c.e. set. Contradiction.

This lemma only depends on $e$ such that $W_e - A = \overline{A}$. But which indices are these?
C.e. sets from the enumeration of $A$

- $W \setminus A = \{x | (\exists s)[x \in W_s \& x \notin A_s] \}$. ($W$ and then maybe $A$.)
- $W \setminus A = (W \setminus A) \cap A$. ($W$ and then $A$.)
- $(W \setminus A) = (W - A) \sqcup (W \setminus A)$.
- $(W - A) = (W \setminus A) \sqcup (W \setminus A)$
- So if $W - A$ is not a c.e. set then $W \setminus A$ is not computable and hence infinite.
Lemma

If $A_0 \sqcup A_1 = A$ and, for all $e$, if $W_e \setminus A$ is infinite then $W_e \setminus A_i$ is infinite, then $A_0, A_1$ is a Friedberg split of $A$.

Proof.

Assume $W - A$ is not a c.e. set but $W - A_0$ is a c.e. set. Let $X = W - A_0$. $X - A = W - A$ is not a c.e. set. So $X \setminus A$ is infinite. Therefore $X \setminus A_0$ is infinite. Contradiction.
Building a Friedberg Split

Theorem (Friedberg)
Every noncomputable set has a Friedberg Split.

Proof.
Use a priority argument to meet the following

\[ R_{e,i,k}: \quad W_e \downarrow A \text{ is infinite} \Rightarrow (\exists x > k)[x \in A_i] \]

Corollary
There is a computable total function \( f(e) = \langle e_0, e_1 \rangle \) such that if \( W_e \) is noncomputable then \( W_{f(e_0)}, W_{f(e_1)} \) is a Friedberg split of \( W_e \).
The Motivating Questions

Question
When does a c.e. set have a nontrivial nonFriedberg split?

Question
Is it possible to uniformly split all noncomputable c.e. sets into a nontrivial nonFriedberg split?
\textbf{D-hhsimple Sets}

\textbf{Definition}

\begin{itemize}
  \item $\mathcal{D}(A) = \{B \mid B - A \text{ is a c.e. set}\}$.
  \item $W$ is \textit{complemented} modulo $\mathcal{D}(A)$ iff there is a c.e. $Y$ such that $W \cup Y \cup A = \omega$ and $(W \cap Y) - A$ is a c.e. set. (Drop modulo $\mathcal{D}(A)$.)
  \item $A$ is \textit{D-hhsimple} iff, for every $W$, if $A \subseteq W$, $W$ is complemented.
  \item A complemented $W$ is 0 (modulo $\mathcal{D}(A)$) iff $W - A$ is a c.e. set.
  \item A complemented $W$ is 1 (modulo $\mathcal{D}(A)$) iff $Y - A$ is a c.e. set (the $Y$ from above). In this case, \textit{WLOG} $Y \cap A = \emptyset$.
  \item $A$ is \textit{D-maximal} iff for every $W$, if $A \subseteq W$, $W$ is complemented and either 0 or 1.
\end{itemize}
Lemma (Cholak, Downey, Herrmann)

All nontrivial splits of a $\mathcal{D}$-maximal set $A$ are Friedberg.

Proof.
Assume that $W - A$ is not a c.e. set (So $W$ is 1). Then, for some $Y$, $W \cup A \cup Y = \ast \omega$ and $Y \cap A = \emptyset$. If $W - A_0$ is c.e. then $A_0 \sqcup ((W - A_0) \cup A_1 \cup Y) = \ast \omega$. So $A_0$ is computable. Contradiction.
There are Nontrivial NonFriedberg Splits

- Let \( R \) be an infinite, co-infinite computable set. Let \( R_K \) be a noncomputable c.e. subset of \( R \).
- Similarly let \( \overline{R}_K \) be a noncomputable c.e. subset of \( \overline{R} \).
- \( R_K \sqcup \overline{R}_K = A \) is a nontrivial nonFriedberg split of \( A \).
- \( R - R_K \) is not a c.e. set but \( \overline{R} - R_K = \overline{R} \) is a c.e. set.
- Here all 3 sets were built simultaneously. We need both \( A \) and \( R \) to construct the split.
A More Difficult Example

Theorem
There is split $A_0, A_1$ of an $r$-maximal set $A$ such that the split is nontrivial and, for all $e$, either $W - A_0$ is a c.e. set or there is a $D$ with $D \cap A_0 = \emptyset$ and $A \cup D \cup W = \ast \omega$.

So $A_0$ is $D$-maximal but there are no restrictions on $A_1$.

Proof.
Sorry, some other talk. But again all 3 sets are built simultaneously.
The Kummer and Herrmann Splitting Theorem

Theorem (Kummer and Herrmann)
If \( A \subseteq X \) is noncomplemented modulo \( \mathcal{D}(A) \) then there are \( X_0 \) and \( X_1 \) such that \( X_i \) is noncomplemented and 
\[ A \subseteq X_0 \sqcup X_1 = X. \]

Corollary
For all noncomputable non-\( \mathcal{D} \)-maximal \( A \), there are disjoint \( X_0 \) and \( X_1 \) such that \( X_i \) is noncomplemented and 
\[ A \subseteq X_0 \sqcup X_1. \]

Proof.
The above theorem applies when \( A \) is not \( \mathcal{D} \)-hh-simple. 
Otherwise \( A \) must have a superset \( W \) which is not 0 or 1. So it’s complement \( Y \) is also not 0 or 1. Let \( X_0 = W \setminus Y \) and \( X_1 = Y \setminus W. \)
Splits of non-$\mathcal{D}$-maximal Sets

**Theorem (Shavrukov)**

Let $A$ be not $\mathcal{D}$-maximal and not computable. Then $A$ has a nontrivial nonFriedberg split.

**Proof.**

There are $X_0, X_1$ such that they are noncomplemented and $A \subseteq X_0 \sqcup X_1$. $X_i - A$ is not a c.e. set (otherwise $X_i$ is 0 and complemented). So $X_i \cap A$ is not computable and $X_i - (X_i \cap A) = X_i$ is a c.e. set. Hence $X_0 \cap A, X_1 \cap A$ is a nontrivial nonFriedberg split. \qed
The Motivating Questions, Again

Question
*When does a c.e. set have a nontrivial nonFriedberg split?*

**Theorem (Shavrukov)**
*All of A’s nontrivial splits are Friedberg iff A is D-maximal.*

**Question**
*Is it possible to uniformly split all noncomputable c.e. sets into a nontrivial nonFriedberg split?*
No.

**Question**
*Is it possible to uniformly split all non D-maximal sets into a nontrivial nonFriedberg split?*
Still no.
Theorem (Cholak)

For every computable $f$ there is an $e$ such that $W_e$ is not computable and if $f(e) = \langle e_0, e_1 \rangle$ then either

- $W_{e_0}, W_{e_1}$ is not a split of $W_e$,
- $W_{e_0}, W_{e_1}$ is a trivial split of $W_e$, or
- $W_{e_0}, W_{e_1}$ is a Friedberg split of $W_e$ and $W_e$ is not $D$-maximal.
The Construction Viewed from $0''$

Build $A = W_e$ via the recursion theorem. Assume that $f(e) = \langle e_0, e_1 \rangle$. Build infinite computable pairwise disjoint sets such that

$$\forall i \left[ W_i \subseteq \bigsqcup_{j \leq i} R_j \text{ or } W_i \cup A \cup \bigsqcup_{j \leq i} R_j = \ast \omega \right]$$

Inside each $R_i$ try to build $A$ to be maximal via Friedberg’s maximal set construction. So $A$ is not computable. Assume that $W_{e_0} = A_0, W_{e_1} = A_1$ is a split (otherwise done). Now in $R_i$ ask is

$$\ast \quad A_0 \cap R_i \text{ infinite?}$$

If no, then we want to focus the construction of $A$ at $R_i$. For $j < i$ dump every ball possible into $A$. For $j > i$, put no balls into $A$. So $A$ is only noncomputable inside $R_i$ and hence $A_0, A_1$ is a trivial split. Similarly, if $A_1 \cap R_i$ is finite.
The Verification

Assume we have positive answers to $\star$ for $e_0$ and $e_1$. So $A$ is maximal inside each $R_i$. The $R_i$ modulo $D(A)$ witness that $A$ is not $D$-maximal. So $A$ has a nontrivial nonFriedberg split. Locally inside each $R_i$, our split $A_0, A_1$ is Friedberg. We must show globally that $A_0, A_1$ is a Friedberg split. Consider $W_i$ and assume $W_i - A$ is not a c.e. set. Now $\#$ holds. If the first clause of $\#$ holds, then $W_i$ is handled locally inside $R_j$ for $j \leq i$ and $W_i - A_l$ is not a c.e. set. Otherwise $R_{i+1} - A \subseteq W_i$. This implies that $(W_i - A_l) \cap R_{i+1}$ is not a c.e. set.