Solovay Functions and the No-gap Phenomena

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We say that function $|x|$ is an *infinitely often tight upper bound* of $C$, up to a constant. How about prefix-free Kolomogrov complexity function $K$?

**Definition**

A function $g$ is a *Solovay function* if $g$ is computable and it holds that

1. $\forall x [K(x) \leq^+ g(x)]$
2. $\exists^\infty x [K(x) \geq^+ g(x)]$

A function $g$ is a *weak Solovay function* if $g$ is right-c.e. and satisfies both 1 and 2.
An equivalent characterization for Solovay functions

Theorem

Let \( f : \mathbb{N} \to \mathbb{N} \) be a right-c.e. function. Then \( f \) is an upper bound of \( K \) iff \( \sum_n 2^{-f(n)} \) is finite.
**Theorem**

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a right-c.e. function. Then $f$ is an upper bound of $K$ iff $\sum_n 2^{-f(n)}$ is finite.

**Theorem (Bienvenu and Downey, 2009)**

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a right-c.e. function. Then $f$ is a weak Solovay function $\iff \sum_n 2^{-f(n)}$ is finite and is a Martin-Löf random real.
### Definition

A sequence $A$ is **K-trivial** if $\forall n \ K(A \upharpoonright n) \leq^+ K(n)$.
# K-triviality and Solovay functions

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Actually, we can replace $K(n)$ in the definition by any weak Solovay function.

## Theorem (Bienvenu, Merkle and Nies, 2011)

If $g$ is a (weak) Solovay function, then

(1) a sequence $A$ is K-trivial iff $\forall n \ K(A \upharpoonright n) \leq^+ g(n)$.

And

(2) turns out to be a characterization of Solovay function among all right-c.e. functions.

## Theorem (Bienvenu, Downey, Nies and Merkle, 2015)

If $g$ is a computable (right-c.e.) function such that for any sequence $A$,

(3) $A$ is K-trivial iff $\forall n \ K(A \upharpoonright n) \leq^+ g(n)$,

then $g$ is a (weak) Solovay function.
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Theorem (Gács-Miller-Yu)

A sequence $A$ is Martin-Löf random iff for all $n \in \omega$, \[ C(A \upharpoonright n) \geq^+ n - K(n). \]
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Weak lowness for $K$ and Solovay functions

Definition

- A sequence $A$ is \textit{weakly low for} $K$ if $\exists^\infty n K^A(n) \geq K(n)$;
- A sequence $A$ is \textit{low for} $\Omega$ if $\Omega$ is Martin-Löf random relative to $A$.
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Miller first showed that these two lowness are equivalent, while Bienvenu noticed a simple proof using Solovay function:
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Miller first showed that these two lowness are equivalent, while Bienvenu noticed a simple proof using Solovay function:

- Function $K$ is right-c.e., it is also right-c.e. relative to $A$.
- And $K$ is also an upper bound for $K^A$ up to an additive constant.
- By definition, $A$ is weakly low for $K$ iff $K$ is a weak Solovay function relative to $A$.
- Relativizing the equivalent characterization of Solovay function, $K$ is a weak Solovay function relative to $A$ iff $\Omega_K = \sum_n 2^{-K(n)}$ is Martin-Löf random relative to $A$.
- So $A$ is weakly low for $K$ iff $A$ is low for $\Omega$. 
Theorem

If $g$ is a weak Solovay function, then a sequence $A$ is weakly low for $K$ iff $\exists^\infty n K^A(n) \geq^+ g(n)$.
Weak lowness for $K$ and Solovay functions

Theorem

*If $g$ is a weak Solovay function, then a sequence $A$ is weakly low for $K$ iff $\exists^\infty n K^A(n) \geq^+ g(n)$.*

- One direction is trivial.
- $A$ is weakly low for $K$, then it is low for $\Omega$.
- $\Omega_g = \sum_n 2^{-g(n)}$ is 1-random and left-c.e., then by Kučera-Slaman Theorem, it is $\Omega$-like.
- Then $\Omega_g$ is 1-random relative to $A$.
- By relativization, $\exists^\infty n K^A(n) \geq^+ g(n)$. 
Theorem

Let $g$ be a right-c.e. function such that for any sequence $A$, $A$ is weakly low for $K$ iff $\exists^\infty n K^A(n) \geq^+ g(n)$, then $g$ is a weak Solovay function.
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Let $g$ be a right-c.e. function such that for any sequence $A$, $A$ is weakly low for $K$ iff $\exists^\infty n K^A(n) \geq^+ g(n)$, then $g$ is a weak Solovay function.

- For all sequence $A$, $\forall n K^A(n) \leq^+ K(n)$.
- If for some sequence $A$, $\exists^\infty n K^A(n) \geq^+ g(n)$, then $\exists^\infty n K(n) \geq^+ g(n)$.
- If $g$ is not an upper bound of $K$, then $\sum_n 2^{-g(n)} = \infty$.
- For all $A$, $\sum_n 2^{-K^A(n)} < \infty$, $\exists^\infty n K^A(n) \geq^+ g(n)$.
Theorem (Miller)

A set $A$ is 2-random iff \( \exists \infty n K(A \upharpoonright n) \geq^+ K(n) + n \).
2-randomness and Solovay functions

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Theorem

If $g$ is a weak Solovay function, then a sequence $A$ is 2-random iff $\exists^\infty n K(A \restriction n) \geq^+ n + f(n)$.
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Theorem

If $g$ is a weak Solovay function, then a sequence $A$ is 2-random iff $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$.

- $A$ is 2-random iff $A$ is 1-random and low for $\Omega$.
- $A$ is 1-random, by Ample Excess Lemma, $\forall n K^A(n) \leq^+ K(A \upharpoonright n) - n$.
- $A$ is low for $\Omega$, by previous result, $\exists^\infty n K^A(n) \geq^+ g(n)$.
- Thus, $\exists^\infty n K(A \upharpoonright n) \geq^+ n + g(n)$.
Theorem

If $f$ is a right-c.e. function, and for any sequence $A$, $A$ is 2-random iff $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$, then $f$ is a weak Solovay function.
Theorem

If $f$ is a right-c.e. function, and for any sequence $A$, $A$ is 2-random iff $\exists^\infty n \; K(A \upharpoonright n) \geq^+ n + f(n)$, then $f$ is a weak Solovay function.

- For all sequence $A$, $\forall n \; K(A \upharpoonright n) \leq^+ n + K(n)$.
- If for some sequence $A$, $\exists^\infty n \; K(A \upharpoonright n) \geq^+ n + f(n)$, then $\exists^\infty n \; K(n) \geq^+ f(n)$.
- If $g$ is not an upper bound of $K$, then for all $A$, $\exists^\infty n \; K^A(n) \geq^+ f(n)$.
- By Ample Excess Lemma, then all 1-random sequences $A$, $\exists^\infty n \; K(A \upharpoonright n) \geq^+ n + f(n)$. 
Infinitely often $K$-triviality and Solovay functions

Definition

A sequence $A$ is *infinitely often $K$-trivial* if there are infinitely many point $n$ such that $K(A \upharpoonright n) \leq^+ K(n)$. It seems very promising that in the definition the function $K(n)$ can be replaced by arbitrary Solovay function, but we will see that it is false.

Theorem

There is a Solovay function $f$ that for some sequence $A$ there are infinitely many point $n$ such that $K(A \upharpoonright n) + f(n)$ but $A$ is not infinitely often $K$-trivial.
A sequence $A$ is *infinitely often $K$-trivial* if there are infinitely many point $n$ such that $K(A \upharpoonright n) \leq^+ K(n)$.

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Theorem

*There is a Solovay function $f$ that for some sequence $A$ there are infinitely many point $n$ such that $K(A \uparrow n) \leq^+ f(n)$ but $A$ is not infinitely often often K-trivial*
Proof.

- Suppose $f$ is a Solovay function, define $f_1$ and $f_2$ as follows:

  \[ f_1(x) = \begin{cases} f(x) & \text{if } x \text{ is odd} \\ 2x & \text{if } x \text{ is even} \end{cases} \quad f_2(x) = \begin{cases} 2x & \text{if } x \text{ is odd} \\ f(x) & \text{if } x \text{ is even} \end{cases} \]

- $\forall x \ K(x) \leq^+ 2|x| \leq 2x$, and $\forall x \ K(x) \leq^+ f(x)$, then $\forall x \ K(x) \leq^+ f_1(x)$ and $K(x) \leq^+ f_2(x)$.

- As there are infinitely many $x$ such that $K(x) \geq^+ f(x)$. then at least for one of $f_i (i = 0, 1)$, there are infinitely many $x$ such that $K(x) \geq^+ f_i(x)$.

- Suppose $i = 1$, then $f_1$ is a Solovay function.

- For any sequence $A$, for all even number $n$, $K(A \upharpoonright n) \leq^+ 2n = f_1(n)$.

\[\Box\]
However, whether the converse is true is still not clear at present. Recently, George and Bauwens independently proved the following theorem.

Theorem

For any function $f$ which goes to infinity, there exists a sequence $A$ such that $A$ is not infinitely often $K$-trivial but $\exists n \in \mathbb{N} (A|n) + K(n) + f(n)$. That's to say, among all right-c.e. functions which are upper bounds of $K$, if for any sequence $A$, $A$ is infinitely often $K$-trivial iff $\exists n \in \mathbb{N} (A|n) + g(n)$, then $g$ is a weak Solovay function.

But whether all computable (right-c.e.) functions which make the equivalence ture should be (weak) Solovay functions is still open.
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But whether all computable (right-c.e.) functions which make the equivalence ture should be (weak) Solovay functions is still open.
Let \( g \) be any weak Solovay function, the following assertions are true.

1. \( \sum_n 2^{-g(n)} \) is a Martin-Löf random real.
2. A sequence \( A \) is \( K \)-trivial iff \( \forall n \ K(A \upharpoonright n) \leq^+ g(n) \).
3. A sequence \( A \) is Martin-Löf random iff \( \forall n \ C(A \upharpoonright n) \geq^+ n - g(n) \).
4. A sequence \( A \) is weakly low for \( K \), iff \( \exists^\infty n \ K^A(n) \geq^+ g(n) \).
5. A sequence \( A \) is 2-random, iff \( \exists^\infty n \ K(A \upharpoonright n) \geq^+ n + g(n) \).

What’s more, among all right-c.e. functions the respective assertion is true exactly for the Solovay functions.
The no-gap phenomena

**Theorem**

Suppose \( f \) is a right-c.e. function, the following are equivalent:

1. \( \forall x [K(x) \leq^+ f(x)] \);
2. \( \sum_n 2^{-f(n)} < \infty \);
3. If \( A \) is \( K \)-trivial, then \( \forall n K(A \upharpoonright n) \leq^+ f(n) \);
4. If \( A \) is 1-random, then \( \forall n C(A \upharpoonright n) \geq^+ n - f(n) \).
5. If \( \exists \infty n K^A(n) \geq^+ f(n) \), then \( A \) is weakly low for \( K \);
6. If \( \exists \infty n K(A \upharpoonright n) \geq^+ n + f(n) \), then \( A \) is 2-random;
The no-gap phenomena

**Theorem**

Suppose $f$ is a right-c.e. function, and is an upper bound for $K$, the following are equivalent:

1. $\exists^\infty x[K(x) \geq^+ f(x)]$;
2. $\sum_n 2^{-f(n)}$ is 1-random;
3. If $\forall n K(A \upharpoonright n) \leq^+ f(n)$, then $A$ is $K$-trivial;
4. If $A$ is weakly low for $K$, then $\exists^\infty n K^A(n) \geq^+ f(n)$;
5. If $A$ is 2-random, then $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$;
6. If $\forall n C(A \upharpoonright n) \geq^+ n - f(n)$, then $A$ is 1-random.
In the proof of our previous theorems, we proved the following so-called “no-gap” theorems:

**No-gap**

There is no function $h : \mathbb{N} \mapsto \mathbb{N}$ which tends to infinity and such that:

1. $C(A \upharpoonright n) \geq^+ n - K(n) - h(n) \implies A$ is Martin-Löf random;
2. $K(A \upharpoonright n) \leq^+ K(n) + h(n) \implies A$ is infinitely often $K$-trivial;

For $K$-triviality, George and Charlotte showed that there is no $\Delta^0_2$ “gap”, but Martijn and George showed there does exist a $\Delta^0_3$ “gap”.