Weihrauch-completeness for layerwise computability\textsuperscript{1}

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\textsuperscript{1}Joint work with George Davie & Willem Fouché (UNISA).
Outline

Definitions

The main result

Examples

A non-example
Layerwise computability

Fix a universal Martin-Löf test $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$.

**Definition**

A (multivalued) function $f : \text{MLR} \Rightarrow X$ is layerwise computable w.r.t. $\mathcal{U}$, iff there exists a computable partial function $F : \subseteq \mathbb{N} \times \text{MLR} \to X$ such that whenever $p \notin U_n$ then $F(n, p) \in f(p)$.

**Theorem (Hölzl & Shafer)**

Layerwise computability does depend on the choice of $\mathcal{U}$ in general, but all optimal Martin-Löf tests yield the same class.
Definition
A finitely-revising machine is a Type-2 machine with the extra capability to erase its output and restart writing it, to be used finitely many times during the computation. A function is computable with finitely many mindchanges, if there this a finitely-revising machine computing it.

Definition
A non-deterministic Type-2 machine with advice space $\mathbb{Z}$ computes a multivalued function $f : X \Rightarrow Y$ as follows:

1. On input $x \in X$, guess some $z \in \mathbb{Z}$.
2. Either: Halt and reject the guess.
3. Or: Run indefinitely, and output some $y \in f(x)$.

Such that for any $x \in X$ there is some $z \in \mathbb{Z}$ leading to case 3.
Observation (Brattka, de Brecht & P.)

Finitely revising machines and non-deterministic machines with advice space \( \mathbb{N} \) are equivalent.

Observation

Any layerwise computable function is computable by non-deterministic machine with advice space \( \mathbb{N} \).
Represented spaces and computability

Definition
A represented space $X$ is a pair $(X, \delta_X)$ where $X$ is a set and $\delta_X : \subseteq \mathbb{N}^\mathbb{N} \rightarrow X$ a surjective partial function.

Definition
$F : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ is a realizer of $f : X \Rightarrow Y$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \delta_X^{-1}($dom$(F))$.

Definition
$f : X \Rightarrow Y$ is called computable (continuous), iff it has a computable (continuous) realizer.
Weihrauch-reducibility

Definition
For \( f : \subseteq X \Rightarrow Y, g : \subseteq V \Rightarrow W \) say

\[
f \leq_W g
\]

iff there are computable \( H, K : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N} \), such that
\( K \langle \text{id}_{\mathbb{N}^\mathbb{N}}, GH \rangle \) is a realizer of \( f \) for every realizer \( G \) of \( g \).

Theorem (Brattka & Gherardi 2011, P. 2010)
\( \mathcal{W} \) is a distributive lattice. The cartesian product \( \times \) is an operation on \( \mathcal{W} \).

Theorem (Higuchi & P. 2013)
For \( A \subseteq \mathbb{N}^\mathbb{N} \), let \( d_A : A \rightarrow \{0\} \). Then \( d : \mathcal{W}^{\text{op}} \rightarrow \mathcal{W} \) is a lattice embedding.
The motivation

1. Identify a theorem

\[ \forall x \in X \ \exists y \in Y . \ D(x) \Rightarrow T(x, y) \]

with the multi-valued function \( T : \subseteq X \Rightarrow Y \), \( \text{dom}(T) = D \) obtained by Skolemization.

2. Then compare theorems via Weihrauch-reducibility to learn about their constructive content.

Similar spirit as (constructive) reverse mathematics, but:

Theorem (Higuchi & P. 2013)
\( \mathcal{W} \) is not a Brouwer algebra.
The degree of $C_N$

**Lemma**

The following are Weihrauch equivalent:

1. $C_N \subseteq A(N) \Rightarrow \mathbb{N}$ be defined via $n \in C_N(A)$ iff $n \in A$

2. $UC_N$, defined via $UC_N = (C_N)|\{A \in A(N) | |A| = 1\}$

3. $\min_A : \subseteq A(N) \rightarrow \mathbb{N}$

4. $\max_O : \subseteq O(N) \rightarrow \mathbb{N}$

5. $\text{Bound} : \subseteq O(N) \Rightarrow \mathbb{N}$, where $n \in \text{Bound}(U)$ iff $\forall m \in U n \geq m$. 

Weihrauch-completeness for layerwise-computability

Definition
Let $\text{LAY}_U : \text{MLR} \Rightarrow \mathbb{N}$ be defined via $n \in \text{LAY}_U(p)$ iff $p \notin U_n$.
Let $\text{rd}_U : \text{MLR} \rightarrow \mathbb{N}$ be defined via
$\text{rd}_U(p) = \min\{n \in \mathbb{N} \mid p \notin U_n\}$.

Observation
$\text{LAY}_U$ is layerwise computable w.r.t. $U$. Whenever $f : \text{MLR} \Rightarrow X$ is layerwise computable w.r.t. $U$, then $f \leq_W \text{LAY}_U$.

- If $f$ is layerwise-computable and $f \equiv_W \text{LAY}_U$, call $f$ Weihrauch-complete for layerwise computability.
- The problems that are Weihrauch-complete for layerwise computability are the most non-computable layerwise-computable problems.
The main theorem

Theorem

\[ \text{LAY}_U \equiv_W \text{rd}_U \equiv_W C_N \times d_{\text{MLR}} \]

Proof.

\[ \text{LAY}_U \leq_W \text{rd}_U \]

Trivial.

\[ \text{rd}_U \leq_W \min_A \times d_{\text{MLR}} \]

We have a random sequence available as input for \( d_{\text{MLR}} \), and the presence of this degree does not matter further. Note that given \( p \) we can compute \( \{ n \mid p \notin U_n \} \in A(\mathbb{N}) \).

\[ \square \]
Proof continued

Proof.

Bound $\times d_{MLR} \leq_W \text{LAY}_U$ The input is an enumeration of some finite set $I \subset \mathbb{N}$ (which we may safely assume to be an interval) and a random sequence $p$. Let $w$ be the current prefix of the output (i.e. the input to LAY$_U$). If we learn that $n \in I$, we consider $w0^N$. As this is not random and $U$ is universal, we know that $w0^N \in U_n$. As $U_n$ is open, there is some – effectively findable – $k \in \mathbb{N}$ such that $w0^k\{0, 1\}^N \subseteq U_n$. We proceed to amend the current output to $w0^k$, and then start outputting $p$ (until we potentially learn $n + 1 \in I$.

As $I$ is finite, the output $q$ will have some tail identical to $p$, and thus is Martin Löf random. By construction, whenever $n \in I$, then $q \in U_n$, thus if $b \in \text{LAY}_U(q)$ then $b \in \text{Bound}(p)$. 

\qed
Corollaries

- $\text{LAY} \prec_W \text{C}_N$
- $\text{LAY} \times \text{LAY} \equiv_W \text{LAY}$ and $\text{LAY} \star \text{LAY} \equiv_W \text{LAY}$
- $\text{LAY} \star \text{C}_N \equiv_W \text{C}_N \star \text{LAY} \equiv_W \text{LAY}$
- $\text{LAY} \prec_W \text{LAY} \equiv_W \text{lim} \times d_{\text{MLR}}$
- $\text{LAY} \prec_W \text{LAY}^* \equiv_W \text{id}_{\text{NN}} + \text{LAY} \prec_W \text{C}_N$
- If $f \preceq_W \text{C}_N$ for $f : \subseteq \text{MLR} \Rightarrow \text{Y}$, then $f \preceq_W \text{LAY}$.
More consequences

Corollary

The following are equivalent for \( f : \subseteq MLR \to Y \) for a computable metric space \( Y \):

1. \( f \) is effectively \( \Delta^0_2 \)-measurable.
2. \( f \) is \( \Pi^0_1 \)-piecewise computable.
3. \( f \leq_W LAY \).

Proof.

By combining the computable Jayne-Rogers theorem (P. & de Brecht 2014) with the main theorem.
Complex oscillations

Definition
The complex oscillations CO are the Martin-Löf random elements of $C_0([0, 1], \mathbb{R})$ equipped with the Wiener measure. Let computable $\eta : \text{MLR} \to \mathbb{R}$ induce the normal distribution $\mathcal{N}(0, 1)$ on $\mathbb{R}$.

Definition
We define the function $\Phi : \text{MLR} \to \text{CO}$ by recursively providing the values $\Phi(\alpha)$ takes on dyadic rationals, and extending it continuously to the interval. Let $\alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_{jn}, \ldots \rangle$, where $n \leq 2^j$. Then we define:

1. $\Phi(\alpha)(1) := \eta(\alpha_0)$
2. $\Phi(\alpha)(\frac{1}{2}) := \frac{1}{2} (\eta(\alpha_0) + \eta(\alpha_1))$
3. $\Phi(\alpha)(\frac{2n+1}{2^{j+1}}) := \frac{1}{2} \left( 2^{-j/2} \eta(\alpha_{jn}) + \Phi(\alpha)(\frac{n+1}{2^j}) + \Phi(\alpha)(\frac{n}{2^j}) \right)$

Theorem (Davie & Fouché)
$\Phi$ is a layerwise computable bijection with computable inverse.
The completeness result

**Theorem**

\[ \Phi \equiv w \text{LAY} \]

**Lemma**

*Given* \( k \in \mathbb{N} \) *and* \( v \in \{0, 1\}^* \) *we can compute some* \( w \in \{0, 1\}^* \) *such that for all* \( \alpha \in MLR \) *we find that* \( k < \sup_{t \in [0,1]} \Phi(vw\alpha)(t) \).
Law of the iterated logarithm

Definition
Let \( \text{LIL} : \text{MLR} \Rightarrow \mathbb{N} \) be defined via \( N \in \text{LIL}(\alpha) \) iff:

\[
\forall n \geq N \quad \left| \sum_{i=0}^{n-1} (2\alpha(i) - 1) \right| < \sqrt{2n \log \log n}
\]

Theorem
\( \text{LIL} \equiv_W \text{LAY} \).

Lemma
Given \( N \in \mathbb{N} \) and \( u \in \{0, 1\}^* \) we can compute some \( v \in \{0, 1\}^* \) such that \( |uv| > N \) and

\[
\left| \sum_{i=0}^{|uv|-1} (2(uv)(i) - 1) \right| > \sqrt{2|uv| \log \log |uv|}.
\]
Birkhoff’s theorem

Definition
Let $S : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N}$ be the usual shift-operator, and $\pi_1 : \{0, 1\}^\mathbb{N} \to \{0, 1\}$ be the projection to the first bit. Let Birkhoff : $\text{MLR} \times \mathbb{N} \Rightarrow \mathbb{N}$ be defined via $N \in \text{Birkhoff}(p, k)$ iff $\forall n \geq N$ we find that:

$$| \left( \frac{1}{n+1} \sum_{i=0}^{n} \pi_1(S^i(p)) \right) - \frac{1}{2} | < 2^{-k}$$

Theorem
Birkhoff $\equiv_{w} LAY$
Lemma

Given $u \in \{0, 1\}^*$ and $k, N \in \mathbb{N}, k > 0$, we can compute some $v \in \{0, 1\}^*$ such that $|uv| \geq N$ and:

$$| \left( \frac{1}{|uv|} \sum_{i=0}^{|uv|-1} \pi_1(S^i(uv)) \right) - \frac{1}{2} | > 2^{-k}$$
Hitting times

Definition
Let $\mathcal{A}_{\lambda>0}(\{0, 1\}^\mathbb{N})$ be the restriction of $\mathcal{A}(\{0, 1\}^\mathbb{N})$ to sets of positive Lebesgue measure. Let $T : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N}$ be the usual shift-operator. Define

$\text{HittingTime}_\mathcal{A} : \text{MLR} \times \mathcal{A}_{\lambda>0}(\{0, 1\}^\mathbb{N}) \to \mathbb{N}$ defined via $\text{HittingTime}_\mathcal{A}(\rho, A) = \min\{n \in \mathbb{N} \mid T^n(\rho) \in A\}$.

Theorem (Kučera)
$\text{HittingTime}_\mathcal{A}$ is well-defined.

Theorem
$\text{HittingTime}_\mathcal{A} \equiv_W \text{LAY}$, but not even $\text{HittingTime}_\mathcal{A}(\cdot, U_{100}^C)$ is layerwise computable.
Some last minute-additions

- Finding the suitable $n$ from the multiple recurrence theorem for Martin-Löf randoms is Weihrauch-equivalent to LAY (but not layerwise computable).
- Computing the time-reversal of a Brownian motion on $[0, \infty)$ should be Weihrauch-reducible to LAY (but what about the other direction)?
Some open questions

- Investigate further layerwise-computable problems.
- Is there a (natural) problem which is non-computable, layerwise computable and strictly below $\text{LAY}$?
A. Pauly, G. Davie and W. Fouché.  
Weihrauch-completeness for layerwise computability  

R. Hölzl and P. Shafer.  
Universality, optimality, and randomness deficiency  