Conditional probability and Van Lambalgen's theorem

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Outline

van Lambalgen's theorem

Philosophically: following tasks are equally hard:

- Finding structure in a pair of sequences
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Theorem (van Lambalgen)

The following are equivalent:

1. (α, β) *is Martin-Löf random,*
2. α *is Martin-Löf random and* β *is Martin-Löf random relative to* α*.*

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Goal: van Lambalgen's theorem for computable measure

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Theorem (?)

The following are equivalent for computable measure P on $\Omega \times \Omega$:

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Theorem (Vovk & Uspensky)

The following are equivalent for all computable bivariate measures P for which $P(\cdot|\alpha)$ is uniformly computable in α:

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Theorem (Folklore)

The following are equivalent for all measures P and Q:

1. (α, β) *is* $(P \times Q)$*-Martin-Löf random relative to a sequence that computes* $(P \times Q)$*,*
2. α *is* P*-Martin-Löf random relative to some sequence that computes* P*, and* β *is* Q*-Martin-Löf random relative to* α *and some sequence that computes* Q*.*

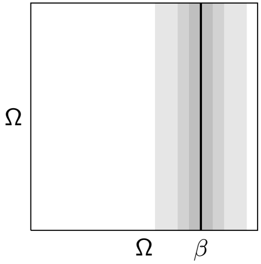
Outline

Conditional probability in continuous space

- P is measure on $\Omega \times \Omega$, let $P(x, y) = P([x], [y])$.

$$v(x|\beta) = \lim_{n\to\infty} \frac{P(x, \beta_1 \ldots \beta_n)}{h(\beta_1 \ldots \beta_n)}, \quad h(y) = P(\Omega, y).$$

- If $P(x, y) = Q(x)R(y)$, then $v(x|\beta) = Q(x)$.
- $v(x|\beta)$ might not exist...
- Computable P, still $v(\cdot|\alpha)$ might not be computable [AFR2011]:



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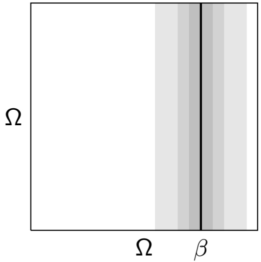

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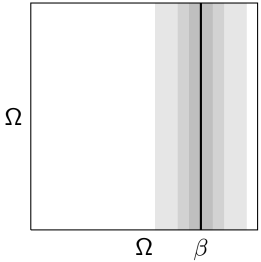

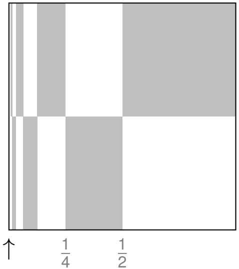

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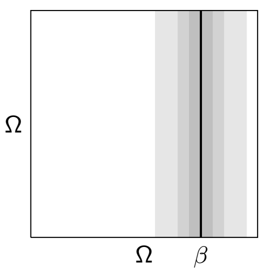

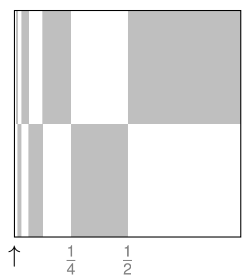

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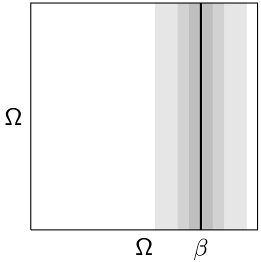

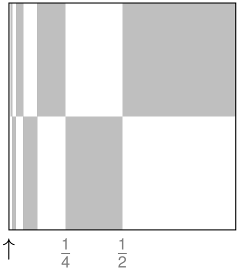

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Theorem (Takahashi)

If β is h-Martin-Löf random, then $v(\cdot|\beta)$ exists and is a measure.

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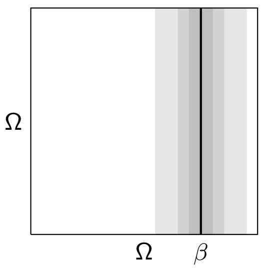

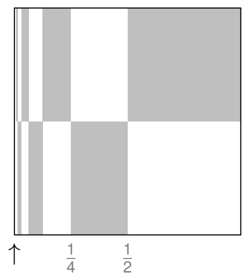

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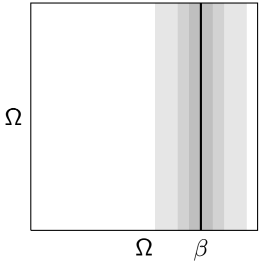

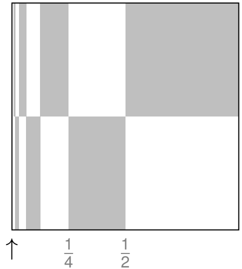

y	λ	0	00	000	...
$\frac{P(0,y)}{P(\lambda,y)}$	2/3	1/3	2/3	1/3	...
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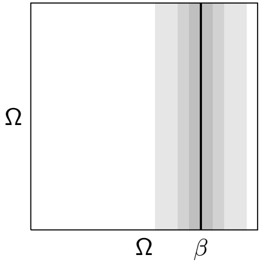

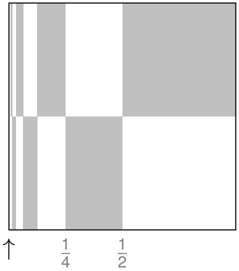

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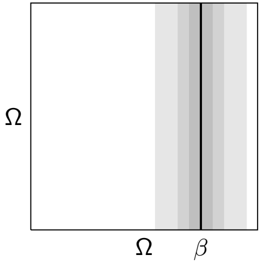

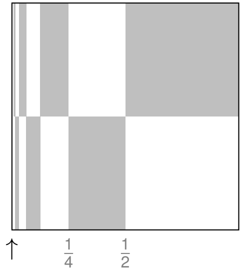

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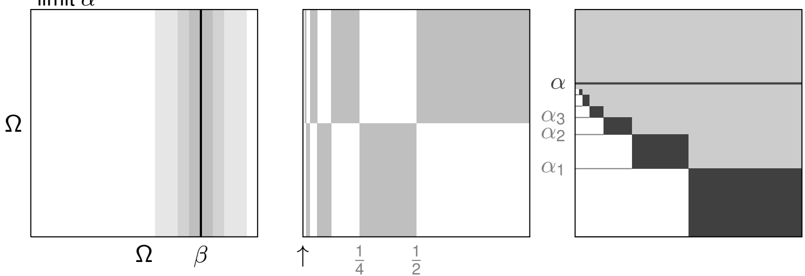

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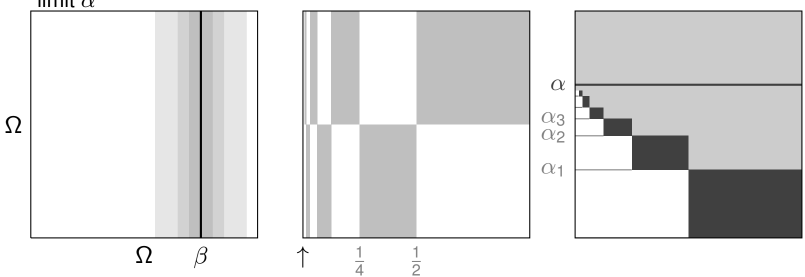

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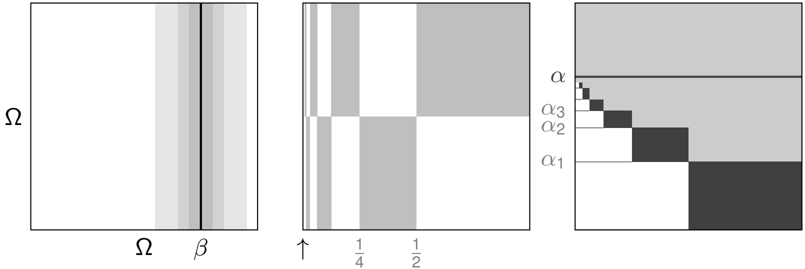

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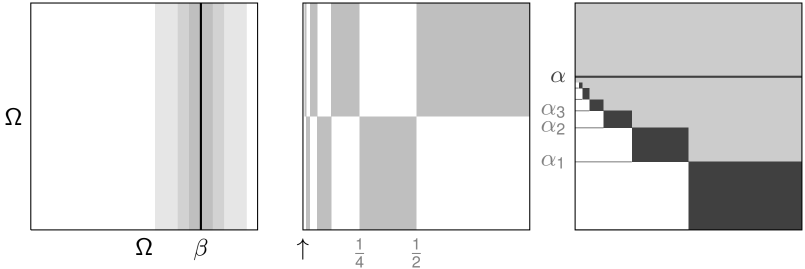

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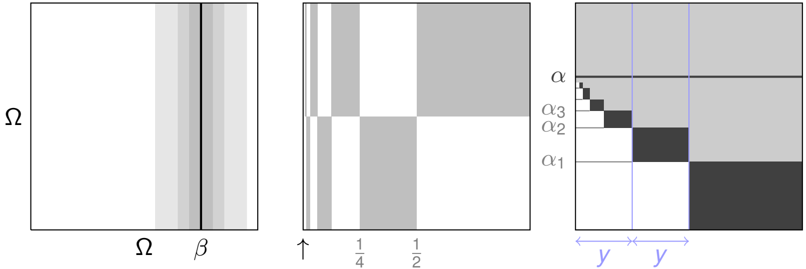

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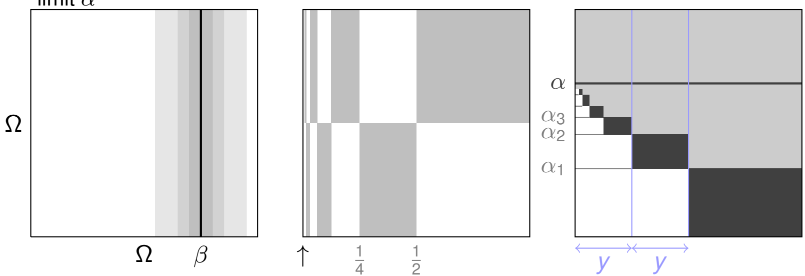

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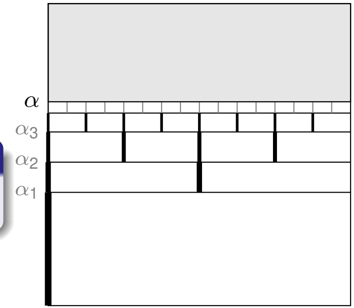

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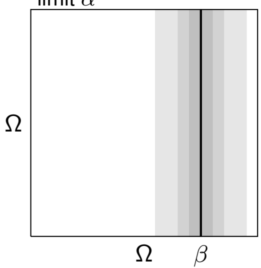

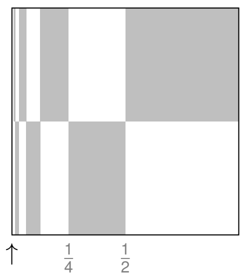

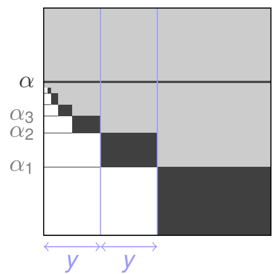

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Outline

Van Lambalgen's theorem fails for computable P

Non-computable measure Q:

- α is *Hippocratic (blind)* Q-random if $\alpha \in \bigcap_n U_n$ for all uniformly effectively open U_n st $Q(U_n) \leq 2^{-n}$
- α is *uniformly* Q-random if there exists a sequence ξ that computes Q, such that $\alpha \in \bigcup_n U_n$ for all uniformly ξ-effectively open U_n with $Q(U_n) \leq 2^{-n}$.

- $R =]\alpha_{|y|}, \alpha_{|y|+1}[\times [y]$?
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Theorem (B.)

There exists a computable P and a pair (α, β) such that

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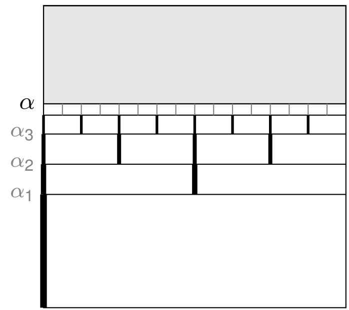

Proof:

- Choose $\alpha_1 \leq \alpha_2 \leq \cdots$ be computable rational sequence with limit Martin-Löf random limit α. P as before.
- $v(\cdot|\beta)$ for β with infinitely many ones:

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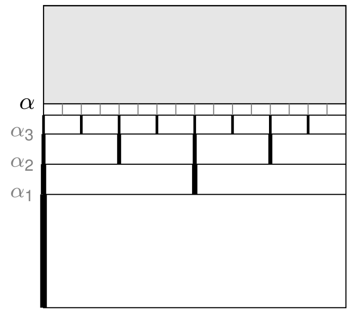

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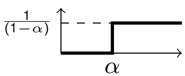

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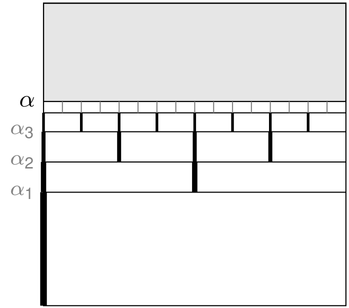

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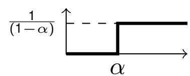
- $v(\cdot|\beta)$ for β with infinitely many ones: 

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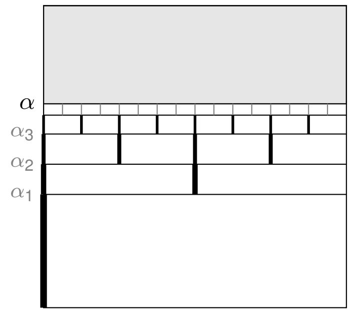

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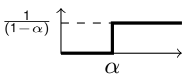

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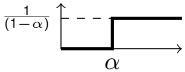

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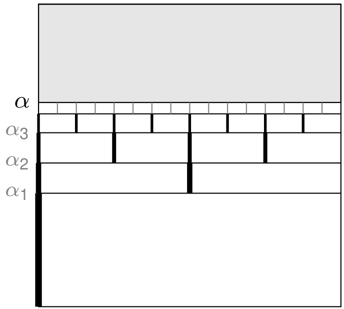

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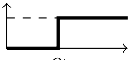
- $v(\cdot|\beta)$ for β with infinitely many ones: 

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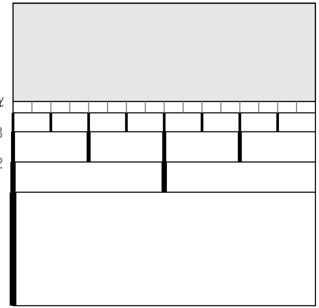

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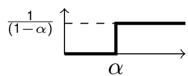

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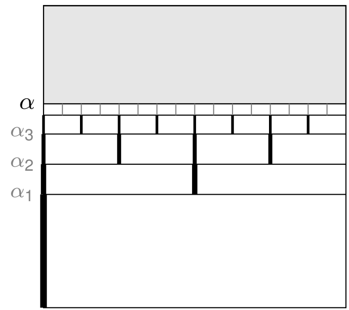

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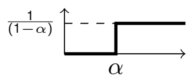

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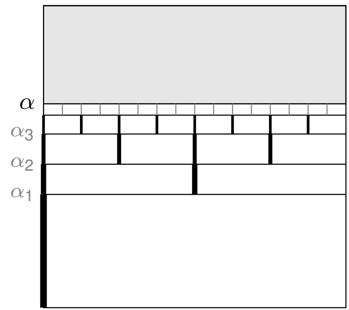

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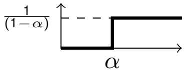

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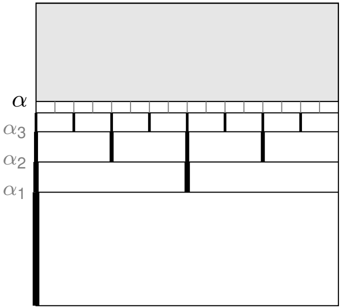

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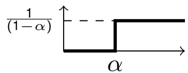

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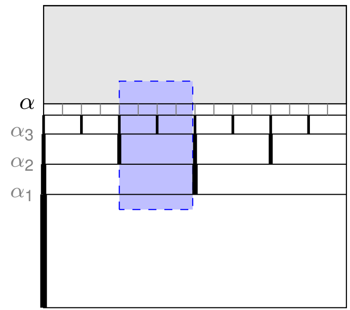

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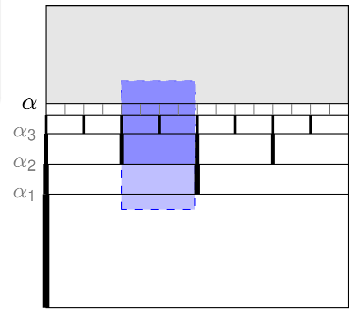

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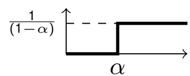

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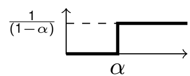
- $v(\cdot|\beta)$ for β with infinitely many ones: 

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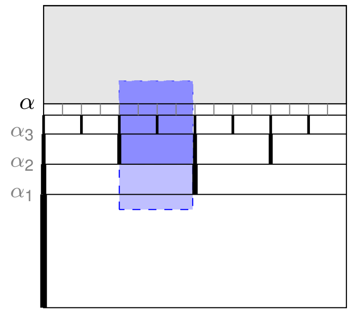

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Outline

1. Conditional probability
2. Van Lambalgen's theorem fails for computable P
3. H. Takahashi's generalization of van Lambalgen's theorem

Generalize van Lambalgen's theorem

Recall $h(y) = P(\text{empty string}, y)$ and $v(x|\beta) = \lim \frac{P(x,\beta_1 \cdots \beta_n)}{h(\beta_1 \cdots \beta_n)}$

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For a computable P and a sequence β for which $v(\cdot|\beta)$ is computable, the following are equivalent:

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Proof that 1 implies 2: original argument, i.e., all uniform measures:

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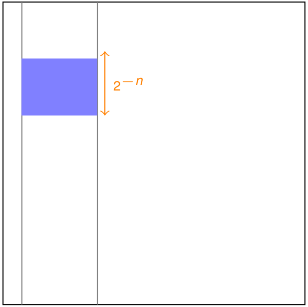

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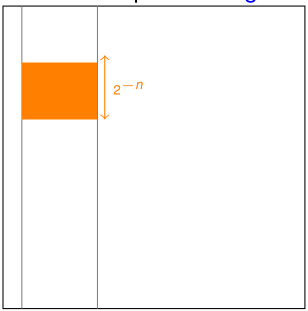

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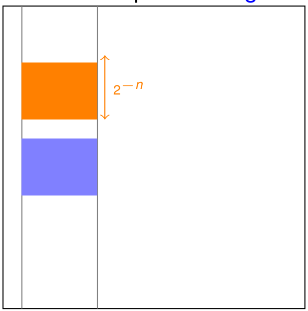

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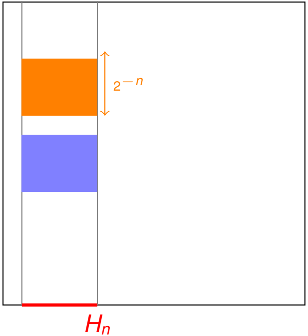

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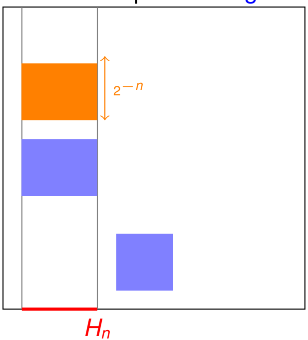

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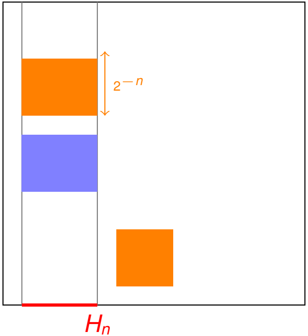

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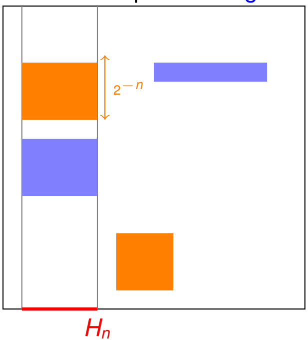

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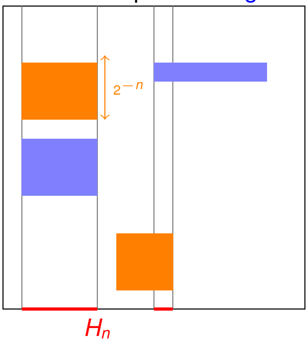

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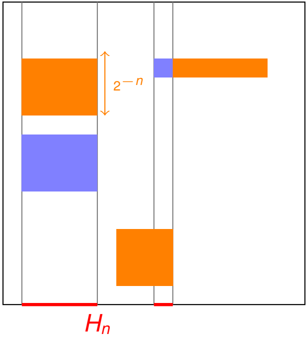

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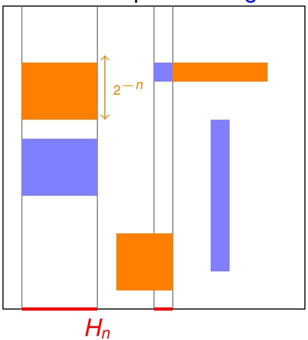

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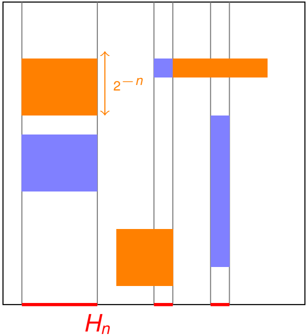

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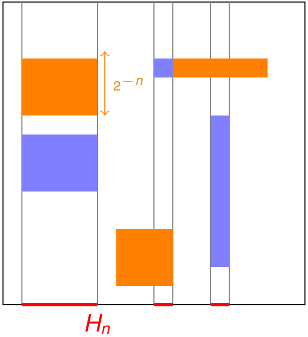

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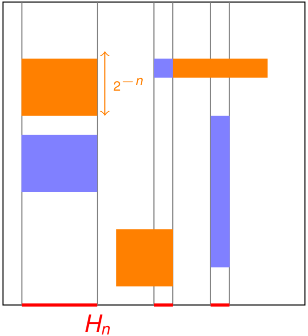

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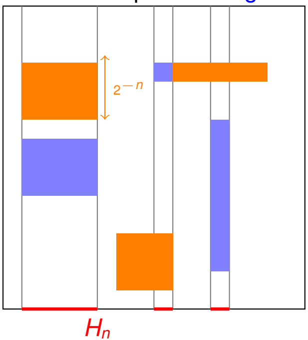

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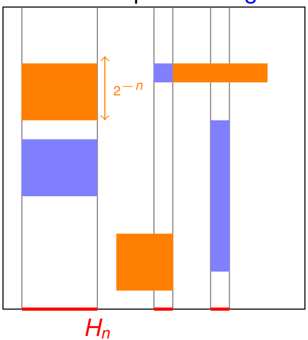

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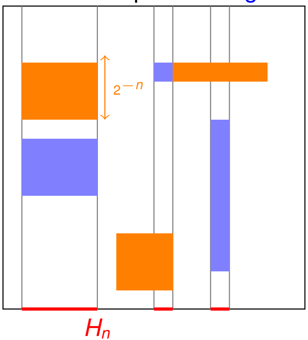

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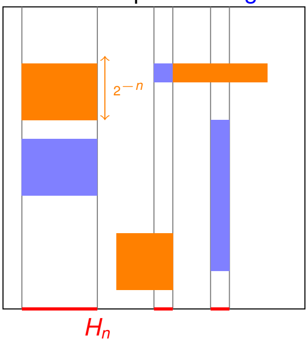

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- Clearly, $v(V_n^\beta|\beta) \leq 2^{-n}$ and 2nd criterium are OK.
- ~~Note $\mu(U_{2n}) \geq 2^{-n}\mu(H_n)$, hence, $2^{-2n}/2^{-n} \geq \mu(H_n)$.~~
- Proof theorem: suppose (α, β) is not random, one of the cases in 2nd criterion happens infinitely often, this is enough for non Martin-Löf randomness