ON THE EXISTENCE AND UNIQUENESS OF A WARPENING FUNCTION IN THE ELASTIC - PLASTIC TORSION OF A CYLINDRICAL BAR WITH MULTIPLY CONNECTED CROSS - SECTION

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0. INTRODUCTION.

In recent years the elastic - plastic torsion of a cylindrical bar has been of considerable interest for many mathematicians and mechanists. In the case of a bar with simply connected cross - section important contributions are due to TING [9,10,11,12] , BREZIS [2] , and BREZIS & STAMPACCHIA [3] . They could show the regularity of the stress components and the existence of a unique displacement vector.

In contrast to the torsion of a bar with simply connected cross - section the elastic - plastic torsion problem is much more involved when the cross - section is multiply connected in view of the non - homogeneous boundary conditions in that case. Using an idea of COURANT [4] LANCHON [7] could determine the stress components in a weak sense as the gradient of the solution to a variational problem with constraints. The $C^{1,\alpha}$ - regularity of that solution has been proved by us in [5] using an abstract regularity theorem of BREZIS & STAMPACCHIA [3] .

Up to now it has been an open problem to determine the displacement vector in the case of a multiply connected cross - section. It is the aim of this paper to solve this problem. The way we do that gives a new physical interpretation of the elastic - plastic torsion problem, namely, we shall treat this problem within the framework of the <u>nonlinear elasti-</u>

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city theory of hardening materials, i.e. the elastic - plastic torsion problem may be looked at as a nonlinear problem without constraints instead of a linear problem with constraints.

1. STATEMENT OF THE PROBLEM AND NOTATIONS.

Let $x = (x^1, x^2, x^3)$ be the Euclidean space coordinates. We consider the torsion of a cylindrical bar of arbitrary cross - section. Let the lower end of the bar be clamped in the plane $x^3 = 0$, and suppose that a constant torque is applied to the other end. Let the x^3 - axis be parallel to the generators of the cylinder. The cross - section Ω is supposed to be a multiply connected domain with finitely many holes Ω_k, $k = 1,\dots,N$. We assume moreover that the respective boundaries $\Gamma_k = \partial\Omega_k$ satisfy $\Gamma_k \cap \Gamma_1 = \emptyset$ for $k \neq 1$.

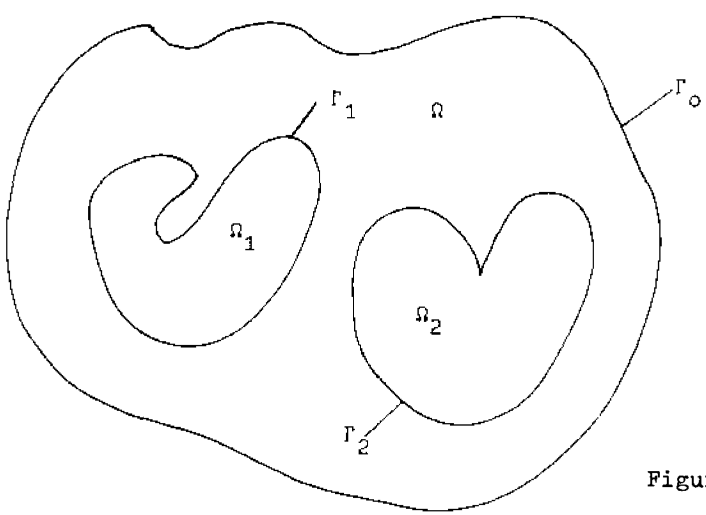

Figure 1

The boundary of Ω is the union of the disjoint family $\{\Gamma_0, \Gamma_1,\dots, \Gamma_N\}$. We assume that $\partial\Omega$ is a Lipschitz boundary which satisfies the following outward sphere condition : for any boundary point x_0 there is a ball B of fixed radius R such that the intersection of $\bar{\Omega}$ and $\bar{B}$ consists of x_0 alone.

For later use we define

(1.1) $$\Omega^{*} = \Omega \cup \bigcup_{k=1}^{N} \overline{\Omega}_k .$$

Following the hypotheses of St. VENANT we assume that the cross - sections of the bar rotate in their planes, but are warped in the direction of the x^3 - axis. Thus, the components of the displacement vector are

(1.2) $$v_1 = -\gamma x^2 x^3, \quad v_2 = \gamma x^1 x^3, \quad v_3 = \gamma w(x^1, x^2),$$

where γ is the torsion per unit length of the bar, and w is an unknown function, the so - called warpening function .

The components of the strain tensor $\varepsilon = (\varepsilon_{ij})$ are defined through the relation

(1.3) $$\varepsilon_{ij} = 1/2 \cdot (D^i v_j + D^j v_i).$$

Hence we obtain

(1.4) $$\varepsilon_{13} = \gamma/2 \cdot (D^1 w - x^2),$$ $$\varepsilon_{23} = \gamma/2 \cdot (D^2 w + x^1),$$

and all components of the diagonal are equal to zero.

The only non - vanishing components of the symmetric stress tensor $\sigma = (\sigma_{ij})$ are $\sigma_{13} = \sigma_{13}(x^1, x^2)$ and $\sigma_{23} = \sigma_{23}(x^1, x^2)$. They satisfy the equilibrium relation

(1.5) $$D^1 \sigma_{13} + D^2 \sigma_{23} = 0$$

in Ω, and the boundary condition

(1.6) $$\sigma_{13} \nu_1 + \sigma_{23} \nu_2 = 0$$

on $\partial\Omega$, where $\nu = (\nu_1, \nu_2)$ is the exterior normal vector of $\partial\Omega$.

We assume the yield criteria of v. MISES [8]

(1.7) $$T^2 = |\sigma_{13}|^2 + |\sigma_{23}|^2 \leq 1,$$

where the elastic range is defined by the strict inequality sign. T is called the <u>tangential stress intensity</u>.

The elastic - plastic torsion problem consists in finding a warpening function w and a stress tensor σ such that the relations (1.4) - (1.7) and

$$(1.8) \qquad \varepsilon_{ij} = \frac{\lambda}{2G}\sigma_{ij}$$

are satisfied, where G is a positive constant, the <u>shear modulus</u>, and $\lambda \in L^{\infty}(\Omega)$ has the property

$$(1.9) \qquad \lambda = \begin{cases} 1, & T < 1 \\ \geq 1, & T = 1. \end{cases}$$

λ is a <u>Lagrange multiplier</u>.

To determine σ LANCHON solved the variational problem

$$(1.10) \quad \int_{\Omega^*} |Dv|^2\, dx - 4G\gamma\cdot\int_{\Omega^*} v\, dx \to \min \text{ in } K_1,$$

where K_1 is the convex set

$$(1.11) \quad K_1 = \{v \in H_o^{1,2}(\Omega^*) : |Dv| \leq 1,\ v_{|\Omega_k} = \text{const},\ k = 1,\dots,N\}.$$

If u is the solution to this problem, then σ is determined through the definition

$$(1.12) \qquad \sigma_{13} = D^2u, \quad \sigma_{23} = -D^1u$$

in Ω; u is called the <u>stress function</u>.

According to the result of [5] u belongs to $H^{2,p}_{loc}(\Omega)$ for any p, $1 \leq p < \infty$; precisely we proved

$$(1.13) \qquad \Delta u \in L^{\infty}(\Omega).$$

The crucial step in solving the elastic - plastic torsion problem completely is to determine w and λ, such that the relations (1.4) and (1.8), or equivalently,

$$(1.14)\quad \begin{aligned} \lambda\cdot D^2u &= G\gamma\cdot(D^1w - x^2) \\ -\lambda\cdot D^1u &= G\gamma\cdot(D^2w + x^1) \end{aligned}$$

are satisfied.

We attack this problem by treating it as the limit case of a sequence of nonlinear problems without constraints, namely, we shall approximate the elastic - plastic behaviour of the material by the behaviour of hardening materials.

Let Γ be the shear strain intensity

$$(1.15)\quad \Gamma^2 = 1/4\cdot(|\varepsilon_{13}|^2 + |\varepsilon_{23}|^2).$$

Then, for elastic - plastic materials, the dependence between T and Γ is given by

$$(1.16)\quad T = \begin{cases} G\Gamma, & \text{if } T < 1 \\ 1, & \text{if } T = 1, \end{cases}$$

i.e. the dependence is not invertible.

For hardening materials Γ can be expressed as a function of T

$$(1.17)\quad \Gamma = \frac{g(T^2)}{G}\cdot T,$$

where the real function $g = g(t)$ satisfies

$$(1.18)\quad g \geq 1 \quad \text{and} \quad \frac{dg}{dt} \geq 0.$$

The relation between ε and σ is then of the form

$$(1.19)\quad \varepsilon_{ij} = \frac{g(T^2)}{2G}\,\sigma_{ij}.$$

For hardening materials the stress function u is determined as the solution of the variational problem

$$(1.20)\quad \int_{\Omega^*}\int_0^{|Dv|^2} g(t)\,dt\,dx - 4G\gamma\cdot\int_{\Omega^*} v\,dx \to \min \text{ in } K_2,$$

$$K_2 = \{v\in H_0^{1,2}(\Omega^*) : v_{|\Omega_k} = \text{const},\ k = 1,\dots,N\}.$$

The solution u then satisfies the nonlinear differential equation in Ω

(1.21) $$- D^i(g(|Du|^2)D^i u) - 2G\gamma = 0$$

and the free boundary conditions

(1.22) $$\int_{\Gamma_k} g(|Du|^2)\cdot Du\cdot\nu \, dH_1 = 2G\gamma\cdot|\Omega_k|$$

for $k = 1,\ldots,N$, where $|\Omega_k|$ is the Lebesgue measure of Ω_k, and where for positive r H_r denotes the r - dimensional Hausdorff measure.

To determine the warpening function w in this case we have to integrate the following system of first order partial differential equations

(1.23) $$\begin{aligned} g(|Du|^2)\cdot D^2 u &= G\gamma\cdot(D^1 w - x^2) \\ -g(|Du|^2)\cdot D^1 u &= G\gamma\cdot(D^2 w + x^1). \end{aligned}$$

But necessary and sufficient conditions to solve this system are just the relations (1.21) and (1.22). Thus, for hardening materials the torsion problem is completely solvable.

Our plan is to approximate the non - invertible relation (1.16) by injective relations valid for hardening materials. This is indicated in Figure 2.

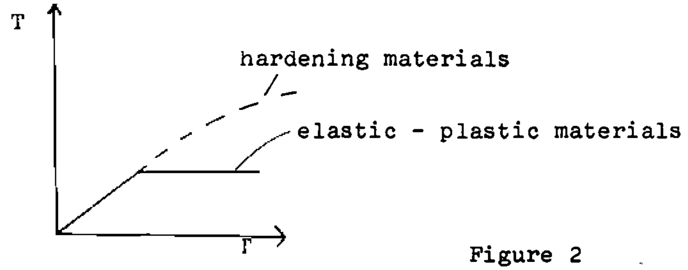

Figure 2

In doing so, we see that the corresponding functions g_ε converge towards the monotone graph

(1.24) $$g_o(t) = \begin{cases} 1, & t < 1 \\ [1,\infty), & t = 1. \end{cases}$$

Let u_ε be the corresponding solutions of the approximating problems. We are going to show in the next section, that we can approximate the graph g_o such that the u_ε's are uniformly bounded in $H^{1,2}(\Omega) \cap H^{2,2}_{loc}(\Omega)$, and that they converge to the solution u of the variational problem (1.10). Moreover, the sequence $g_\varepsilon(|Du_\varepsilon|^2)$ is uniformly bounded and a subsequence converges weakly in $L^2(\Omega)$ to a function λ satisfying $\lambda(x) \in g_o(|Du|^2)$ for a.e. x.

Furthermore, if w_ε are the corresponding warpening functions, so that the relations

$$(1.25)\quad \begin{aligned} g_\varepsilon(|Du_\varepsilon|^2)\cdot D^2 u_\varepsilon &= G\gamma\cdot(D^1 w_\varepsilon - x^2) \\ -g_\varepsilon(|Du_\varepsilon|^2)\cdot D^1 u_\varepsilon &= G\gamma\cdot(D^2 w_\varepsilon + x^1) \end{aligned}$$

are valid, then it follows from the preceding estimates that a subsequence of the w_ε's converges uniformly to a Lipschitz function w being the warpening function for the elastic - plastic torsion problem.

The stress function u also solves the differential equation

$$(1.26)\qquad - D^i(\lambda D^i u) - 2G\gamma = 0,$$

or writing it more suggestively,

$$(1.27)\qquad - D^i(g_o(|Du|^2)D^i u) - 2G\gamma \ni 0.$$

2. THE EXISTENCE OF A LAGRANGE MULTIPLIER.

In this section we make the same assumptions as before, except that Ω is supposed to be a bounded domain of $\mathbb{R}^n$, $n \geq 2$.

Let g_ε be the sequence of functions

$$(2.1)\qquad g_\varepsilon(t) = \begin{cases} 1, & t \leq 1 \\ e^{m/\varepsilon\cdot(t-1)}, & t \geq 1, \end{cases}$$

where the positive constant m is to determined later. The g_ε's satisfy the condition (1.18) and approximate the monotone graph g_o.

Consider the variational problems

(2.2) $\int_{\Omega^*} \int_0^{|Dv|^2} g_\varepsilon(t)\, dtdx - 4G\gamma \cdot \int_{\Omega^*} v\, dx \to \min$ in K_2.

Let u_ε be the solutions of these problems. Then the sequence $\{u_\varepsilon\}$ is bounded in $H_0^{1,2}(\Omega^*)$, and hence a subsequence, not relabled, converges weakly to some element $u_0 \in K_2$. We assert

(2.3) $$u_0 = u,$$

where u is the solution of problem (1.10).

To prove this, we observe that

(2.4) $$u_\varepsilon \to u_0 \text{ in } L^2(\Omega^*).$$

Thus, for each k, k = 1,...,N,

(2.5) $$c_k^\varepsilon = u_{\varepsilon|\Omega_k}$$

is bounded. Hence, we deduce

(2.6) $$|c_k^\varepsilon - c_l^\varepsilon| \le \text{const} \cdot \text{dist}(\Gamma_k, \Gamma_l)$$

for k, l = 0,...,N, where c_0^ε is equal to zero.

From the estimate (2.6) we immediately conclude (cf. Theorem 2.1 below for similar considerations) that $|Du_\varepsilon|_{\partial\Omega}$ is uniformly bounded, and hence $|Du_\varepsilon|_\Omega$.

We finally affirm

(2.7) $$|Du_0|_\Omega \le 1.$$

Suppose for a moment that this estimate would be valid. Then u_0 would belong to K_1, and would therefore be a solution of problem (1.10). The assertion (2.3) would then follow from the uniqueness of the solution.

To prove (2.7), choose $\rho > 1$ and $\alpha > 0$ arbitrarily. Then it follows from the minimum property of u_ε that

(2.8) $$|E_\varepsilon| = |\{|Du_\varepsilon| > \rho\}| < \alpha \quad \text{for a.e.}\,\varepsilon.$$

Hence, we conclude

$$(2.9)\qquad \int_\Omega \max\{|Du_\varepsilon| - \rho, 0\}\, dx = \int_E \{|Du_\varepsilon| - \rho\}\, dx \le \text{const}\cdot\alpha$$

for those values of ε, on account of the uniform boundedness of $|Du_\varepsilon|_\Omega$.

Since the function $t \to \max\{t - \rho, 0\}$ is convex, we then obtain

$$(2.10)\qquad \int_\Omega \max\{|Du_0| - \rho, 0\}\, dx \le \text{const}\cdot\alpha,$$

from which the result follows in view of the arbitrariness of α and ρ.

This result implies especially

$$(2.11)\qquad c_k^\varepsilon \to c_k \quad \text{for } k = 0,\dots,N,$$

where c_k are the boundary values of u on Γ_k.

We now make the following fundamental assumption, namely, we suppose

$$(2.12)\qquad |c_k - c_1| < \text{dist}(\Gamma_k, \Gamma_1) \quad \text{for } k \neq 1.$$

This inequality is trivially satisfied if we replace the strict inequality sign by " $\le$ ".

The strict inequality means physically that there are no " plastic arcs " connecting two different components of the boundary of Ω. This condition is always satisfied if the cross - section is simply connected, or if the torsion angle is sufficiently small, since the plastic parts spread out from the boundary and vary continuously with γ.

Combining the relations (2.11) and (2.12) we immediately conclude that the inequality

$$(2.13)\qquad |c_k^\varepsilon - c_1^\varepsilon| \le \text{dist}(\Gamma_k, \Gamma_1)$$

is valid for a.e. ε.

Now we are able to prove the main theorem

<u>THEOREM 2.1</u>. - <u>Let</u> Ω <u>satisfy an outward sphere condition of radius</u> R. <u>Let</u> u <u>resp</u>. u_ε <u>be the solutions to the variational problems</u> (1.10) <u>resp</u>. (2.2) <u>and suppose inequality</u> (2.12) <u>to be valid</u>. <u>Then, we can demonstrate the following propositions</u>

(i) $\limsup |Du_\varepsilon|_\Omega \leq 1$,
(ii) $g_\varepsilon(|Du_\varepsilon|^2) \leq$ const,
(iii) $||u_\varepsilon|| \leq$ const <u>in</u> $H^{2,2}_{loc}(\Omega)$,
(iv) $u_\varepsilon \to u$ <u>in</u> $H^{1,2}(\Omega)$,
(v) $g_\varepsilon(|Du_\varepsilon|^2) \rightharpoonup \lambda \in L^\infty(\Omega)$ <u>in</u> $L^2(\Omega)$,

<u>where</u>

(vi) $\lambda(x) \in g_o(|Du|^2)$ <u>for</u> a.e. x.

<u>From these relations we finally conclude</u>

$$- D^i(\lambda D^i u) - 2G\gamma = 0. \tag{2.14}$$

<u>PROOF</u> : To prove (i) it will be sufficient to estimate $\limsup |Du_\varepsilon|_{\partial\Omega}$.

For each boundary point $x_o \in \partial\Omega$ we shall construct barrier functions δ_ε^-, δ_ε^+ satisfying

$$- D^i(g_\varepsilon(|D\delta_\varepsilon^+|^2)D^i\delta_\varepsilon^+) - 2G\gamma \geq 0, \tag{2.15}$$

$$- D^i(g_\varepsilon(|D\delta_\varepsilon^-|^2)D^i\delta_\varepsilon^-) - 2G\gamma \leq 0, \tag{2.16}$$

$$\limsup \max\{|D\delta_\varepsilon^-|_\Omega, |D\delta_\varepsilon^+|_\Omega\} \leq 1, \tag{2.17}$$

$$\delta_\varepsilon^-(x) \leq u_\varepsilon(x) \leq \delta_\varepsilon^+(x) \quad \text{for all } x \in \partial\Omega, \tag{2.18}$$

and

$$\delta_\varepsilon^-(x_o) = \delta_\varepsilon^+(x_o) = u_\varepsilon(x_o). \tag{2.19}$$

From the maximum principle we then conclude

$$|Du_\varepsilon|_\Omega \leq |Du_\varepsilon|_{\partial\Omega} \leq \max\{|D\delta_\varepsilon^-|_\Omega, |D\delta_\varepsilon^+|_\Omega\}. \tag{2.20}$$

To construct the barriers let $x_o \in \partial\Omega$ be arbitrary, and let B be a ball of radius R touching Ω at x_o from the exterior. We suppose that B is centered in the origin. Let d be a constant such that

$$d \geq |x| - R \quad \text{for all } x \in \Omega. \tag{2.21}$$

We then define

$$\delta_\varepsilon^+(x) = \delta_\varepsilon(x) + u_\varepsilon(x_o), \tag{2.22}$$

where

$$\delta_\varepsilon(x) = \frac{e^{\varepsilon d}}{\varepsilon}\{1 - e^{-\varepsilon(|x| - R)}\}. \tag{2.23}$$

δ_ε^- is similar defined

$$\delta_\varepsilon^-(x) = - \delta_\varepsilon(x) + u_\varepsilon(x_o). \tag{2.24}$$

In the following we shall only consider the upper barrier δ_ε^+; the considerations for δ_ε^- are identical.

From the definition of δ_ε we immediately conclude

$$D^i\delta_\varepsilon(x) = e^{\varepsilon(d + R - |x|)}\cdot x^i|x|^{-1} \tag{2.25}$$

and

$$D^iD^j\delta_\varepsilon = e^{\varepsilon(d + R - |x|)}\cdot\{\frac{\delta^{ij}}{|x|} - \frac{x^ix^j}{|x|^3} - \varepsilon\cdot\frac{x^ix^j}{|x|^2}\} \tag{2.26}$$

Thus, we obtain

$$|D\delta_\varepsilon^+| = e^{\varepsilon(d + R - |x|)} \tag{2.27}$$

and

$$D^iD^j\delta_\varepsilon^+\cdot D^i\delta_\varepsilon^+\cdot D^j\delta_\varepsilon^+ = - \varepsilon\cdot e^{3\varepsilon(d + R - |x|)}. \tag{2.28}$$

We therefore deduce

$$- D^i(g_\varepsilon(|D\delta_\varepsilon^+|^2)D^i\delta_\varepsilon^+) - 2G\gamma = g_\varepsilon\cdot\{- \Delta\delta_\varepsilon^+ + 2m\cdot e^{3\varepsilon(d + R - |x|)}\} - 2G\gamma \geq 2m - n/R - 2G\gamma > 0 \tag{2.29}$$

for small values of ε and sufficiently large m.

In view of these relations the conditions (2.15) - (2.19) are satisfied. Hence, the estimates (2.20) and (2.27) yield

$$|Du_\varepsilon|_\Omega \leq e^{\varepsilon d} \tag{2.30}$$

from which proposition (ii)

$$\limsup g_\varepsilon(|Du_\varepsilon|^2) \leq e^{2md} \tag{2.31}$$

is easily derived.

Proposition (iii) is an immediate consequence of the following lemma

LEMMA 2.1. - Let g satisfy the condition (1.18), and let $u \in C^3(\Omega) \cap C^{0,1}(\bar{\Omega})$ be a solution of the partial differential equation

$$- D^i(g(|Du|^2)D^i u) - \mu = 0, \tag{2.32}$$

where $\mu \in L^\infty(\Omega)$ is given. Then, for any compact domain Ω', $\Omega' \subset\subset \Omega$, the estimate

$$||u||_{2,2,\Omega'} \leq \text{const} \tag{2.33}$$

is valid, where the constant depends on Ω', $|g(|Du|^2)|_\Omega$, $|Du|_\Omega$, and $|\mu|_\Omega$.

PROOF OF LEMMA 2.1 : Let $\xi \in H^{1,2}_{loc}(\Omega)$ and $\eta \in C^\infty_c(\Omega)$. Multiplying the equation (2.32) with $\xi\eta^2$ and integrating by part yield

$$\int_\Omega gD^i u\{D^i\xi\eta^2 + 2\xi D^i\eta\eta\}\, dx = \int_\Omega \mu\xi\eta^2\, dx. \tag{2.34}$$

Setting $\xi = - D^r D^r u$ for some fixed number r, $1 \leq r \leq n$, we derive

$$\int_\Omega g\{|DD^r u|^2\eta^2 + 2\eta D^i u(D^r\eta D^i D^r u - D^i\eta D^r D^r u)\}\, dx + \\ + \int_\Omega 2g'|D^i D^r u D^i u|^2\eta^2\, dx = - \int_\Omega \mu D^r D^r u\eta^2\, dx \tag{2.35}$$

The assertion is now obvious in view of (1.18).

To prove (iv) we observe that u_ε converges uniformly to u and that

$$- D^i(g_\varepsilon(|Du|^2 D^i u) = - \Delta u. \tag{2.36}$$

Thus, we obtain

$$\int_\Omega |D(u - u_\varepsilon)|^2\, dx \leq \int_\Omega \{g_\varepsilon(|Du|^2)D^i u - g_\varepsilon(|Du_\varepsilon|^2)D^i u_\varepsilon)\}D^i(u - u_\varepsilon)\, dx \\ \leq \int_\Omega |\Delta u - 2G\gamma|\cdot|u - u_\varepsilon|\, dx + \text{const}\cdot\int_{\partial\Omega}|u - u_\varepsilon|\, dH_{n-1} \tag{2.37}$$

from which the result follows.

The assertion (v) is an immediate consequence of (ii). The crucial step is to prove (vi) : Let $G = \{x \in \Omega : |Du(x)| < 1\}$. Then

(2.38) $$g_\varepsilon(|Du_\varepsilon|^2) \to 1 \quad \text{for a.e. } x \in G$$

in view of (iv). The result now follows from Lebesgue's dominated convergence theorem.

The final relation (2.14) is derived from (iv) and (v).

3. THE EXISTENCE OF A WARPENING FUNCTION.

Let us return to the physical case n = 2. As we have seen in Section 1 there are functions w_ε such that

(3.1) $$\begin{aligned} g_\varepsilon(|Du_\varepsilon|^2)D^2u_\varepsilon &= G\gamma(D^1w_\varepsilon - x^2) \\ -\, g_\varepsilon(|Du_\varepsilon|^2)D^1u_\varepsilon &= G\gamma(D^2w_\varepsilon + x^1). \end{aligned}$$

Thus, $|Dw_\varepsilon|$ is uniformly bounded. On the other hand we know that the w_ε's itselves are uniformly bounded, since they can be expressed as integrals of bounded functions. Therefore, a subsequence converges to some function w satisfying the system (1.14).

4. THE UNIQUENESS OF THE LAGRANGE MULTIPLIER.

We shall show that the condition (2.12) guaranteeing the existence of a Lagrange multiplier λ will also serve for proving the uniqueness of λ The warpening function is then uniquely determined up to an additive constant.

The proof is a slight modification of BREZIS' proof [1] who derived the same result in the case of a simply connected domain.

We assume in the following that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and that $\partial\Omega$ is of class C^2, so that $u \in H^{2,2}(\Omega)$ (extend the result of Theorem 2.1 up to the boundary, or cf. [5]).

Suppose there were two different multipliers λ_1 and λ_2. Set $\mu = \lambda_1 - \lambda_2$. Let $E = \{|Du| < 1\}$ and $P = \{|Du| = 1\}$. Then μ is identically zero on E, and we have to show that this is also true on P.

We deduce from (2.14)

$$(4.1) \qquad - D^i(\mu D^i u) = 0.$$

Let k, $0 \leq k \leq N$, be given, and let U_k be neighbourhood of Γ_k. We are going to prove

$$(4.2) \qquad \mu_{|\Omega \cap U_k} \equiv 0$$

if U_k is sufficiently small.

Let h be an arbitrary smooth function with support in U_k, and set

$$(4.3) \qquad \zeta_k(x) = \int_{c_k}^{u(x)} h(x + (t - u)Du)\, dt.$$

Then $\zeta_k = 0$ on Γ_j for each j : for j = k this is trivial, and for $j \neq k$ we observe

$$(4.4) \qquad x + (t - u(x))Du(x) \notin \Gamma_k$$

if $x \in \Gamma_j$ and $t \in [c_k, c_j]$ in view of (2.12). Thus, $\zeta_k \in H_0^{1,2}(\Omega)$ and BREZIS' proof is applicable yielding

$$(4.5) \qquad \mu_{|\Omega \cap U} \equiv 0$$

in some neighbourhood U of $\partial\Omega$.

Finally, let h be a smooth function and η, $0 \leq \eta \leq 1$, be such that $\eta = 1$ on $\Omega - U$ and $\eta = o$ on $\partial\Omega$. Set

$$(4.6) \qquad \zeta(x) = \int_0^{u(x)} h(x + (t - u)Du)\, dt.$$

Then, multiplying (4.1) with $\zeta\eta$ we immediately conclude

$$(4.6) \qquad \int_\Omega \mu \cdot D^i u \cdot D^i \zeta \, dx = 0,$$

hence the result (cf. [1]).

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