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THE QUANTIZATION OF GRAVITY INTERACTING WITH A YANG-MILLS AND HIGGS FIELD

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1. Introduction

A unified quantum theory incorporating the four fundamental forces of nature is one of the major open problems in physics. The Standard Model combines electro-magnetism, the strong force and the weak force, but ignores gravity. The quantization of gravity is therefore a necessary first step to achieve a unified quantum theory.

The Einstein equations are the Euler-Lagrange equations of the Einstein-Hilbert functional and quantization of a Lagrangian theory requires to switch from a Lagrangian view to a Hamiltonian view. In a ground breaking paper, Arnowitt, Deser and Misner [1] expressed the Einstein-Hilbert Lagrangian in a form which allowed to derive a corresponding Hamilton function by applying the Legendre transformation. However, since the Einstein-Hilbert Lagrangian is singular, the Hamiltonian description of gravity is only correct if two additional constraints are satisfied, namely, the Hamilton constraint and the diffeomorphism constraint. Dirac [4] proved how to quantize a constrained Hamiltonian system—at least in principle—and his method has been applied to the Hamiltonian setting of gravity, cf. the paper of [3] and the monographs by Kiefer [17] and Thiemann [18]. In the general case, when arbitrary globally hyperbolic spacetime metrics are allowed, the problem turned out to be extremely difficult and solutions could only be found by assuming a high degree of symmetry.

However, in a series of papers we achieved the quantization of gravity for general hyperbolic spacetimes, cf. [7, 8, 6, 9, 10, 11, 13] as well as [16], and developed a mathematical model which describes the quantized interaction of gravity with a Yang-Mills and Higgs field. We like to give an overview of the results in the following.

Deriving the Einstein equations by a Hamiltonian setting requires a global time function x^0 and foliation of spacetime by its level hypersurfaces. Thus, we consider a spacetime $N = N^{n+1}$ with metric $(\bar g_{\alpha\beta})$, $0 \le \alpha, \beta \le n$, assuming the existence of a global time function x^0 which will also define the time coordinate. Furthermore, as we have proved in [7, Theorem 3.2], we may only consider metrics that can be split by the time function, i.e., the metrics can be expressed in the form

$$d\bar s^2 = -w^2(dx^0)^2 + g_{ij}dx^i dx^j, \tag{1.1}$$

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where $w > 0$ is a smooth function and $g_{ij}(x^0, x)$ are Riemannian metrics. Let

$$M(t) = \{x^0 = t\}, \qquad t \in x^0(N) \equiv I, \tag{1.2}$$

be the coordinate slices, then the g_{ij} are the induced metrics. Moreover, let $\mathcal{G}$ be a compact, semi-simple, connected Lie group with Lie algebra $\mathfrak{g}$, and let

$$E_1 = (N, \mathfrak{g}, \pi, \mathrm{Ad}(\mathcal{G})) \tag{1.3}$$

be the corresponding adjoint bundle with bas space N. Then we consider the functional

$$J = \int_N (\bar{R} - 2\Lambda) + \int_N (\alpha_1 L_{YM} + \alpha_2 L_H), \tag{1.4}$$

where the α_i, $i = 1, 2$, are positive coupling constants, $\bar{R}$ the scalar curvature, Λ a cosmological constant, L_{YM} the energy of a connection in E_1 and L_H the energy of a Higgs field with values in $\mathfrak{g}$. The integration over N is to be understood symbolically, since we shall always integrate over an open precompact subset $\tilde{\Omega} \subset N$.

In the paper [8] we already considered a canonical quantization of the above action and proved that it will be sufficient to only consider connections A^a_μ satisfying the Hamilton gauge

$$A^a_0 = 0, \tag{1.5}$$

thereby eliminating the Gauß constraint, such that the only remaining constraint is the Hamilton constraint, cf. [8, Theorem 2.3].

Using the ADM partition (1.2) of N, cf. [1], such that

$$N = I \times \mathcal{S}_0, \tag{1.6}$$

where $\mathcal{S}_0$ is the Cauchy hypersurface $M(0)$ and applying canonical quantization we obtained a Hamilton operator $\mathcal{H}$ which was a normally hyperbolic operator in a fiber bundle E with base space $\mathcal{S}_0$ and fibers

$$F(x) \times (\mathfrak{g} \otimes T^{0,1}_x(\mathcal{S}_0)) \times \mathfrak{g}, \qquad x \in \mathcal{S}_0, \tag{1.7}$$

where $F(x)$ is the space of Riemannian metrics. We quantized the action by looking at the Wheeler-DeWitt equation in this bundle. The fibers of E are equipped with a Lorentzian metric such that they are globally hyperbolic and the transformed Hamiltonian $\mathcal{H}$, which is now a hyperbolic operator, is a normally hyperbolic operator acting only in the fibers.

The Wheeler-DeWitt equation has the form

$$\mathcal{H}u = 0, \tag{1.8}$$

with $u \in C^\infty(E, \mathbb{C})$ and we defined with the help of the Green's operator a symplectic vector space and a corresponding Weyl system.

The Wheeler-DeWitt equation seems to be the obvious quantization of the Hamilton condition. However, $\mathcal{H}$ acts only in the fibers and not in the base space which is due to the fact that the derivatives are only ordinary covariant derivatives and not functional derivatives, though they are supposed to be functional derivatives, but this property is not really invoked when a functional derivative is applied to u, since the result is the same as applying a partial derivative.

Therefore, we shall discard the Wheeler-DeWitt equation and express the Hamilton condition differently by looking at the evolution equation of the mean curvature of the foliation hypersurfaces $M(t)$ and implementing the Hamilton condition on the right-hand side of this evolution equation. The left-hand side, a time derivative, we shall express with the help of the Poisson brackets. After canonical quantization the Poisson brackets become a commutator and now we can employ the fact that the derivatives are functional derivatives, since we have to differentiate the scalar curvature of a metric. As a result we obtain an elliptic differential operator in the base space, the main part of which is the Laplacian of the metric.

On the right-hand side of the evolution equation the interesting term is H^2, the square of the mean curvature. It will be transformed to a second time derivative and will be the only remaining derivative with respect to a fiber variable, since the differentiations with respect to the other variables cancel each other.

The resulting quantized equation is then a wave equation

$$\begin{aligned}(1.9)\qquad &\frac{1}{32}\frac{n^2}{n-1}\ddot u-(n-1)t^{2-\frac4n}\varDelta u-\frac n2 t^{2-\frac4n}Ru+\alpha_1\frac n8 t^{2-\frac4n}F_{ij}F^{ij}u\\ &+\alpha_2\frac n4 t^{2-\frac4n}\gamma_{ab}\sigma^{ij}\varPhi^a_i\varPhi^b_i u+\alpha_2\frac n2 mt^{2-\frac4n}V(\varPhi)u+nt^2\varLambda u=0,\end{aligned}$$

in a globally hyperbolic spacetime

$$Q=(0,\infty)\times\mathcal{S}_0 \tag{1.10}$$

describing the interaction of a given complete Riemannian metric σ_{ij} in $\mathcal{S}_0$ with a given Yang-Mills and Higgs field; V is the potential of the Higgs field and m a positive constant. The existence of the time variable, and its range, is due to the Lorentzian metric in the fibers of E.

Remark 1.1. For the results and arguments it is completely irrelevant that the values of the Higgs field $\varPhi$ lie in a Lie algebra, i.e., $\varPhi$ could also be just an arbitrary scalar field, or we could consider a Higgs field as well as an another arbitrary scalar field. Hence, let us stipulate that the Higgs field could also be just an arbitrary scalar field. It will be used to produce a mass gap simply by interacting with the gravitation ignoring the Yang-Mills field.

If $\mathcal{S}_0$ is compact we proved a spectral resolution of equation (1.9) by first considering a stationary version of the hyperbolic equation, namely, the elliptic eigenvalue equation

$$\begin{aligned}(1.11)\qquad &-(n-1)\varDelta v-\frac n2 Rv+\alpha_1\frac n8 F_{ij}F^{ij}v\\ &+\alpha_2\frac n4\gamma_{ab}\sigma^{ij}\varPhi^a_i\varPhi^b_i v+\alpha_2\frac n2 mV(\varPhi)v=\mu v.\end{aligned}$$

It has countably many solutions (v_i,μ_i) such that

$$\mu_0<\mu_1\le\mu_2\le\cdots, \tag{1.12}$$

$$\lim\mu_i=\infty. \tag{1.13}$$

Let v be an eigenfunction with eigenvalue $\mu>0$, then we look at solutions of (1.9) of the form

$$u(x,t)=w(t)v(x). \tag{1.14}$$

u is then a solution of (1.9) provided w satisfies the implicit eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot w-\mu t^{2-\frac4n}w-nt^2\varLambda w=0, \tag{1.15}$$

where $\varLambda$ is the eigenvalue.

This eigenvalue problem has countably many solutions $(w_i,\varLambda_i)$ with finite energy, i.e.,

$$\int_0^\infty\{|\dot w_i|^2+(1+t^2+\mu t^{2-\frac4n})|w_i|^2\}<\infty. \tag{1.16}$$

More precisely, we proved, cf. [9, Theorem 6.7],

Theorem 1.2. *Assume $n\ge2$ and $\mathcal{S}_0$ to be compact and let (v,μ) be a solution of the eigenvalue problem (1.11) with $\mu>0$, then there exist countably many solutions $(w_i,\varLambda_i)$ of the implicit eigenvalue problem (1.15) such that*

$$\varLambda_i<\varLambda_{i+1}<\cdots<0, \tag{1.17}$$

$$\lim_i \Lambda_i = 0, \tag{1.18}$$

and such that the functions

$$u_i = w_i v \tag{1.19}$$

are solutions of the wave equation (1.9). The transformed eigenfunctions

$$\tilde{w}_i(t) = w_i(\lambda_i^{\frac{n}{4(n-1)}} t), \tag{1.20}$$

where

$$\lambda_i = (-\Lambda_i)^{-\frac{n-1}{n}}, \tag{1.21}$$

form a basis of $L^2(\mathbb{R}_+^*, \mathbb{C})$ *and also of the Hilbert space* H *defined as the completion of* $C_c^\infty(\mathbb{R}_+^*, \mathbb{C})$ *under the norm of the scalar product*

$$\langle w, \tilde{w} \rangle_1 = \int_0^\infty \{\bar{w}'\tilde{w}' + t^2 \bar{w}\tilde{w}\}, \tag{1.22}$$

where a prime or a dot denotes differentiation with respect to t*.*

For compact $\mathcal{S}_0$ we can also proved a mass gap.

Theorem 1.3. *Let* $\mathcal{S}_0$ *be compact and let* V *satisfy*

$$V(\Phi) > 0 \qquad \text{a.e.}, \tag{1.23}$$

then there exists m_0 *such that for all* $m \geq m_0$ *the first eigenvalue* μ_0 *of equation (1.11) is strictly positive with an a priori bound from below depending on the data.*

However, for non-compact $\mathcal{S}_0$ one has to use a different approach in order to quantize the wave equation (1.9). Let us first consider the temporal eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda| t^2 w = \lambda t^{2-\frac{4}{n}} w \tag{1.24}$$

in the Sobolev space

$$H_0^{1,2}(\mathbb{R}_+^*). \tag{1.25}$$

Here,

$$\Lambda < 0 \tag{1.26}$$

is the cosmological constant.

The eigenvalue problem (1.24) can be solved by considering the generalized eigenvalue problem for the bilinear forms

$$B(w, \tilde{w}) = \int_{\mathbb{R}_+^*} \{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}' + n|\Lambda| t^2 \bar{w}\tilde{w}\} \tag{1.27}$$

and

$$K(w, \tilde{w}) = \int_{\mathbb{R}_+^*} t^{2-\frac{4}{n}} \bar{w}\tilde{w} \tag{1.28}$$

in the Sobolev space $\mathcal{H}$ which is the completion of

$$C_c^\infty(\mathbb{R}_+^*, \mathbb{C}) \tag{1.29}$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$B(w, \varphi) = \lambda K(w, \varphi) \quad \forall \varphi \in \mathcal{H} \tag{1.30}$$

which is equivalent to (1.24).

Theorem 1.4. *The eigenvalue problem (1.30) has countably many solutions (w_i, λ_i) such that*

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \tag{1.31}$$

$$\lim \lambda_i = \infty, \tag{1.32}$$

and

$$K(w_i, w_j) = \delta_{ij}. \tag{1.33}$$

The w_i are complete in $\mathcal{H}$ as well as in $L^2(\mathbb{R}_+^*)$.

Secondly, let A be the elliptic operator on the left-hand side of (1.11), assuming that its coefficients are smooth and bounded in any

$$C^m(\mathcal{S}_0), \qquad m \in \mathbb{N}, \tag{1.34}$$

then A is self-adjoint in $L^2(\mathcal{S}_0, \mathbb{C})$ and, if $\mathcal{S}_0$ is asymptotically Euclidean, i.e., if it satisfies the very mild conditions in [13, Assumptions 3.1], then the Schwartz space $\mathscr{S}$ of rapidly decreasing functions can also be defined in $\mathcal{S}_0$,

$$\mathscr{S} = \mathscr{S}(\mathcal{S}_0), \tag{1.35}$$

such that

$$\mathscr{S} \subset L^2(\mathcal{S}_0) \subset \mathscr{S}' \tag{1.36}$$

is a Gelfand triple and the eigenvalue problem in $\mathscr{S}'$

$$Af = \lambda f \tag{1.37}$$

has a solution for any $\lambda \in \sigma(A)$, cf. [13, Theorem 2.4]. Let

$$(\mathscr{E}_\lambda)_{\lambda \in \sigma(A)} \tag{1.38}$$

be the set of eigendistributions in $\mathscr{S}'$ satisfying

$$Af(\lambda) = \lambda f(\lambda), \qquad f(\lambda) \in \mathscr{E}_\lambda, \tag{1.39}$$

then the $f(\lambda)$ are actually smooth functions in $\mathcal{S}_0$ with polynomial growth, cf. [11, Theorem 3]. Moreover, due to a result of Donnelly [5], we know that

$$[0, \infty) \subset \sigma_{\text{ess}}(A), \tag{1.40}$$

hence, any temporal eigenvalue λ_i in Theorem 1.4 is also a spatial eigenvalue of A in $\mathscr{S}'$

$$Af(\lambda_i) = \lambda_i f(\lambda_i). \tag{1.41}$$

Since the eigenspaces $\mathscr{E}_{\lambda_i}$ are separable we deduce that for each i there is an at most countable basis of eigendistributions in $\mathscr{E}_{\lambda_i}$

$$v_{ij} \equiv f_j(\lambda_i), \qquad 1 \le j \le n(i) \le \infty, \tag{1.42}$$

satisfying

$$Av_{ij} = \lambda_i v_{ij}, \tag{1.43}$$

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{1.44}$$

The functions

$$u_{ij} = w_i v_{ij} \tag{1.45}$$

are then smooth solutions of the wave equations. They are considered to describe the quantum development of $(\mathcal{S}_0, \sigma_{ij})$.

Let us summarize this result as a theorem:

Theorem 1.5. *Let A satisfy the conditions in (1.34), assume that $\mathcal{S}_0$ is asymptotically Euclidean, and let w_i resp. v_{ij} be the countably many solutions of the temporal resp. spatial eigenvalue problems for selfadjoint operators H_0 resp. H_1, then*

$$u_{ij} = w_i v_{ij} \tag{1.46}$$

are smooth solutions of the wave equation (1.9) which can be expressed in the form

$$t^{2-\frac{4}{n}}(H_0 u - H_1 u) = 0, \tag{1.47}$$

where $u = u(t,x)$. The special solutions in (1.46) describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

We used a similar approach to describe the quantum development of the event horizons of blackholes, see [12, 15]. Moreover, we were able to apply quantum statistics to these quantized systems: In [14] we defined an abstract Hilbert space $\mathcal{H}$, where the eigendistributions of H_1 form an ONB, such that H_0 and H_1 have the same eigenvalues but with different multiplicities. H_1 is also essentially self-adjoint in $\mathcal{H}$. Let $\tilde{H}_1$ be the unique self-adjoint extension of H_1, namely its closure, then we proved that for any $\beta > 0$

$$e^{-\beta \tilde{H}_1} \tag{1.48}$$

is of trace class in $\mathcal{H}$. In addition $\tilde{H}_1$ satisfies

$$\tilde{H}_1 \geq \lambda_0 I, \qquad \lambda_0 > 0. \tag{1.49}$$

Let

$$H \equiv d\Gamma(\tilde{H}_1) \tag{1.50}$$

be the canonical extension of $\tilde{H}_1$ to the symmetric Fock space

$$\mathscr{F} = \mathscr{F}_+(\mathcal{H}), \tag{1.51}$$

then

$$e^{-\beta H} \tag{1.52}$$

is of trace class in $\mathscr{F}$ because of (1.48) and (1.49), cf. [2, Prop. 5.2.27]. Hence we could define the partition function

$$Z = \operatorname{tr}(e^{-\beta H}), \tag{1.53}$$

the density operator

$$\rho = Z^{-1} e^{-\beta H} \tag{1.54}$$

and the von Neumann entropy

$$S = -\operatorname{tr}(\rho \log \rho) = \log Z + \beta E, \tag{1.55}$$

where E is the average energy and $\beta > 0$ the inverse temperature

$$\beta = T^{-1}. \tag{1.56}$$

Here is a summary of the results in [14]:

Theorem 1.6. (i) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{1.57}$$

we have

$$\lim_{\Lambda \to 0} E = \infty \tag{1.58}$$

as well as

$$\lim_{\Lambda \to 0} S = \infty, \tag{1.59}$$

where the limites are uniform in β.

(ii) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{1.60}$$

we have

$$\lim_{|\Lambda| \to \infty} E = 0 \tag{1.61}$$

as well as

$$\lim_{|\Lambda| \to \infty} S = 0, \tag{1.62}$$

where the limites are uniform in β.

The behaviour of Z with respect to Λ is described in the theorem:

Theorem 1.7. *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{1.63}$$

we have

$$\lim_{\Lambda \to 0} Z = \infty \tag{1.64}$$

and for any

$$\beta_0 \le \beta \tag{1.65}$$

the relation

$$\lim_{|\Lambda| \to \infty} Z = 1 \tag{1.66}$$

is valid. The convergence in both limites is uniform in β.

Remark 1.8. The first part of Theorem 1.6 reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density we consider the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho\eta = Z^{-1}\eta, \tag{1.67}$$

i.e., the dark energy density should be proportional to Z^{-1}.

Furthermore, we also applied quantum statistics to the quantized version of a Friedmann universe and proved:

Theorem 1.9. *The results in the last two theorems and the conjectures in the remark above are also valid, if the quantized spacetime $N = N^{n+1}$, $n \geq 3$, is a Friedmann universe without matter but with a negative cosmological constant Λ and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian H_1 all have multiplicity one.*

Remark 1.10. Let us also mention that we used Planck units in this paper i.e.,

$$c = G = \hbar = K_B = 1. \tag{1.68}$$

References

[1] R. Arnowitt, S. Deser, and C. W. Misner, *The dynamics of general relativity*, Gravitation: an introduction to current research (Louis Witten, ed.), John Wiley, New York, 1962, pp. 227–265.

[2] Ola Bratteli and Derek W. Robinson, *Operator algebras and quantum statistical mechanics. 2*, second ed., Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997, Equilibrium states. Models in quantum statistical mechanics.

[3] Bryce S. DeWitt, *Quantum Theory of Gravity. I. The Canonical Theory*, Phys. Rev. **160** (1967), 1113–1148, doi:10.1103/PhysRev.160.1113.

[4] Paul A. M. Dirac, *Lectures on quantum mechanics*, Belfer Graduate School of Science Monographs Series, vol. 2, Belfer Graduate School of Science, New York, 1967, Second printing of the 1964 original.

[5] Harold Donnelly, *Exhaustion functions and the spectrum of Riemannian manifolds*, Indiana Univ. Math. J. **46** (1997), 505–528, doi:10.1512/iumj.1997.46.1338.

[6] Claus Gerhardt, *A unified quantum theory II: gravity interacting with Yang-Mills and spinor fields*, 2013, arXiv:1301.6101.

[7] ______, *The quantization of gravity in globally hyperbolic spacetimes*, Adv. Theor. Math. Phys. **17** (2013), no. 6, 1357–1391, arXiv:1205.1427, doi:10.4310/ATMP.2013.v17.n6.a5.

[8] ______, *A unified quantum theory I: gravity interacting with a Yang-Mills field*, Adv. Theor. Math. Phys. **18** (2014), no. 5, 1043–1062, arXiv:1207.0491, doi:10.4310/ATMP.2014.v18.n5.a2.

[9] ______, *The quantization of gravity*, (2015), arXiv:1501.01205, to appear in ATMP (2018).

[10] ______, *A unified field theory II: Gravity interacting with a Yang-Mills and Higgs field*, (2016), arXiv:1602.07191.

[11] ______, *Deriving a complete set of eigendistributions for a gravitational wave equation describing the quantized interaction of gravity with a Yang-Mills field in case the Cauchy hypersurface is non-compact*, (2016), arXiv:1605.03519.

[12] ______, *The quantization of a black hole*, (2016), arXiv:1608.08209.

[13] ______, *The quantum development of an asymptotically Euclidean Cauchy hypersurface*, (2016), arXiv:1612.03469.

[14] ______, *Trace class estimates and applications*, (2017), pdf file.

[15] ______, *The quantization of a Kerr-AdS black hole*, Advances in Mathematical Physics **vol. 2018** (2018), Article ID 4328312, 10 pages, arXiv:1708.04611, doi:10.1155/2018/4328312.

[16] ______, *The quantization of gravity*, Fundamental Theories of Physics, vol. 194, Springer, Cham, 2018, doi:10.1007/978-3-319-77371-1.

[17] Claus Kiefer, *Quantum Gravity*, 2nd ed., International Series of Monographs on Physics, Oxford University Press, 2007.

[18] Thomas Thiemann, *Modern canonical quantum general relativity*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2007, With a foreword by Chris Isham.

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