# CLOSED IMMERSED UMBILIC HYPERSURFACES IN $\mathbb{R}^{n+1}$ ARE SPHERES

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## 0. Main Theorem

The following theorem is well-known, though I don't know who proved it first. It certainly follows from Alexandrov's result that embedded closed hypersurfaces of constant mean curvature are spheres. It also holds locally, see [1, Vol. IV, p. 11].

0.1. **Theorem.** *Let $M$ be a closed, connected, umbilic immersion into $\mathbb{R}^{n+1}$ of class $C^3$, then $M$ is a sphere and the immersion is an embedding.*

*Proof.* Since every point of $M$ is umbilic, we have

$$h_{ij} = \kappa g_{ij}, \tag{0.1}$$

where $\kappa \in C^1(M)$. Using the Codazzi equations we deduce

$$\kappa_i = h^j_{i;j} = h^j_{j;i} = H_i = n\kappa_i \tag{0.2}$$

and hence $\kappa$ is constant.

Let

$$x : M_0 \to \mathbb{R}^{n+1}, \qquad x = x(\xi) \tag{0.3}$$

be the immersion of $M$ and $\xi_0 \in M_0$ be a point such that

$$\tfrac{1}{2}|x(\xi_0)|^2 = \sup_{M_0} \tfrac{1}{2}|x|^2, \tag{0.4}$$

then we deduce

$$0 \geq \langle x_{ij}, x\rangle + \langle x_i, x_j\rangle = -h_{ij}\langle x, \nu\rangle + g_{ij} \tag{0.5}$$

and we infer further, since $\langle x, \nu\rangle = |x|$,

$$\kappa \geq |x|^{-1} > 0. \tag{0.6}$$

Thus, $M$ is an immersed, closed, strictly convex hypersurface and hence embedded, in view of Hadamard's theorem, and it bounds a strictly convex body $\hat{M}$.

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After a translation we may assume that $0 \in \operatorname{int} \hat{M}$. Let $u = \langle x, \nu\rangle$ be the support function of $M$. Using the Gauß map we assume that $u$ is defined on $S^n$ and then it satisfies the equation

$$\kappa g_{ij} = h_{ij} = u_{ij} + u\sigma_{ij}. \tag{0.7}$$

where $\sigma_{ij}$ is the metric in $S^n$ and the covariant derivatives of $u$ are defined with respect to that metric.

Moreover, $\sigma_{ij}$ can be expressed as the induced metric of the embedding of $S^n$ via the Gauß map, i.e.,

$$\sigma_{ij} = \langle \nu_i, \nu_j\rangle = h_i^k h_{kj} = \kappa^2 g_{ij}, \tag{0.8}$$

where we used the Weingarten equations

$$\nu_i = h_i^k x_k. \tag{0.9}$$

Hence we conclude

$$g_{ij} = \kappa^{-2}\sigma_{ij} \tag{0.10}$$

and equation (0.7) can be rewritten as

$$\kappa^{-1}\sigma_{ij} = u_{ij} + u\sigma_{ij}, \tag{0.11}$$

or equivalently

$$(u - \kappa^{-1})\sigma_{ij} = -u_{ij}. \tag{0.12}$$

Setting

$$\varphi = u - \kappa^{-1} \tag{0.13}$$

we conclude that $\varphi$ is a spherical harmonic of degree 1, since

$$-\Delta\varphi = n\varphi \quad \wedge \quad \int_{S^n} \varphi = 0, \tag{0.14}$$

hence $\varphi$ can be written as a linear combination of $\nu^\alpha$ the coordinate functions of $\mathbb{R}^{n+1}$ restricted to $S^n$

$$\varphi = a_\alpha \nu^\alpha \tag{0.15}$$

and thus we obtain

$$\tilde{u} = u - a_\alpha\nu^\alpha = \kappa^{-1} = \text{const}, \tag{0.16}$$

which can be looked at as the support function of the translated hypersurface $x - (a^\alpha)$, the image of which we still denote by $M$. Let $x = x(\xi)$ be the embedding of $M$, then

$$\tilde{u} = \langle x, \nu\rangle \tag{0.17}$$

and differentiating $\varphi = \frac{1}{2}|x|^2$ covariantly we deduce

$$\varphi_{ij} = g_{ij} - h_{ij}\langle x, \nu\rangle = g_{ij} - \kappa g_{ij}\kappa^{-1} = 0, \tag{0.18}$$

i.e., $\varphi = \text{const}$ and $M$ is a sphere with center in the origin. □

## References

[1] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I-V. 2nd ed.*, Publish Perish, Inc., Berkeley, 1979.

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