THE INVERSE MEAN CURVATURE FLOW IN ROBERTSON-WALKER SPACES AND ITS APPLICATION TO COSMOLOGY

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Abstract. We consider the inverse mean curvature flow in Robertson-Walker spacetimes that satisfy the Einstein equations and have a big crunch singularity and prove that under natural conditions the rescaled inverse mean curvature flow provides a smooth transition from big crunch to big bang. We also construct an example showing that in general the transition flow is only of class C^3.

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0. Introduction

In a recent paper [2] we proved that the inverse mean curvature flow, properly rescaled, did provide a transition from big crunch to big bang. The transition flow was of class C^3. The underlying $(n+1)$-dimensional spacetime N was fairly general, a cosmological spacetime satisfying some structural conditions, we called these spacetimes ARW spaces.

In this paper we shall show that in general the differentiability class C^3 is the best posible for the transition flow. If it should be of class C^∞, then additional assumptions have to be satisfied.

We shall consider the problem in Robertson-Walker spaces $N = I \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a spaceform with curvature $\tilde{\kappa} = -1, 0, 1$, it may be compact or

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not, and the metric in N is of the form

$$d\bar{s}^2 = e^{2f}(-(dx^0)^2 + \sigma_{ij}(x)dx^i dx^j), \tag{0.1}$$

where $x^0 = \tau$ is the time function, (σ_{ij}) the metric of $\mathcal{S}_0$, $f = f(\tau)$, and x^0 ranges between $-a < x^0 < 0$. We assume that there is a big crunch singularity in $\{x^0 = 0\}$, i.e., we assume

$$\lim_{\tau \to 0} f(\tau) = -\infty \quad \text{and} \quad \lim_{\tau \to 0} -f' = \infty. \tag{0.2}$$

The Einstein equations should be valid with a cosmological constant Λ

$$G_{\alpha\beta} + \Lambda \bar{g}_{\alpha\beta} = \kappa T_{\alpha\beta}, \quad \kappa > 0, \tag{0.3}$$

or equivalently,

$$G_{\alpha\beta} = \kappa(T_{\alpha\beta} - \sigma \bar{g}_{\alpha\beta}), \quad \sigma = \tfrac{\Lambda}{\kappa}. \tag{0.4}$$

If $(T_{\alpha\beta})$ is the stress-energy tensor of a perfect fluid

$$T^0_0 = -\rho, \quad T^\alpha_i = p\delta^\alpha_i \tag{0.5}$$

with an equation of state

$$p = \tfrac{\omega}{n}\rho, \tag{0.6}$$

then the equation (0.4) is equivalent to the Friedmann equation

$$|f'|^2 = -\tilde{\kappa} + \tfrac{2\kappa}{n(n-1)}(\rho + \sigma)e^{2f}, \tag{0.7}$$

which can be easily derived by looking at the component $\alpha = \beta = 0$ in (0.4).

Assuming that ω is of the form

$$\omega = \omega_0 + \lambda(f), \quad \omega_0 = \text{const}, \tag{0.8}$$

where $\lambda = \lambda(t)$ is smooth satisfying

$$\lim_{t \to -\infty} \lambda(t) = 0, \tag{0.9}$$

such that there exists a primitive $\tilde{\mu} = \tilde{\mu}(t)$, $\tilde{\mu}' = \lambda$, with

$$\lim_{t \to -\infty} \tilde{\mu}(t) = 0, \tag{0.10}$$

then ρ obeys the conservation law

$$\rho = \rho_0 e^{-(n+\omega_0)f} e^{-\tilde{\mu}}, \tag{0.11}$$

cf. [2, Lemma 0.2]. Hence we deduce from (0.7)

$$|f'|^2 = -\tilde{\kappa} + \tfrac{2\kappa}{n(n-1)}(\rho_0 e^{-(n+\omega_0)f} e^{-\tilde{\mu}} + \sigma)e^{2f}. \tag{0.12}$$

The main result of the paper can be summarized in the following theorem

0.1. **Theorem.** *Let $\tilde{\gamma} = \frac{1}{2}(n+\omega_0-2) > 0$, and assume that λ satisfies the condition* (1.2) *and that μ can be viewed as a smooth and even function in the variable $(-r)^{\tilde{\gamma}}$, where $r = -e^f < 0$, or that it can be extended to a smooth and even function on $(-\tilde{\gamma}^{-1}, \tilde{\gamma}^{-1})$, then the transition flow $y = y(s, \xi)$, as defined in* (2.16) *and* (2.17), *is smooth in $(-\tilde{\gamma}^{-1}, \tilde{\gamma}^{-1}) \times \mathcal{S}_0$, if either*

$$\omega_0 \in \mathbb{R} \quad \text{and} \quad \sigma = 0, \tag{0.13}$$

or

$$\omega_0 = 4 - n \quad \text{and} \quad \sigma \in \mathbb{R}. \tag{0.14}$$

Let us emphasize that the smooth transition from big crunch to big bang does not constitute the existence of a cyclic universe, cf. the end of Section 2 for a detailed discussion.

In Section 3 we prove that in general the transition flow is only of class C^3 by constructing a counter example.

We believe that the results and even more the proofs indicate strongly that the inverse mean curvature flow is the right vehicle to offer a smooth transition from big crunch to big bang in case of abstract spacetimes that are not embedded in a bulk spacetime.

We refer to [1, Section 2] for a description of our notations and conventions.

1. The Friedmann equation

We want to solve the Friedmann equation (0.12) in an interval $I = (-a, 0)$ such that the resulting spacetime N is an ARW space, cf. [4, Definition 0.8] for a definition that applies to closed spacetimes as well as to non-closed.

In a slightly different setting we proved in [2, Section 9] that a cosmological spacetime satisfying the Einstein equations for a perfect fluid with an equation of state (0.6), $\omega = \text{const}$, is an ARW space, if

$$\tilde{\gamma} = \tfrac{1}{2}(n + \omega - 2) > 0. \tag{1.1}$$

This result will also be valid in the present situation.

1.1. **Lemma.** *Let $\tilde{\gamma} = \frac{1}{2}(n + \omega_0 - 2)$ be positive and assume that λ and $\tilde{\mu}$ satisfy the conditions stated in the previous section, and in addition suppose*

$$|D^m \lambda(t)| \le c_m \qquad \forall\, m \in \mathbb{N}. \tag{1.2}$$

Then the Friedmann equation (0.12) *can be solved in an interval $I = (-a, 0)$ such that $f \in C^\infty(I)$ and the relations* (0.2) *are valid. Moreover, N is an ARW space.*

Proof. We want to apply the existence result [4, Theorem 3.1]. Multiply equation (0.12) by $e^{2\tilde{\gamma} f}$ and set

$$\varphi = e^{\tilde{\gamma} f} \tag{1.3}$$

and

$$r = -e^f. \tag{1.4}$$

Then φ satisfies the differential equation

$$\tilde{\gamma}^{-2}\dot{\varphi}^2 = -\tilde{\kappa}e^{2\tilde{\gamma}f} + \tfrac{2\kappa}{n(n-1)}(\sigma e^{2(\tilde{\gamma}+1)f} + \rho_0 e^{-\mu}), \tag{1.5}$$

where we defined $\mu = \mu(r)$ by

$$\mu(r) = \tilde{\mu}(\log(-r)). \tag{1.6}$$

Suppose the Friedmann equation were solvable with f satisfying (0.2), then the right-hand side of (1.5) would tend to $\frac{2\kappa}{n(n-1)}\rho_0$, if $\tau \to 0$. Thus, we see that solving (0.12) and (0.2) is equivalent to solving

$$\tilde{\gamma}^{-1}\dot{\varphi} = -\sqrt{F(\varphi)} \tag{1.7}$$

with initial value $\varphi(0) = 0$, where

$$F(\varphi) = -\tilde{\kappa}\varphi^2 + \tfrac{2\kappa}{n(n-1)}(\rho_0 e^{-\mu} + \sigma\varphi^{2(1+\tilde{\gamma}^{-1})}) \tag{1.8}$$

and μ should be considered to depend on

$$\mu(r) = \mu(-\varphi^{\tilde{\gamma}^{-1}}). \tag{1.9}$$

We can now apply the existence result in [4, Theorem 3.1] to conclude that (1.7) has a solution $\varphi \in C^1((-a,0]) \cap C^\infty((-a,0))$, where, if we choose a maximal, a is determined by the requirement

$$\lim_{\tau\to -a}\varphi = \infty \quad \text{or} \quad \lim_{\tau\to -a}F(\varphi) = 0. \tag{1.10}$$

Set $f = \tilde{\gamma}^{-1}\log\varphi$, then f satisfies (0.2), since $F(0) > 0$.

Moreover, differentiating (1.5) with respect to τ and dividing the resulting equation by $2f'e^{\tilde{\gamma}f}$ we obtain

$$f'' + \tilde{\gamma}|f'|^2 = -\tilde{\kappa}\tilde{\gamma} + \tfrac{\kappa}{n(n-1)}(2\sigma(\tilde{\gamma}+1)e^{2f} - \lambda\rho_0 e^{-\mu}), \tag{1.11}$$

from which we conclude that N is an ARW space, in view of (0.9), (0.10) and (1.2). $\square$

2. The transition flow

Let M_0 be a spacelike hypersurface with positive mean curvature with respect to the past directed normal, then the inverse mean curvature flow with initial hypersurface M_0 is given by the evolution equation

$$\dot{x} = -H^{-1}\nu, \tag{2.1}$$

where ν is the past directed normal of the flow hypersurfaces $M(t)$ which are locally defined by an embedding

$$x = x(t,\xi), \quad \xi = (\xi^i), \tag{2.2}$$

cf. [2] for details.

In general, even in Robertson-Walker spaces, this evolution problem can only be solved, if $\mathcal{S}_0$ is compact. However, if, in the present situation, we assume that M_0 is a coordinate slice $\{x^0 = \text{const}\}$, then the fairly complex parabolic system (2.1) is reduced to a scalar ordinary differential equation.

Look at the component $\alpha = 0$ in (2.1). Writing the hypersurfaces $M(t)$ as graphs over $\mathcal{S}_0$

$$M(t) = \{\, (u, x) \colon x \in \mathcal{S}_0 \,\}, \tag{2.3}$$

we see that u only depends on t, $u = u(t)$, and u satisfies the differential equation

$$\dot{u} = \frac{1}{-nf'}, \tag{2.4}$$

where $f = f(u)$, with initial value $u(0) = u_0$, cf. [2, Section 2]. The mean curvature of the slices $M(t)$ is given by

$$H = e^{-f}(-nf'). \tag{2.5}$$

From (2.4) we immediately deduce

$$\tfrac{d}{dt}(nf + t) = nf'\dot{u} + 1 = 0, \tag{2.6}$$

and hence

$$e^{nf}e^{t} = \text{const} = c, \tag{2.7}$$

or equivalently,

$$e^{\tilde{\gamma} f}e^{\gamma t} = c, \tag{2.8}$$

where $\gamma = \frac{1}{n}\tilde{\gamma}$, and where the symbol c may represent different constants.

The conservation law (2.8) can be viewed as the integrated version of the inverse mean curvature flow.

In [2, Theorem 3.6] we proved that there are positive constants c_1, c_2 such that

$$-c_1 \le \tilde{u} \le -c_2 < 0. \tag{2.9}$$

The old proof also works in the present situation, where $\mathcal{S}_0$ is not necessarily compact, since u doesn't depend on x.

Moreover,

$$\lim_{t\to\infty} \tilde{u} \quad \text{exists}, \tag{2.10}$$

cf. [2, Lemma 7.1].

We shall define a new spacetime $\hat{N}$ by reflection and time reversal such that the IMCF in the old spacetime transforms to an IMCF in the new one.

By switching the light cone we obtain a new spacetime $\hat{N}$. The flow equation in N is independent of the time orientation, and we can write it as

$$\dot{x} = -H^{-1}\nu = -(-H)^{-1}(-\nu) \equiv -\hat{H}^{-1}\hat{\nu}, \tag{2.11}$$

where the normal vector $\hat{\nu} = -\nu$ is past directed in $\hat{N}$ and the mean curvature $\hat{H} = -H$ negative.

Introducing a new time function $\hat{x}^0 = -x^0$ and formally new coordinates $(\hat{x}^\alpha)$ by setting

$$\hat{x}^0 = -x^0, \quad \hat{x}^i = x^i, \tag{2.12}$$

we define a spacetime $\hat{N}$ having the same metric as N—only expressed in the new coordinate system—such that the flow equation has the form

$$\dot{\hat{x}} = -\hat{H}^{-1}\hat{\nu}, \tag{2.13}$$

where $M(t) = \operatorname{graph}\hat{u}(t)$, $\hat{u} = -u$.

The singularity in $\hat{x}^0 = 0$ is now a past singularity, and can be referred to as a big bang singularity.

The union $N \cup \hat{N}$ is a smooth manifold, topologically a product $(-a, a) \times \mathcal{S}_0$—we are well aware that formally the singularity $\{0\} \times \mathcal{S}_0$ is not part of the union; equipped with the respective metrics and time orientation it is a spacetime which has a (metric) singularity in $x^0 = 0$. The time function

$$\hat{x}^0 = \begin{cases} x^0, & \text{in } N, \\ -x^0, & \text{in } \hat{N}, \end{cases} \tag{2.14}$$

is smooth across the singularity and future directed.

Using the time function in (2.14) the inverse mean curvature flows in N and $\hat{N}$ can be uniformly expressed in the form

$$\dot{\hat{x}} = -\hat{H}^{-1}\hat{\nu}, \tag{2.15}$$

where (2.15) represents the original flow in N, if $\hat{x}^0 < 0$, and the flow in (2.13), if $\hat{x}^0 > 0$.

In [2] we then introduced a new flow parameter

$$s = \begin{cases} -\gamma^{-1}e^{-\gamma t}, & \text{for the flow in } N, \\ \gamma^{-1}e^{-\gamma t}, & \text{for the flow in } \hat{N}, \end{cases} \tag{2.16}$$

and defined the flow $y = y(s)$ by $y(s) = \hat{x}(t)$. $y = y(s)$ is then defined in $[-\gamma^{-1}, \gamma^{-1}] \times \mathcal{S}_0$, smooth in $\{s \neq 0\}$, and satisfies the evolution equation

$$y' \equiv \tfrac{d}{ds} y = \begin{cases} -\hat{H}^{-1}\hat{\nu}e^{\gamma t}, & s < 0, \\ \hat{H}^{-1}\hat{\nu}e^{\gamma t}, & s > 0, \end{cases} \tag{2.17}$$

or equivalently, if we only consider the salar version with $\eta = \eta(s)$ representing y^0

$$\eta' = \tfrac{d}{ds}\eta = \begin{cases} \dot{u}e^{\gamma t}, & s < 0, \\ -\dot{\hat{u}}e^{\gamma t}, & s > 0. \end{cases} \tag{2.18}$$

According to the results in [2, Theorem 8.1] y, and hence η, are of class C^3 across the singularity.

Now, looking at the relation (2.8) we see that the new parameter s could just as well be defined by

$$s = \begin{cases} -\tilde{\gamma}^{-1} e^{\tilde{\gamma} f}, & s < 0, \\ \tilde{\gamma}^{-1} e^{\tilde{\gamma} f}, & s > 0, \end{cases} \tag{2.19}$$

where in N as well as in $\hat{N}$ f is considered to be a function of $u(t)$, $f = f(u(t))$.

Defining s by (2.19) we deduce for $s < 0$

$$\eta' = \dot{u}\tfrac{dt}{ds} = \dot{u}\frac{1}{-f' e^{\tilde{\gamma} f}\dot{u}} = \frac{1}{-f' e^{\tilde{\gamma} f}} \equiv \varphi^{-1}. \tag{2.20}$$

The same relation is also valid for $s > 0$.

Suppose now that φ, or equivalently, φ^2,

$$\varphi^2 = |f'|^2 e^{2\tilde{\gamma} f} = -\tilde{\kappa} e^{2\tilde{\gamma} f} + \tfrac{2\kappa}{n(n-1)}(\sigma e^{2(\tilde{\gamma}+1)f} + \rho_0 e^{-\mu}), \tag{2.21}$$

can be viewed as an even function in $e^{\tilde{\gamma} f}$, or equivalently, an even function in s, then η would be of class C^∞ across the singularity, and hence the transition flow $y = y(s)$ would be smooth.

We have thus proved

2.1. **Theorem.** *Let $\tilde{\gamma} = \frac{1}{2}(n + \omega_0 - 2) > 0$, and assume that λ satisfies the condition (1.2) and that μ is a smooth and even function in the variable $(-r)^{\tilde{\gamma}}$, $r < 0$, or can be extended to a smooth and even function on $(-\tilde{\gamma}^{-1}, \tilde{\gamma}^{-1})$, then the transition flow $y = y(s, \xi)$ is smooth in $(-\tilde{\gamma}^{-1}, \tilde{\gamma}^{-1}) \times \mathcal{S}_0$, if either*

$$\omega_0 \in \mathbb{R} \quad \text{and} \quad \sigma = 0, \tag{2.22}$$

or

$$\omega_0 = 4 - n \quad \text{and} \quad \sigma \in \mathbb{R}. \tag{2.23}$$

If $n = 3$ and (2.23) is valid, this means that we consider a radiation dominated universe.

Let us also emphasize that in the preceding theorem we have only proved a smooth transition from big crunch to big bang. This does not necessarily mean that we have a cyclic universe—the same observation also applies to the transition results we obtained in [4] in a brane cosmology setting.

It could well be that the following scenario holds: The spacetime N exists in $-\infty < \tau < 0$ with the only singularity in $\tau = 0$, a big crunch; the mean curvature of the slices $\{x^0 = \text{const}\}$ is always positive and

$$\lim_{\tau \to -\infty} e^f = \infty. \tag{2.24}$$

After a smooth transition through the singularity the mirror image $\hat{N}$ develops.

Such a pair of universes $(N, \hat{N})$ can be easily constructed, in fact, this will always be the case, if the right-hand side of equation (1.8) never vanishes and grows at most quadratically in φ, which will be the case, if $\sigma = 0$, since then equation (1.7) will be solvable in an interval $(-a, 0]$, where a is determined by the requirement

$$\lim_{\tau \to -a} F(\varphi) = 0. \tag{2.25}$$

Hence, if $F(\varphi)$ never vanishes, the solution of (1.7) will exist in $(-\infty, 0]$. Moreover, in $\tau = -\infty$ there cannot be a singularity, a big bang, since this would require that the mean curvature of the coordinate slices tend to $-\infty$. But this impossible, since H never changes sign, there exist no maximal hypersurfaces in N.

To give an explicit example set $\sigma = \mu = 0$ and assume $\tilde{\kappa} = 0, -1$. Then equation (1.5) has the form

$$\dot{\varphi}^2 = -\tilde{\kappa}\tilde{\gamma}^2\varphi^2 + \tfrac{2\kappa\tilde{\gamma}^2}{n(n-1)}\rho_0. \tag{2.26}$$

If $\tilde{\kappa} = -1$, we deduce

$$\varphi = \lambda \sinh(c\tau), \quad \lambda < 0, \quad c > 0, \tag{2.27}$$

and if $\tilde{\kappa} = 0$, then

$$\dot{\varphi} = -c^2, \tag{2.28}$$

hence

$$\varphi = -c^2\tau, \tag{2.29}$$

i.e.,

$$e^f = (-c^2\tau)^{\tilde{\gamma}^{-1}}. \tag{2.30}$$

3. A counter example

We shall show that, even in the case of Robertson-Walker spaces, the transition flow is in general only of class C^3, by constructing a counter example.

3.1. **Theorem.** *Let $\omega = \omega_0$ be such that*

$$\tilde{\gamma} = \tfrac{1}{2}(n + \omega - 2) \geq 2, \tag{3.1}$$

and assume $\sigma \neq 0$. Then the Friedmann equation (0.12) has a solution in the interval $(-a, 0)$ such that corresponding spacetime is an ARW space. The transition flow $y = y(s)$, however, is only of class C^3. If $\tilde{\gamma} = 2$, then y is of class $C^{3,1}$, but, if $\tilde{\gamma} > 2$, then

$$\lim_{s \uparrow 0} \left|\frac{d^4\eta}{(ds)^4}\right| = \infty, \tag{3.2}$$

where $\eta = \eta(s)$ is defined as in (2.18).

Proof. Due to Lemma 1.1 the Friedman equation is solvable and the resulting spacetime is an ARW space.

Notice that

$$s = \begin{cases} -ce^{\tilde{\gamma} f}, & s < 0, \\ ce^{\tilde{\gamma} f}, & s > 0, \end{cases} \tag{3.3}$$

and hence we conclude

$$\eta'(s) = \varphi^{-1}, \tag{3.4}$$

cf. equation (2.20), where φ^2 can be expressed as

$$\varphi^2 = -\tilde{\kappa} c_1 s^2 + c_2 \rho_0 + c_3 \sigma (s^2)^{1+\tilde{\gamma}^{-1}} \tag{3.5}$$

with positive constants c_i.

The proof of the theorem can now be completed by elementary calculations. $\square$

References

1. Claus Gerhardt, *Hypersurfaces of prescribed curvature in Lorentzian manifolds*, Indiana Univ. Math. J. **49** (2000), 1125–1153, pdf file.
2. ______, *The inverse mean curvature flow in ARW spaces - transition from big crunch to big bang*, 2004, 39, pages, math.DG/0403485.
3. ______, *The inverse mean curvature flow in cosmological spacetimes*, 2004, 24 pages, math.DG/0403097.
4. ______, *Transition from big crunch to big bang in brane cosmology*, 2004, 19 pages, gr-qc/0404061.
5. Justin Khoury, *A briefing on the ekpyrotic/cyclic universe*, 2004, 8 pages, astro-ph/0401579.
6. Neil Turok and Paul J. Steinhardt, *Beyond inflation: A cyclic universe scenario*, 2004, 27 pages, hep-th/0403020.

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