Global $C^{1,1}$-Regularity for Solutions of Quasilinear Variational Inequalities

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Communicated by J. C. C. NITSCHE

0. Introduction

We shall consider solutions u of variational inequalities of the form

$$u \in K; \quad \langle Au + Hu, v - u \rangle \geqq 0 \quad \forall v \in K, \tag{0.1}$$

where $K = \{v \in H^{1,\infty}(\Omega): v \geqq \psi, v_{|\partial\Omega} = \varphi\}$, where

$$Au = -D_i(a^i(x, u, Du)) \tag{0.2}$$

is an *elliptic* operator, and where

$$Hu = H(x, u, Du). \tag{0.3}$$

Throughout the paper Ω denotes a bounded open set in $\mathbb{R}^n$, $n \geqq 2$.

It is well-known that the solutions of (0.1) are of class $H^{2,p}(\Omega)$ for any finite p, if the data and the coefficients are smooth enough. More precisely, one can show that

$$Au \in L^\infty(\Omega), \tag{0.4}$$

from which the L^p-estimates are easily derived. It is also known that the second derivatives of the solutions can at most be bounded. This border-line result has been proved by FREHSE [3, 4] for linear operators and in our paper [5] for general quasilinear operators, if the obstacle ψ is strictly below the prescribed boundary data φ near the boundary. The method of proof in [5] also yields interior estimates without any specific assumption on ψ near the boundary. Another proof for the local $C^{1,1}$-regularity has been given by BRÉZIS and KINDERLEHRER in [1] for a special class of non-linear operators.

The first global result has recently been obtained by JENSEN [7] for linear operators L of the form

$$Lu = -a^{ij} D_i D_j u \tag{0.5}$$

with $C^{2,\alpha}$ coefficients a^{ij}. His idea was to choose new coordinates $\bar{x} = \bar{x}(x)$ such that the new coefficients $\bar{a}^{ij}$ split on the boundary, namely

$$\bar{a}^{\sigma n} = 0 \text{ on } \partial\Omega, \tag{0.6}$$

for $1 \leqq \sigma \leqq n-1$. Since the old coefficients enter into the transformation $\Phi: x \to \bar{x}$, it is evident that Φ is of class C^{m+1} if the a^{ij}'s are of class C^m with a corresponding relation for the weak differentiability.

In the non-linear case the corresponding transformation Φ would be only of class $H^{2,p}$ for any finite p, and its second derivatives could only be estimated by the second derivatives of the solution u. Thus, even if it were possible to estimate the second derivatives of u in the new coordinate system (which seems doubtful) this would give no bound of the second derivatives in the original coordinate system.

To derive $C^{1,1}$-estimates in the nonlinear case, we therefore do not transform the coordinates so that the coefficients matrix split in the new coordinates. Instead, using the method of penalization we approximate any given solution of the variational inequality by smooth functions, and prove uniform estimates for the global $C^{1,1}$-norm. Unfortunately, our method of proof does not yield local estimates near the boundary. Precisely, we prove the following result.

0.1 Theorem. *Let $\partial\Omega$ be of class $C^{3,\alpha}$, $\varphi \in C^3(\bar{\Omega})$, $\psi \in C^2(\bar{\Omega})$. Assume that the a^i's are of class C^2 in x and u and of class C^3 in the p variables, and that H is of class C^1 in all its arguments. Then any solution of the variational inequality* (0.1) *is of class $H^{2,\infty}(\Omega)$.*

Of course, C^m can everywhere be replaced by $C^{m-1,1}$.

1. Preliminaries

The a priori estimates which we shall derive in the next section are for solutions u of approximating problems. To be sure that approximating solutions exist, we shall replace A and H by operators $\tilde{A}$ and $\tilde{H}$ satisfying appropriate growth conditions so that corresponding Dirichlet problems are always solvable.

To be precise, let u_0 be a solution of the variational inequality (0.1). Then, by assumption, u_0 is Lipschitz continuous and therefore of class $H^{2,p}(\Omega)$ for any finite p. Let M be such that

$$1 + |u_0| + |Du_0| \leqq M. \tag{1.1}$$

We then change the operator as follows: first let ϑ be a smooth real valued function such that

$$\vartheta(t) = \begin{cases} t, & |t| \leqq M \\ M+1, & |t| \geqq M+1 \end{cases}. \tag{1.2}$$

Secondly, let ω, g be smooth functions such that

$$(1.3)\qquad \omega(t) = \begin{cases} 1, & 0 \leqq t \leqq 2M \\ 0, & t \geqq 3M \end{cases}$$

and g is convex satisfying

$$(1.4)\qquad g(t) = \begin{cases} 0, & 0 \leqq t \leqq M \\ c \cdot t, & t \geqq 2M, \end{cases}$$

where $c > 0$.

We then define

$$(1.5)\qquad \tilde{a}^i(x, t, p) = a^i(x, \vartheta(t), p) \cdot \omega(|p|^2) + k \cdot g'(|p|^2) \cdot p^i$$

and

$$(1.6)\qquad \tilde{H}(x, t, p) = H(x, \vartheta(t), p) \cdot \omega(|p|^2).$$

If we choose k large enough, the $\tilde{a}^i$ then form a uniformly elliptic vector field such that

$$(1.7)\qquad \tilde{A}u_0 + \tilde{H}u_0 = Au_0 + Hu_0;$$

for details, see [6; Appendix II].

Moreover, for large γ the operator

$$(1.8)\qquad \tilde{A}u + \tilde{H}u + \gamma u$$

is *uniformly monotone* in the sense of MINTY, that is, there exists $c > 0$ such that

$$(1.9)\qquad \langle \tilde{A}u_1 + \tilde{H}u_1 + \gamma u_1 - \tilde{A}u_2 - \tilde{H}u_2 - \gamma u_2, u_1 - u_2 \rangle \geqq c \cdot \|u_1 - u_2\|_{1,2}^2,$$

for any $u_1, u_2 \in H^{1,2}(\Omega)$ which agree on the boundary. In the following we shall omit the tilde, assuming that A and H have all the attributes stated above.

For the approximation process we also need slightly sharper differentiability assumptions on the a^i's and on H: for simplicity we shall assume for the moment that the a^i's and H are of class C^4 in all their arguments.

We then consider solutions u of the Dirichlet problem

$$(1.10)\qquad Au + Hu + \gamma u + \mu\beta(u - \psi) = \gamma u_0, \quad u_{|\partial\Omega} = \varphi,$$

where $\mu > 0$ is a parameter tending to infinity and β is a monotone function satisfying

$$(1.11)\qquad \beta(t) = \begin{cases} 0, & t \geqq 0 \\ <0, & t < 0 \end{cases}.$$

For our purpose it is convenient to choose the special function

$$(1.12)\qquad \beta(t) = \begin{cases} 0, & t \geqq 0 \\ -t^2, & t < 0, \end{cases}$$

which is of class $C^{1,1}(\mathbb{R})$.

In view of our assumptions, the boundary value problem (1.10) always has a solution u of class $C^{3,\alpha}(\bar{\Omega})$. We shall show that the second derivatives of u are uniformly bounded, independently of μ. In the limit, u will tend to the given solution u_0 of the variational inequality, after having removed the additional differentiability assumptions on the a^i's and on H.

The following lemma is an immediate consequence of the maximum principle.

1.1 Lemma. *Let u be a solution of* (1.10). *Then* $u - \psi \geqq c \cdot \mu^{-\frac{1}{2}}$ *and*

$$\mu \cdot |\beta(u-\psi)| \leqq c^2, \tag{1.13}$$

where

$$c^2 := \sup_{\Omega} |A\psi + H\psi|. \tag{1.14}$$

Hence we conclude that

$$Au \in L^\infty(\Omega) \tag{1.15}$$

with a uniform bound, and

$$\|u\|_{2,p} \leqq c \quad \forall\, 1 \leqq p < \infty, \tag{1.16}$$

where the constant depends on p, $\|\psi\|_{2,\infty}$, $\|\varphi\|_{2,\infty}$, $\partial\Omega$, and on other known quantities.

Next, let x_0 be an arbitrary but fixed point on the boundary. We may "smooth" a small portion of the boundary near x_0, and so may assume without loss of generality that $x_0 = 0$, that the equation (1.10) is defined in the upper half-ball $B_1^+(0)$, and that the part of the hyperplane which is defined through

$$\Gamma = \{(\hat{x}, x^n) : |\hat{x}| \leqq 1, x^n = 0\} \tag{1.17}$$

represents the "smoothed" portion of $\partial\Omega$.

For later use, we note that coordinates with Greek indices like x^σ refer to coordinates in the hyperplane $1 \leqq \sigma \leqq n-1$. The differential operators D_σ, $D_\sigma D_\varrho$, etc., are similarly defined. We further remark that the coefficients a^{ij}, a^{ijk}, etc. are obtained from a^i by differentiation with respect to the p variable alone. The symbol "$_{,k}$" is defined through the following rule

$$a^i_{,k}(x, u, Du) = \frac{\partial a^i}{\partial x^k} + \frac{\partial a^i}{\partial u} \cdot D_k u. \tag{1.18}$$

A similar definition applies for "$_{,kl}$". Finally we denote by f^i any vector field such that

$$\|f^i\|_p \leqq c(1 + \|u\|_{2,p})^m \tag{1.19}$$

for any $1 \leqq p \leqq \infty$, where c and m are arbitrary constants depending on p; similarly f will denote any function which can be estimated in the same way.

2. The a priori estimates

Following the ideas in [2] and [7], we shall estimate the quantity

$$(2.1)\qquad \lambda \cdot a^{kl}\, D_k D_l u \pm D_r D_s u, \quad \lambda > 0$$

from below, first for $1 \leqq r, s \leqq n-1$ and then for $1 \leqq r \leqq n-1$, $s = n$. Since $a^{kl}\, D_k D_l u$ is bounded, this gives an estimate for all second derivatives of u except for $D_n D_n u$. But the normal derivatives can then be bounded by using the equation and the uniform ellipticity of the a^{ij}'s.

Let us differentiate equation (1.10) with respect to D_s to obtain

$$(2.2)\qquad - D_i(a^{ij}\, D_j D_s u + a^i_{,s}) + \mu \cdot \beta'\, D_s(u-\psi) + D_s H + \gamma\, D_s u = \gamma\, D_s u_0 .$$

Differentiating further, this time with respect to D_r, yields

(2.3)

$$-D_i(a^{ij}\, D_j D_r D_s u) + \mu \cdot \beta''\, D_r(u-\psi)\, D_s(u-\psi) + \gamma\, D_r D_s u + \mu \cdot \beta'\, D_r D_s(u-\psi) = D_i f^i .$$

On the other hand, let us consider the relation

$$\begin{aligned}(2.4)\qquad & -D_i(a^{ij}\, D_j(a^{kl}\, D_k D_l u)) \\ &= -D_i(a^{ij} a^{kl}\, D_j D_k D_l u + a^{ij} a^{klm}\, D_m D_j u \cdot D_k D_l u + a^{ij} a^{kl}_{,j}\, D_k D_l u) \\ &= -D_i(a^{ij}\, D_j D_k D_l u)\, a^{kl} + D_i f^i - a^{ij}\, D_j D_k D_l u (a^{klm}\, D_i D_m u + a^{kl}_{,i}).\end{aligned}$$

We observe that the last expression on the right-hand side of (2.4) is equal to

$$\begin{aligned}(2.5)\qquad & -D_j(a^{ij}\, D_i D_m u \cdot a^{klm}\, D_k D_l u) + a^{ij}\, D_i D_j D_m u \cdot a^{klm}\, D_k D_l u + D_i f^i + f \\ &= D_m(a^{ij}\, D_i D_j u) \cdot a^{klm}\, D_k D_l u + D_i f^i + f,\end{aligned}$$

so that finally we deduce that

$$\begin{aligned}(2.6)\qquad & -D_i(a^{ij}\, D_j(a^{kl}\, D_k D_l u)) \\ &= -D_i(a^{ij}\, D_j D_k D_l u)\, a^{kl} + \mu \cdot \beta'\, D_m(u-\psi) \cdot a^{klm}\, D_k D_l u + D_i f^i + f,\end{aligned}$$

where we have also used the differentiated form of (1.10), namely

$$(2.7)\qquad -a^{ij}\, D_i D_j u - a^i_{,i} + Hu + \mu \cdot \beta(u-\psi) + \gamma u = \gamma u_0 .$$

Combining (2.3) and (2.6), we conclude that

$$\begin{aligned}(2.8)\qquad & -D_i(a^{ij}\, D_j(\lambda a^{kl}\, D_k D_l u + D_r D_s u) \\ & + \gamma\{\lambda a^{kl}\, D_k D_l u + D_r D_s u\} \\ & + \mu \cdot \beta''\{\lambda a^{kl}\, D_k(u-\psi) \cdot D_l(u-\psi) + D_r(u-\psi)\, D_s(u-\psi)\} \\ & + \mu \cdot \beta'\{\lambda a^{kl}\, D_k D_l(u-\psi) + D_r D_s(u-\psi)\} \\ & - \mu \cdot \beta' \cdot \lambda \cdot D_m(u-\psi) \cdot a^{klm}\, D_k D_l u = f + D_i f^i .\end{aligned}$$

Bearing in mind that the a^{kl} are uniformly elliptic, we now choose λ so large that

(2.9) $\lambda a^{kl} D_k(u-\psi) D_l(u-\psi) + D_r(u-\psi) D_s(u-\psi) \geqq 2 \cdot |D(u-\psi)|^2.$

Furthermore, we note that

(2.10) $$\beta'' = -2 \text{ where } u < \psi; \quad \beta'' = 0 \text{ elsewhere},$$

and

(2.11) $$\beta' = |u-\psi| \text{ where } u < \psi; \quad \beta' = 0 \text{ elsewhere}.$$

Hence we get the estimate

(2.12) $$\mu \cdot \beta' D_m(u-\psi) \cdot a^{klm} D_k D_l u \geqq \mu \cdot \beta'' |D(u-\psi)|^2 + \mu \cdot \beta \cdot f.$$

But $\mu \cdot \beta$ is bounded, so finally

(2.13) $$\begin{aligned} &-D_i(a^{ij} D_j(\lambda \cdot a^{kl} D_k D_l u + D_r D_s u)) \\ &+ \gamma\{\lambda a^{kl} D_k D_l u + D_r D_s u\} \\ &+ \mu \cdot \beta' \cdot \{\lambda a^{kl} D_k D_l(u-\psi) + D_r D_s(u-\psi)\} \geqq f + D_i f^i. \end{aligned}$$

Let w and $\overline{w}$ be defined through

(2.14) $$w = \lambda a^{kl} D_k D_l u + D_r D_s u$$

and

(2.15) $$\overline{w} = \lambda a^{kl} D_k D_l \psi + D_r D_s \psi.$$

We may then rewrite (2.13) as

(2.16) $$-D_i(a^{ij} D_j w) + \gamma \cdot w + \mu \cdot \beta' \cdot (w - \overline{w}) \geqq f + D_i f^i.$$

We note for later reference that the same formula is also valid if we replace $D_r D_s u$ by $-D_r D_s u$.

We are now in the position to estimate the second derivatives of u. We first consider the tangential part of these derivatives.

2.1 Theorem. *The second tangential derivatives of u can be estimated by*

(2.17) $$\sup_{B^+_{\frac{1}{2}}(0)} |D_\sigma D_\varrho u| \leqq \sup_{B^+_1(0)} |D_\sigma D_\varrho \varphi| + c,$$

where c depends on $\|u\|_{2,p}$ for sufficiently large values of p, and on $\|H\|_\infty$, $\|\psi\|_{2,\infty}$.

Proof. Let η be a cut-off function such that $\eta = 1$ in $B^+_{\frac{1}{2}}(0)$ and $\operatorname{supp} \eta \subset B^+_1(0)$. Choose

(2.18) $$k_0 = \sup_{\Gamma} |D_\sigma D_\varrho \varphi| + \sup_{B^+_1(0)} |\overline{w}|,$$

where we have used (2.14), (2.15) with $s = \sigma$ and $r = \varrho$.

Next we multiply inequality (2.16) by $w_k \cdot \eta^2 := \min(w \cdot \eta^2 + k, 0) \cdot \eta^2$, where $k \geqq k_0$, and integrate to obtain

$$\int_{B_1^+} a^{ij} D_j w \cdot \eta^2 \cdot D_i w_k + \gamma \cdot \int_{B_1^+} w_k^2 \tag{2.19}$$
$$\leqq \int_{B_1^+} f \cdot w_k \cdot \eta^2 - \int_{B_1^+} f^i \cdot D_i(w_k \cdot \eta^2) - \int_{B_1^+} a^{ij} D_j w\, D_i \eta^2 \cdot w_k,$$

where we have used the fact that w_k vanishes on Γ.

Let $A(k)$ be the set where $w_k \neq 0$, and let $|A(k)|$ be its Lebesgue measure. Then in view of the uniform ellipticity of the a^{ij}'s we conclude from (2.19) that

$$\int_{B_1^+} |Dw_k|^2 + \gamma \cdot \int_{B_1^+} w_k^2 \leqq c \cdot \left\{ \int_{A(k)} f^i \cdot f_i + \int_{A(k)} |f| \cdot |w| \right\}, \tag{2.20}$$

where c also depends on $|D\eta|$. Applying the Sobolev inequality we derive

$$\left(\int_{B_1^+} |w_k| \right)^{\frac{n-1}{n}} \leqq c \cdot |A(k)|^a, \tag{2.21}$$

where $0 < a < 1$ can be chosen arbitrarily close to 1, and where c depends on a and powers of $\|u\|_{2,p}$ for corresponding values of p. Choosing a so that

$$a + \frac{1}{n} = 1 + \frac{1}{2n},$$

we deduce for $h > k$ that

$$|h-k| \cdot |A(h)| \leqq \int_{B_1^+} |w_k| \leqq c \cdot |A(k)|^{1+1/2n}. \tag{2.22}$$

Hence

$$w \cdot \eta^2 \geqq -k_0 - c \cdot |A(k_0)|^{1/2n} \tag{2.23}$$

in view of a well-known lemma due to Stampacchia [8; Lemme 4.1].

The same estimate is valid if instead of

$$w = \lambda a^{kl} D_k D_l u + D_\sigma D_\varrho u \tag{2.24}$$

we choose

$$w = \lambda a^{kl} D_k D_l u - D_\sigma D_\varrho u \tag{2.25}$$

and define $\bar{w}$ accordingly.

In view of the result of Lemma 1.1 this proves the theorem.

Let us next find a bound for $D_\sigma D_n u$.

2.2 Theorem. *We have*

$$\sup_{B_{1/2}^+} |D_\sigma D_n u| \leqq c \cdot (1 + \|u\|_{2,\infty})^\varepsilon \tag{2.26}$$

for any ε, $0 < \varepsilon < 1$, *where* c *depends on* ε, *on* $\|u\|_{2,p}$ *for large* p, *and on known quantities.*

Proof. We consider inequality (2.16) with

$$w = \lambda a^{kl} D_k D_l u + D_\sigma D_n u, \tag{2.27}$$

and multiply it with

$$w_k \cdot \eta^2 := \min(w \cdot \eta^2 + k, 0) \cdot \eta^2,$$

where η is defined as in the proof of Theorem 2.1 and $k \geqq k_0 \geqq \sup_U |\bar{w}| + 1$. Integrating by parts, we obtain

$$\int_{B_1^+} |Dw_k|^2 + \gamma \cdot \int_{B_1^+} w_k^2 \tag{2.28}$$
$$\leqq c(1 + \|u\|_{2,\infty})^m |A(k)| + \int_\Gamma |f^n \cdot w_k| + \left| \int_\Gamma a^{nj} D_j w \cdot \eta^2 \cdot w_k \right|,$$

where we have used the ellipticity of the a^{ij}'s, and where the constant c also depends on $|D\eta|$.

To estimate the boundary integrals, let us first observe that

$$\int_\Gamma |f^n \cdot w_k| \leqq \|f^n\|_\infty \cdot \int_\Gamma |w_k| \leqq \|f^n\|_\infty \cdot \int_{B_1^+} |Dw_k|. \tag{2.29}$$

To estimate the second integral, we note that $\beta = \beta' = 0$ on Γ so that

$$w = F + D_\sigma D_n u, \tag{2.30}$$

where $D_j F = f$, a quantity which can be estimated without difficulty. Thus, the crucial term is

$$a^{nj} D_j D_\sigma D_n u. \tag{2.31}$$

Let us first consider the part

$$a^{nn} D_n D_\sigma D_n u. \tag{2.32}$$

From the equation we conclude that

$$a^{nn} D_\sigma D_n D_n u = f - a^{\varrho\varkappa} D_\varkappa D_\varrho D_\sigma u - a^{n\varrho} D_\varrho D_\sigma D_n u - a^{\varrho n} D_\varrho D_\sigma D_n u \tag{2.33}$$

where the second term on the right-hand side is bounded due to the assumption $\varphi \in C^3$. Hence, we deduce that (2.31) is a sum of f and terms of the form

$$a^{\varrho n} D_\varrho D_\sigma D_n u, \tag{2.34}$$

or equivalently, terms of the form

$$a^{\varrho n} D_\varrho w \tag{2.35}$$

in view of (2.30).

Therefore, we finally obtain

$$a^{nj} D_j w \cdot \eta^2 \cdot w_k = f \cdot w_k + \tfrac{1}{2} a^{\varrho n} D_\varrho w_k^2, \tag{2.36}$$

where f also depends on $|D\eta|$ and where, to be absolutely precise, we should also include terms of the form $\frac{1}{2} a^{n\varrho} D_\varrho w_k^2$.

From the identity

$$\int_\Gamma a^{\varrho n} D_\varrho w_k^2 = -\int_\Gamma D_\varrho a^{\varrho n} w_k^2 = \int_\Gamma f \cdot w_k^2 \tag{2.37}$$

it follows that the second boundary integral in (2.28) can be estimated in the same way as the first one, so that we arrive at an estimate of the form

$$\int_{B_1^+} |Dw_k|^2 + \gamma \cdot \int_{B_1^+} |w_k|^2 \leqq c(1 + \|u\|_{2,\infty})^m \cdot |A(k)|. \tag{2.38}$$

Arguing now as in the proof of Theorem 2.1, we conclude that

$$w \cdot \eta^2 \geqq -k_0 - c(1 + \|u\|_{2,\infty})^m \cdot |A(k_0)|^{1/n} \tag{2.39}$$

with some new integer m. The value $1/n$ for the exponent of $|A(k_0)|$ is simply a coincidence, and it is not essential at all.

We now optimize the right-hand side of (2.39) by appropriate choice of k_0. First, we observe that $a^{kl} D_k D_l u$ is bounded, and hence

$$A(k_0) \subset \{x \in B_1^+ : |D_\sigma D_n u| > c \cdot k_0\} \tag{2.40}$$

for a suitable constant c, if k_0 is sufficiently large. The volume of the larger set is estimated by

$$c^{-p} \cdot k_0^{-p} \cdot \int_{B_1^+} |D_\sigma D_n u|^p \tag{2.41}$$

for any finite p, thus we obtain

$$w \cdot \eta^2 \geqq -k_0 - c(1 + \|u\|_{2,\infty})^m \cdot k_0^{-p/n} \tag{2.42}$$

where c now depends on p and on $\|u\|_{2,p}$.

Choosing

$$k_0 = 1 + \sup_{B_1^+} |\overline{w}| + (1 + \|u\|_{2,\infty})^{\frac{m \cdot n}{p+n}} \tag{2.43}$$

we obtain

$$w \cdot \eta^2 \geqq -c \cdot k_0. \tag{2.44}$$

The same estimate also holds if we replace $D_\sigma D_n u$ by $-D_\sigma D_n u$ in the definition of w. The theorem is therefore proved if p is chosen appropriately.

To obtain an a priori bound for all second derivatives we note that by using the equation the normal derivatives $D_n D_n u$ can be expressed in terms of $D_\sigma D_n u$ and $D_\sigma D_\varrho u$. Thus, we conclude that

$$\|u\|_{2,\infty,B_{1/2}^+} \leqq c_\varepsilon (1 + \|u\|_{2,\infty})^\varepsilon \tag{2.45}$$

for any ε, $0 < \varepsilon < 1$. Since $\partial\Omega$ is compact this estimate also holds in a boundary neighbourhood N. A slightly revised version of the proof of Theorem 2.1 then yields

$$\|u\|_{2,\infty,\Omega \setminus N} \leqq c_\varepsilon (1 + \|u\|_{2,\infty})^\varepsilon, \tag{2.46}$$

so that finally we get an a priori estimate for $\|u\|_{2,\infty,\Omega}$ depending only on the quantities mentioned in Theorem 0.1.

Letting μ tend to infinity, we derive the existence of a function $u \in H^{2,\infty}(\Omega)$ solving the variational inequality

$$\langle Au + Hu + \gamma(u - u_0), v - u\rangle \geqq 0 \quad \forall\, v \in K, \tag{2.47}$$

where A, H, and φ satisfy the stronger differentiability assumptions. Since the estimates are independent of these assumptions, a simple approximation argument shows that the variational problem (2.4) has a solution $u \in H^{2,\infty}(\Omega)$ assuming only the weaker conditions. Uniqueness of the solution (cf. (1.9)) then yields $u = u_0$.

Note. This work has been supported by the Deutsche Forschungsgemeinschaft.

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(Received March 15, 1982)