Regularity of Solutions of Nonlinear Variational Inequalities

CLAUS GERHARDT

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Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let A be a quasilinear differential operator in divergence form,

$$A = -D^i(a_i(x, u, p)), \tag{1}$$

with

$$a_i \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \tag{2}$$

and

$$\left(\frac{\partial a_i}{\partial p^j}\right) \xi^i \xi^j > 0 \qquad \text{for all } \xi \in \mathbb{R}^n - \{0\}. \tag{3}$$

Suppose that $u \in \boldsymbol{K}$ is a solution of the variational inequality

$$\langle Au + H, v - u \rangle \geqq 0 \qquad \text{for all } v \in \boldsymbol{K}, \tag{4}$$

where

$$\boldsymbol{K} = \{v \in H^{1,\infty}(\Omega) \colon v \geqq \psi,\, v_{|\partial\Omega} = f\},$$

and where H, f, and ψ are given functions such that

$$H \in H^{1,\infty}(\Omega), \tag{5}$$

$$f \in C^2(\bar{\Omega}), \tag{6}$$

and

$$\psi \in H^{2,\infty}(\Omega), \qquad \psi_{|\partial\Omega} < f. \tag{7}$$

Moreover, assume that

$$Au \text{ is essentially bounded} \tag{8}$$

and hence

$$u \in H^{2,p}(\Omega) \qquad \text{for any } p, 1 < p < \infty.\,[1] \tag{9}$$

[1] For a short proof of this well-known conclusion the reader is referred to the Appendix.

The aim of this paper is to show that the second derivatives of u are locally bounded. Up to now results in this direction have been given by J. FREHSE [2], [3] and D. KINDERLEHRER [6] for linear variational inequalities. Moreover, in the case of two dimensions and a concave obstacle ψ, D. KINDERLEHRER has announced regularity results for the (nonlinear) minimal surface operator (to appear in Proc. Symp. Pure Math. Vol. 23).

We shall show that Frehse's method of proof continues to work in the general case of a nonlinear operator.

Theorem. *Under the assumptions stated above the variational inequality* (4) *has a solution* $u \in H^{2,\infty}_{\mathrm{loc}}(\Omega)$.

Proof. Define the coincidence set I by

$$I = \{x \in \Omega : u(x) = \psi(x)\}.$$

Since $u, \psi \in C^0(\bar{\Omega})$ it is clear that I is closed, and since $u - \psi > 0$ on $\partial\Omega$ there exists an open set Ω' with $I \subset\subset \Omega' \subset\subset \Omega$. Choose $h \in \mathbb{R}^n - \{0\}$ such that $\Omega' \pm h \subset\subset \Omega$. Then for any non-negative $\phi \in C_c^\infty(\Omega' - I)$ there exists a value $\varepsilon_0 = \varepsilon_0(\phi, h)$ such that for $0 < \varepsilon \leqq \varepsilon_0$

$$u_\varepsilon = u + \varepsilon \phi_h \in K \tag{10}$$

where

$$\phi_h(x) = |h|^{-2}\{\phi(x+h) - 2\phi(x) + \phi(x-h)\}. \tag{11}$$

Setting $v = u_\varepsilon$ in (4), we obtain

$$\langle Au + H, \phi_h \rangle \geqq 0. \tag{12}$$

First of all, let us consider the term

$$\int a_i(x, u, Du) D^i\{\phi(x-h) - \phi(x)\}\, dx,$$

which can be written also in the form

$$\int \{a_i(x+h, u(x+h), Du(x+h)) - a_i(x, u(x), Du(x))\} D^i \phi\, dx. \tag{13}$$

Since the expression in the braces is

$$\int_0^1 \frac{d}{dt} a_i(x + th, tu(x+h) + (1-t)u(x), tDu(x+h) + (1-t)Du(x))\, dt, \tag{14}$$

we can write (13) in the form

$$\int \{a_{ij}(h) D^j[u(x+h) - u(x)] + b_i(h)[u(x+h) - u(x)] + c_{ij}(h) h^j\} D^i \phi\, dx, \tag{15}$$

where we have set

$$a_{ij}(h) = \int_0^1 \frac{\partial a_i}{\partial p^j}(\ldots)\, dt,$$

$$b_i(h) = \int_0^1 \frac{\partial a_i}{\partial u}(\ldots)\, dt, \qquad c_{ij}(h) = \int_0^1 \frac{\partial a_i}{\partial x^j}(\ldots)\, dt$$

and where the dots denote the arguments in (14).

Let w be a solution of the equation

$$-\Delta w = H \tag{16}$$

in any ball B, $\Omega \subset\subset B$. Then using the preceding results we derive from the inequality (12)

$$\begin{aligned} \int \{ & a_{ij}(h) D^i u_h + |h|^{-2} [a_{ij}(-h) - a_{ij}(h)] D^j [u(x-h) - u(x)] \\ & + b_i(h) u_h + |h|^{-2} [b_i(-h) - b_i(h)] [u(x-h) - u(x)] \\ & + |h|^{-2} [c_{ij}(h) - c_{ij}(-h)] h^j + |h|^{-2} D^i w_h \} D^i \phi \, dx \geqq 0. \end{aligned} \tag{17}$$

From the assumption (5) and from the Calderon-Zygmund Inequalities we conclude that w belongs to $H^{3,p}(\Omega)$ for any p, $1<p<\infty$. Moreover, the a_{ij}'s are Lipschitz functions and the second derivatives of u are p-summable for any p, $1<p<\infty$. Therefore, the lower order terms in (17) belong to $L^p(\Omega)$ for any p, $1<p<\infty$. Hence it follows from a theorem due to STAMPACCHIA (see [8], Théorème 4.1) that

$$u_h \geqq \min(0, \min_{\partial(\Omega'-I)} u_h) - c \quad \text{in } \Omega' - I, \tag{18}$$

where the constant c does not depend on h.

Since $\partial\Omega' \subset \{u > \psi\}$ we know that u_h is bounded on $\partial\Omega'$, while for $x \in I$, $u_h(x)$ is estimated from below by $\psi_h(x)$. Thus

$$u_h(x) \geqq -\text{const.} \quad \text{for all } x \in \Omega'. \tag{19}$$

Now take a fixed $x \in \Omega' - I$. Since $u \in C^2(\Omega' - I)$ we obtain from (19)

$$D^2 u(x) \xi^2 \geqq -\text{const.} \quad \text{for all } \xi \in \mathbb{R}^n, \ |\xi| = 1. \tag{20}$$

To complete the proof we need the following

Lemma. *Let $B = (b_{ij})_{i,j=1,\ldots,n}$ and $C = (c_{ij})_{i,j=1,\ldots,n}$ be symmetric matrices such that $B \geqq 0$ and $C > 0$. Then*

$$\|B\| \leqq \operatorname{tr}(BC) \cdot \|C\| \cdot \|C^{-1}\|^2. \tag{21}$$

Suppose the lemma to be valid. Then we take for B the matrix $D^2 u + \text{const.} I$ and for C the coefficient matrix (a_{ij}), $a_{ij} = \partial a_i / \partial p^j$, and from (8), (20), and (21) conclude that $D^2 u$ is bounded on $\Omega' - I$. Since $D^2 u = D^2 \psi$ a.e. on I, the theorem is completely proved.

Proof of the Lemma. Let $\xi \in \mathbb{R}^n - \{0\}$ and define η by $C\eta = \xi$; then

$$\langle B\xi, \xi \rangle = \langle BC\eta, C\eta \rangle = \langle CBC\eta, \eta \rangle. \tag{22}$$

We may assume CBC to be diagonalized with corresponding nonnegative eigenvalues λ_i; thus

$$\langle B\xi, \xi \rangle = \sum \lambda_i |\eta^i|^2 \leqq \operatorname{tr}(CBC) \cdot |\eta|^2. \tag{23}$$

Since B, C are non-negative and symmetric, we obtain

$$\operatorname{tr}(CBC) = \operatorname{tr}(BC^2) \leqq \operatorname{tr}(BC) \cdot \|C\|, \tag{24}$$

and hence

$$\langle B\xi, \xi\rangle \leqq \mathrm{tr}(BC)\cdot \|C\| \cdot \|C^{-1}\|^2 \cdot |\xi|^2 \tag{25}$$

from which the assertion follows.

The problem of the boundedness of the second derivatives of solutions to nonlinear variational inequalities with obstacles has also been solved by H. BREZIS and D. KINDERLEHRER [9]. They showed that the solutions of certain approximating differential equations have locally uniformly bounded second derivatives.

Appendix

We shall give a brief outline of the proof of the assertion (9).

Theorem. *Let u be a Lipschitz function such that Au belongs to $L^p(\Omega)$ for $p>n$, and such that $u_{|\partial\Omega}=f$, $f\in C^2(\bar{\Omega})$. Then*

$$u\in H^{2,p}(\Omega). \tag{26}$$

Proof. From [10], Théorème 11.1 and Théorème 11.2, we conclude that u belongs to $H^{2,2}(\Omega)\cap C^{1,\alpha}(\bar{\Omega})$ for any α, $0<\alpha<1$. Hence we can differentiate formally to obtain

$$\Sigma\, a_{ij} D^i D^j u = F, \tag{27}$$

where $F\in L^p(\Omega)$ and the coefficients a_{ij} are Hölder continuous. Subtracting $\Sigma\, a_{ij} D^i D^j f$ from the equation, we may restrict ourselves to the case $u_{|\partial\Omega}=0$.

Assuming that the (unique) solution u of (27), with $u_{|\partial\Omega}=0$, lies in the Sobolev space $H^{2,p}(\Omega)$, then the following a priori estimate holds (see [11]):

$$\|u\|_{2,p} \leqq \text{const.}\cdot\{\|F\|_p + \|u\|_p\}, \tag{28}$$

where the constant depends on p, n, Ω, the ellipticity constant of the a_{ij}'s, and on the modulus of continuity of the coefficients.

Conversely, approximating F by smooth functions F_ε, we can solve the equations

$$\begin{aligned} \Sigma\, a_{ij} D^i D^j u_\varepsilon &= F_\varepsilon \\ u_{\varepsilon|\partial\Omega} &= 0 \end{aligned} \tag{29}$$

in $C^{2,\alpha}(\bar{\Omega})$, $0<\alpha<1$, since the coefficients are Hölder continuous. The assertion (26) now follows from the a priori estimate, since the equation (27) has a unique solution $u\in H^{2,2}(\Omega)\cap C_0^1(\bar{\Omega})$.

References

1. BREZIS, H., & G. STAMPACCHIA, Sur la régularité de la solution d'inéquations elliptiques. Bull. Soc. Math. France **96**, 153–180 (1968)
2. FREHSE, J., On the regularity of the solution of a second order variational inequality. Boll. Un. Mat. Ital. **6**, 312–315 (1972)
3. FREHSE, J., On the regularity of solutions of linear elliptic variational inequalities. (Unpublished paper)

4. GERHARDT, C., Hypersurfaces of prescribed mean curvature over obstacles. Math. Z. (to appear)
5. HARTMANN, P., & G. STAMPACCHIA, On some nonlinear elliptic differential functional equations. Acta Math. **115**, 271–310 (1966).
6. KINDERLEHRER, D., The coincidence set of solutions of certain variational inequalities. Arch. Rational Mech. Analysis **40**, 231–250 (1971)
7. LEWY, H., & G. STAMPACCHIA, On existence and smoothness of solutions of some noncoercive variational inequalities. Arch. Rational Mech. Anal. **41**, 241–253 (1971)
8. STAMPACCHIA, G., Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier **XV**, 189–258 (1965)
9. BREZIS, H., & D. KINDERLEHRER, The smoothness of solutions to nonlinear variational inequalities (to appear)
10. STAMPACCHIA, G., Equations elliptiques du second ordre à coefficients discontinus. Sém. Math. Sup. Université de Montréal 1966
11. GRECO, D., Nuove formule integrali di maggiorazione per le soluzione di un'equazione lineare di tipo ellitico ed applicazioni alla teoria del potenziale. Ricerce Mat. **5**, 126–149 (1956)

Fachbereich Mathematik
Universität Mainz
D-6500 Mainz, Saarstraße 21
Federal Republic of Germany

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