

# SMOOTHNESS OF LIPSCHITZ SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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### 1. LIPSCHITZ SOLUTIONS ARE SMOOTH

We consider the Dirichlet problem

$$(1.1) \quad \begin{aligned} Au + H(x, u, Du) &= 0, \\ u|_{\partial\Omega} &= \varphi, \end{aligned}$$

where

$$(1.2) \quad Au = -D_i(a^i(x, u, Du))$$

is an elliptic quasilinear operator. The vector field  $a^i$  and  $H$  should satisfy

$$(1.3) \quad a^i \in C^{m-1, \alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n),$$

and

$$(1.4) \quad H \in C^m(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n),$$

such that  $2 \leq m \in \mathbb{N}$ ,  $0 < \alpha < 1$ .

Then we can prove:

**1.1. Theorem.** *Let  $\Omega \Subset \mathbb{R}^n$  be open, connected with  $\partial\Omega \in C^{m, \alpha}$ ,  $2 \leq m \in \mathbb{N}$ ,  $0 < \alpha < 1$ , and let  $\varphi \in C^{m, \alpha}(\bar{\Omega})$ . Then any weak solution  $u \in C^{0,1}(\bar{\Omega})$  of (1.1) is of class  $C^{m, \alpha}(\bar{\Omega})$ .*

*Proof.* (i) Let  $u_0 \in C^{0,1}(\bar{\Omega})$  be a weak solution such that

$$(1.5) \quad 1 + |u_0| + |Du_0| \leq M.$$

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From [2, Section 1] and [1, Appendix II] we conclude that there exists a uniformly elliptic vector field  $\tilde{a}^i \in C^{m-1,\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$  and a function  $\tilde{H} \in C^m(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$  such that

$$(1.6) \quad |\tilde{a}^{ij}| + \left| \frac{\partial \tilde{a}^i}{\partial x^j} \right| + \left| \frac{\partial \tilde{a}^i}{\partial t} \right| \leq c,$$

$$(1.7) \quad |\tilde{a}^i| \leq c(1 + |\xi|),$$

and

$$(1.8) \quad |\tilde{a}^{ij}|_{m-2,\alpha} \leq c.$$

Furthermore, we have

$$(1.9) \quad |\tilde{H}|_{m,\alpha} \leq c,$$

and

$$(1.10) \quad Au_0 + H(x, u_0, Du_0) = \tilde{A}u_0 + \tilde{H}(x, u_0, Du_0),$$

Hence, there exists a constant  $\gamma > 0$  such that the operator

$$(1.11) \quad u \rightarrow \tilde{A}u + H(x, u, Du) + \gamma u$$

is coercive, i.e., there exists a constant  $m > 0$  such that for any functions  $u_i \in H^{1,2}(\Omega)$ ,  $i = 1, 2$ , with equal boundary values there holds

$$(1.12) \quad \langle \tilde{A}u_2 + \tilde{H}(x, u_2, Du_2) + \gamma u_2 - \tilde{A}u_1 - \tilde{H}(x, u_1, Du_1) - \gamma u_1, u_2 - u_1 \rangle \\ \geq m \|u_2 - u_1\|_{1,2}^2,$$

and we conclude that the Dirichlet problem

$$(1.13) \quad \begin{aligned} \tilde{A}u + \tilde{H}(x, u, Du) + \gamma(u - u_0) &= 0, \\ u|_{\partial\Omega} &= \varphi, \end{aligned}$$

has the unique solution  $u = u_0$ .

(ii) Since  $\partial\Omega \in C^{m,\alpha}$ , we may extend  $u_0$  in a neighbourhood of  $\bar{\Omega}$  such that the extension is as smooth as  $u_0$  is up to the maximal regularity  $C^{m,\alpha}$ . This will be important when we improve the regularity of  $u_0$  successively. The improved regularity will then be reflected in the boundedness of the corresponding norms of a mollification of  $u_0$ , cf. the implication (1.17).

Let  $v_\epsilon$  be a mollification of  $u_0$ , then we shall show in Lemma 1.2 below that the Dirichlet problem

$$(1.14) \quad \begin{aligned} \tilde{A}u_\epsilon + \tilde{H}(x, u_\epsilon, Du_\epsilon) + \gamma(u_\epsilon - v_\epsilon) &= 0, \\ u_\epsilon|_{\partial\Omega} &= \varphi, \end{aligned}$$

has a unique solution  $u_\epsilon \in C^{m,\alpha}(\bar{\Omega})$  such that

$$(1.15) \quad |u_\epsilon|_{m,\alpha,\bar{\Omega}} \leq \text{const},$$

where the constant depends on

$$(1.16) \quad |v_\epsilon|_{m-2,\alpha,\bar{\Omega}} + |v_\epsilon|_{1,\Omega}$$

but not explicitly on  $\epsilon$ .

Thus, we conclude

$$(1.17) \quad u_0 \in C^{m-2,\alpha}(\bar{\Omega}) \cap C^{0,1}(\bar{\Omega}) \implies |u_\epsilon|_{m,\alpha,\bar{\Omega}} \leq \text{const}$$

uniformly in  $\epsilon$ , and the  $u_\epsilon$  converge in  $C^2(\bar{\Omega})$  to a solution  $u \in C^{m,\alpha}(\bar{\Omega})$  of (1.13), and we deduce further that

$$(1.18) \quad u = u_0,$$

i.e.,

$$(1.19) \quad u_0 \in C^{m-2,\alpha}(\bar{\Omega}) \cap C^{0,1}(\bar{\Omega}) \implies u_0 \in C^{m,\alpha}(\bar{\Omega}).$$

By assumption  $u_0$  is Lipschitz and therefore  $u_0 \in C^{2,\alpha}(\bar{\Omega})$ . A simple induction argument then yields the final result.  $\square$

**1.2. Lemma.** *Under the assumptions of the preceding theorem the Dirichlet problem (1.14) has a solution  $u_\epsilon \in C^{m,\alpha}(\bar{\Omega})$  satisfying the estimate (1.15).*

*Proof.* For simplicity we drop the subscript  $\epsilon$  in  $u_\epsilon$  resp.  $v_\epsilon$ . Linearizing the operator in (1.14), the equation can be written in the form

$$(1.20) \quad -\tilde{a}^{ij}u_{ij} + a(x, u, Du) + \gamma(u - v) = 0,$$

where

$$(1.21) \quad \tilde{a}^{ij} = \tilde{a}^{ij}(x, u, Du)$$

is uniformly elliptic,  $a \in C^{m-2,\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$

$$(1.22) \quad |a(x, t, \xi)| \leq c(1 + |\xi|),$$

and

$$(1.23) \quad \frac{\partial a}{\partial t} + \gamma > 0,$$

i.e., the maximum principle can be applied to yield uniqueness of the linearized equation.

(i) Now, let  $u \in C^{m,\alpha}(\bar{\Omega})$  be a solution (1.20) with boundary values  $\varphi$ , then we first deduce a priori estimates in the  $C^0$ -norm because of the maximum principle, and then a priori estimates in the  $C^1$ -norm,

$$(1.24) \quad |Du| \leq c(|v|_1),$$

cf. [4, Theorem 15.2].

(ii) Since  $u$  is also a solution of (1.14) we obtain  $L^2$ -estimates

$$(1.25) \quad \|u\|_{2,2} \leq c(|v|_1),$$

cf. PDE II, Kap. 1.6.

(iii) The De Giorgi-Nash estimates then yield a priori estimates in  $C^{1,\lambda}(\bar{\Omega})$  for some  $0 < \lambda < 1$ . For simplicity we shall write  $\alpha$  for  $\lambda$  having in mind that

this  $\alpha$  might not be the original  $\alpha$ . However, once we have proved uniform  $C^2$ -estimates it will be obvious that we can switch back to the original meaning of  $\alpha$ . name they neighbour

(iv) Applying then the Schauder estimates to the solution of (1.20), we infer

$$(1.26) \quad |u|_{2,\beta,\bar{\Omega}} \leq c(|u|_{1,\alpha}, |v|_{0,\beta}, |v|_1)$$

for some

$$(1.27) \quad 0 < \beta \leq \alpha.$$

Applying the Schauder estimates again, but this time using the fact that  $u \in C^2(\bar{\Omega})$ , we deduce

$$(1.28) \quad |u|_{2,\alpha,\bar{\Omega}} \leq c(|u|_{1,\alpha}, |v|_{0,\alpha}, |v|_1).$$

Using induction we then conclude

$$(1.29) \quad |u|_{m,\alpha,\bar{\Omega}} \leq c(|u|_{1,\alpha}, |v_{m-2,\alpha}|, |v|_1).$$

(v) The preceding a priori estimates are also valid, if, instead of the Dirichlet problem (1.14), we consider solutions of the problems

$$(1.30) \quad \begin{aligned} \tilde{A}u + \sigma \tilde{H}(x, u, Du) + (1 - \sigma) \frac{\partial \tilde{a}^i(x, u, Du)}{\partial x^i} + \gamma(u - \sigma v) &= 0, \\ u|_{\partial\Omega} &= \sigma\varphi, \end{aligned}$$

for fixed but arbitrary  $0 < \sigma < 1$ .

Choosing  $\gamma > 0$  large enough, the new operator is also coercive independently of  $\sigma$ , because of (1.6), and the preceding a priori estimates are also independent of  $\sigma$ .

Then we conclude from [3, Theorem 3.1.2] that (1.14) has a solution  $u_\epsilon \in C^{2,\alpha}(\bar{\Omega})$ , and we deduce further that  $u_\epsilon$  is also of class  $C^{m,\alpha}$ , because of the Schauder theory and the uniqueness, satisfying the estimate (1.29).  $\square$

Using the  $L^p$ -estimates instead of the Schauder estimates we can prove:

**1.3. Theorem.** *Let  $\Omega \Subset \mathbb{R}^n$  be open, connected with  $\partial\Omega \in C^2$ , and let  $\varphi \in H^{2,p}(\Omega)$ ,  $n < p < \infty$ . Then any weak solution  $u \in C^{0,1}(\bar{\Omega})$  of (1.1) is of class  $H^{2,p}(\Omega)$ .*

*Proof.* Exercise; note that [3, Theorem 3.1.2] can also be applied in the present situation.  $\square$

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