A PARTITION FUNCTION FOR SCHWARZSCHILD-ADS AND KERR-ADS BLACK HOLES AND FOR QUANTIZED GLOBALLY HYPERBOLIC SPACETIMES WITH A NEGATIVE COSMOLOGICAL CONSTANT

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Abstract. We apply quantum statistics to our quantized versions of Schwarzschild-AdS and Kerr-AdS black holes and also to the quantized globally hyperbolic spacetimes having an asymptotically Euclidean Cauchy hypersurface by first proving, for the temporal Hamiltonian H_0, that $e^{-\beta H_0}$, $\beta > 0$, is of trace class and then, that this result is also valid for the spatial Hamiltonian H_1, which has the same eigenvalues but with larger multiplicities. Since the lowest eigenvalue is strictly positive the extension of $e^{-\beta H_1}$ to the corresponding symmetric Fock space is also of trace class and we are thus able to define a partition function Z, the operator density ρ, the entropy S, and the average energy E. We prove that S and E tend to infinity if the cosmological constant Λ tends to 0 and vanish if $|\Lambda|$ tends to infinity. We also conjecture that E is the source of the dark matter and that the dark energy density is a multiple of the eigenvalue of ρ with respect to the vacuum vector which is Z^{-1}.

Contents

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1. Introduction

In three recent papers we applied our model of quantum gravity to a globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface [5] and to a Schwarzschild-AdS [4] resp. Kerr-AdS black hole [6]. In all three cases the quantized model had the same structure, namely, it consisted of special solutions to a wave equation

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{1.1}$$

in a quantum spacetime

$$N = \mathbb{R}_+ \times \mathcal{S}_0, \tag{1.2}$$

where $\mathcal{S}_0$ is a n-dimensional, $n \geq 3$, Cauchy hypersurface of the original spacetime. The Laplacian and the scalar curvature correspond to the metric σ_{ij} in $\mathcal{S}_0$, cf. [3, Theorem 6.9], where we derived this wave equation after a canonical quantization process. The special solutions are a sequence of smooth functions which are a product of temporal and spatial eigenfunctions, where the spatial eigenfunctions are eigendistributions.

In case of the globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface the solutions to the wave equation can be expressed in the form

$$u_{ij} = w_i v_{ij}, \qquad i \in \mathbb{N},\ 1 \leq j \leq m \leq \infty, \tag{1.3}$$

where the w_i are the eigenfunctions of a temporal Hamilton operator H_0

$$H_0 w_i = \lambda_i w_i \tag{1.4}$$

and the λ_i have multiplicity one such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{1.5}$$

and for each fixed i the at most countably many v_{ij} generate an eigenspace

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{1.6}$$

of a spatial Hamiltonian H_1, i.e.,

$$H_1 v_{ij} = \lambda_i v_{ij}. \tag{1.7}$$

We have

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{1.8}$$

In the two remaining cases of the black holes the special solutions are labelled by three indices

$$u_{ijk} = w_i \zeta_{ijk} \varphi_j, \tag{1.9}$$

where the w_i are the same temporal eigenfunctions as before, the φ_j are the eigenfunctions of an elliptic operator A on a smooth compact Riemannian manifold (M, σ_{ij}), where topologically

$$M \simeq \mathbb{S}^{n-1}, \tag{1.10}$$

at least the physically interesting cases, i.e.,

$$A\varphi_j = \tilde{\mu}_j \varphi_j, \tag{1.11}$$

$$\tilde{\mu}_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots \tag{1.12}$$

The φ_j form a mutually orthogonal basis of $L^2(M)$. For a Schwarzschild-AdS black hole we know that

$$\tilde{\mu}_0 \le 0, \tag{1.13}$$

and for a Kerr-AdS black hole this condition can be assured by assuming that the rotational parameter a is small enough such that the scalar curvature of σ_{ij} is positive. Let us emphasize that we considered in [6] Kerr-AdS black holes of odd dimensions

$$\dim N = 2m + 1, \qquad m \ge 2, \tag{1.14}$$

and assumed that all rotational parameters a_i are equal

$$a_i = a \ne 0 \qquad \forall\, 1 \le i \le m. \tag{1.15}$$

The ζ_{ijk} are eigendistributions in $\mathscr{S}'(\mathbb{R})$ satisfying

$$-\zeta''_{ijk} = \omega_{ij}^2 \zeta_{ijk}, \qquad k = 1, 2, \tag{1.16}$$

where

$$\zeta_{ij1}(\tau) = \frac{1}{\sqrt{2\pi}} e^{i\omega_{ij}\tau} \tag{1.17}$$

and

$$\zeta_{ij2}(\tau) = \frac{1}{\sqrt{2\pi}} e^{-i\omega_{ij}\tau}, \tag{1.18}$$

where

$$\omega_{ij} \ge 0 \tag{1.19}$$

is defined by the relation

$$\lambda_i = \tilde{\mu}_j + \omega_{ij}^2, \tag{1.20}$$

i.e., for any $i \in \mathbb{N}$ we look for all j satisfying

$$\tilde{\mu}_j \le \lambda_i \tag{1.21}$$

and then choose $\omega_{ij} \ge 0$ satisfying (1.20). Let N_i be the set of integers such that the $\tilde{\mu}_j$ satisfy (1.21), then the smooth functions

$$\zeta_{ijk}\varphi_j \tag{1.22}$$

are mutually orthogonal in $L^2(M, \sigma_{ij})$—for fixed i and k; note that we only have two different eigendistributions ζ_{ijk}, if

$$\omega_{ij} > 0, \tag{1.23}$$

otherwise we have only one. The eigendistributions ζ_{ij1} and ζ_{ij2} are also considered to be „orthogonal“ since their Fourier transforms

$$\hat{\zeta}_{ijk} = \delta_{\pm\omega_{ij}} \tag{1.24}$$

have disjoint supports.

Finally, the smooth functions u_{ijk} in (1.9) can be considered to be mutually orthogonal since u_{ijk} and $u_{i'j'k'}$ are mutually orthogonal in

$$L^2(\mathbb{R}_+, d\mu) \otimes L^2(M), \tag{1.25}$$

where

$$d\mu = t^{2-\frac{4}{n}} dt, \tag{1.26}$$

if

$$\omega_{ij} = \omega_{i'j'} \quad \wedge \quad k = k' \tag{1.27}$$

and as tempered distributions otherwise.

The u_{ijk} are eigendistributions for both the temporal Hamiltonian H_0 as well as for the spatial Hamiltonian H_1 with the same eigenvalues λ_i, where now the eigenvalues have finite multiplicities different from 1 by definition of the eigendistributions and the u_{ijk} also solve the wave equation, since the wave equation can be expressed as

$$\varphi_0(H_0 u - H_1 u) = 0, \tag{1.28}$$

where $u = u(t,x)$ is a smooth function

$$x \in \mathcal{S}_0 = \mathbb{R} \times M \tag{1.29}$$

and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{1.30}$$

In Section 3 we shall prove that we can define an abstract Hilbert space $\mathcal{H}$, where the eigendistributions u_{ijk} resp. u_{ij} in (1.3) form a basis of mutually orthogonal unit vectors such that the Hamiltonian H_1 can be defined on the dense subspace, which is the algebraic span of the basis vectors, as an essentially self-adjoint operator. Let $\tilde{H}_1$ be its unique self-adjoint extension, namely its closure, then we shall prove that for any $\beta > 0$

$$e^{-\beta \tilde{H}_1} \tag{1.31}$$

is of trace class in $\mathcal{H}$. In addition $\tilde{H}_1$ satisfies

$$\tilde{H}_1 \geq \lambda_0 I, \qquad \lambda_0 > 0. \tag{1.32}$$

Let

$$H \equiv d\Gamma(\tilde{H}_1) \tag{1.33}$$

be the canonical extension of $\tilde{H}_1$ to the symmetric Fock space

$$\mathscr{F} = \mathscr{F}_+(\mathcal{H}), \tag{1.34}$$

then

$$e^{-\beta H} \tag{1.35}$$

is of trace class in $\mathscr{F}$ because of (1.31) and (1.32), cf. [1, Prop. 5.2.27]. Hence we can define the partition function

$$Z = \operatorname{tr}(e^{-\beta H}), \tag{1.36}$$

the density operator

$$\rho = Z^{-1}e^{-\beta H} \tag{1.37}$$

and the von Neumann entropy

$$S = -\operatorname{tr}(\rho \log \rho) = \log Z + \beta E, \tag{1.38}$$

where E is the average energy and $\beta > 0$ the inverse temperature

$$\beta = T^{-1}. \tag{1.39}$$

Here is a summary of the results derived in Section 3:

1.1. **Theorem.** (i) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{1.40}$$

we have

$$\lim_{\Lambda \to 0} E = \infty \tag{1.41}$$

as well as

$$\lim_{\Lambda \to 0} S = \infty, \tag{1.42}$$

where the limites are uniform in β.

(ii) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{1.43}$$

we have

$$\lim_{|\Lambda| \to \infty} E = 0 \tag{1.44}$$

as well as

$$\lim_{|\Lambda| \to 0} S = 0, \tag{1.45}$$

where the limites are uniform in β.

The behaviour of Z with respect to Λ is described in the theorem:

1.2. **Theorem.** *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{1.46}$$

we have

$$\lim_{\Lambda \to 0} Z = \infty \tag{1.47}$$

and for any

$$\beta_0 \le \beta \tag{1.48}$$

the relation

$$\lim_{|\Lambda|\to\infty} Z = 1 \tag{1.49}$$

is valid. The convergence in both limites is uniform in β.

1.3. **Remark.** The first part of Theorem 1.1 reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density we consider the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho\eta = Z^{-1}\eta, \tag{1.50}$$

i.e., the dark energy density should be proportional to Z^{-1}.

In Section 4 we also applied quantum statistics to the quantized version of a Friedmann universe and proved:

1.4. **Theorem.** *The results in the theorems and the conjectures in the remark above are also valid, if the quantized spacetime $N = N^{n+1}$, $n \ge 3$, is a Friedmann universe without matter but with a negative cosmological constant Λ and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian H_1 all have multiplicity one.*

1.5. **Remark.** Let us also mention that we use Planck units in this paper, i.e.,

$$c = G = \hbar = K_B = 1. \tag{1.51}$$

2. Trace class estimates

We want to apply quantum statistics to the system described by the wave equation and its special solutions. Therefore, we need a separable Hilbert space $\mathcal{H}$ and a Hamiltonian H such that

$$H \ge \lambda_0 > 0 \tag{2.1}$$

and

$$e^{-\beta H}, \qquad \beta > 0, \tag{2.2}$$

is of trace class in $\mathcal{H}$.

A natural candidate is the temporal Hamiltonian H_0 mentioned in the introduction which corresponds to a generalized eigenvalue problem that has been considered in [5, Section 4]: Define the bilinear forms

$$B(w,\tilde w) = \int_{\mathbb{R}_+^*} \{\frac{1}{32}\frac{n^2}{n-1}\bar w'\tilde w' + n|\Lambda|t^2\bar w\tilde w\} \tag{2.3}$$

and

$$K(w,\tilde w)=\int_{\mathbb{R}_+^*} t^{2-\frac{4}{n}}\bar w\tilde w \tag{2.4}$$

in the Sobolev space $\mathcal{H}_1$ which is the completion of

$$C_c^\infty(\mathbb{R}_+^*,\mathbb{C}) \tag{2.5}$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$B(w,\varphi)=\lambda K(w,\varphi)\quad \forall\,\varphi\in\mathcal{H}_1. \tag{2.6}$$

The following theorem was proved in the former paper.

2.1. **Theorem.** *The eigenvalue problem (2.6) has countably many solutions (w_i,λ_i) such that*

$$0<\lambda_0<\lambda_1<\lambda_2<\cdots, \tag{2.7}$$

$$\lim\lambda_i=\infty, \tag{2.8}$$

and

$$K(w_i,w_j)=\delta_{ij}. \tag{2.9}$$

The w_i are complete in $\mathcal{H}_1$ as well as in $L^2(\mathbb{R}_+^*)$.

The eigenfunctions w_i solve the ordinary differential equation

$$Aw_i=-\frac{1}{32}\frac{n^2}{n-1}\ddot w_i+n|\varLambda|t^2w_i=\lambda_i t^{2-\frac{4}{n}}w_i. \tag{2.10}$$

Let $\varphi_0=\varphi_0(t)$ be defined by

$$\varphi_0(t)=t^{2-\frac{4}{n}}, \tag{2.11}$$

then the operator

$$\tilde A=\varphi_0^{-1}A \tag{2.12}$$

is symmetric in

$$\mathcal{H}=L^2(\mathbb{R}_+,d\mu),\qquad d\mu=\varphi_0 dt, \tag{2.13}$$

and the w_i are eigenfunctions of $\tilde A$

$$\tilde Aw_i=\lambda_i w_i. \tag{2.14}$$

The equation (2.10) is equivalent to

$$\varphi_0\tilde Aw_i=\lambda_i\varphi_0 w_i \tag{2.15}$$

and $\tilde A$ with domain

$$D(\tilde A)=\langle w_i: i\in\mathbb{N}\rangle\subset\mathcal{H} \tag{2.16}$$

is essentially self-adjoint as will be proved later, Lemma 3.1 on page 19, in a more general setting. We denote its unique self-adjoint extension by H_0.

We shall now prove that

$$e^{-\beta H_0}, \qquad \beta > 0, \tag{2.17}$$

is of trace class in $\mathcal{H}$.

First, we need two lemmata:

2.2. **Lemma.** *The embedding*

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 = L^2(\mathbb{R}_+, d\tilde{\mu}), \tag{2.18}$$

where

$$d\tilde{\mu} = (1+t^2)^{-2} dt, \tag{2.19}$$

is Hilbert-Schmidt.

Proof. Maurin was the first to prove that the embedding

$$H^{m,2}(\Omega) \hookrightarrow L^2(\Omega), \tag{2.20}$$

where

$$\Omega \subset \mathbb{R}^n \tag{2.21}$$

is a bounded domain, is Hilbert-Schmidt provided

$$m > \frac{n}{2}, \tag{2.22}$$

cf. [8, Theorem 1, p. 336]. We adapt his proof to the present situation.

Let $w \in \mathcal{H}_1$, then, assuming w is real valued,

$$\begin{aligned} |w(t)|^2 = 2\int_0^t \dot{w}w \le 2\int_o^\infty |\dot{w}|^2 + \frac{1}{2}\int_0^\infty |w|^2 \\ \le c\|w\|_1^2 \end{aligned} \tag{2.23}$$

for all $t > 0$, where $\|\cdot\|_1$ is the norm in $\mathcal{H}_1$. To derive the last inequality in (2.23) we used

$$\int_0^1 |w|^2 \le 2\int_0^1 |\dot{w}|^2 + \frac{1}{2}\int_0^1 |w|^2 \tag{2.24}$$

which is easily be deduced from the equation in (2.23). The estimate

$$|w(t)| \le c_0\|w\|_1 \qquad \forall\, t > 0 \tag{2.25}$$

is of course also valid for complex valued functions from which infer that, for any $t > 0$, the linear form

$$w \to w(t), \qquad w \in \mathcal{H}_1, \tag{2.26}$$

is continuous, hence it can be expressed as

$$w(t) = \langle \varphi_t, w \rangle, \tag{2.27}$$

where

$$\varphi_t \in \mathcal{H}_1 \tag{2.28}$$

and

$$\|\varphi_t\|_1 \le c_0. \tag{2.29}$$

Now, let

$$e_i \in \mathcal{H}_1 \tag{2.30}$$

be an ONB, then

$$\sum_{i=0}^{\infty} |e_i(t)|^2 = \sum_{i=0}^{\infty} |\langle \varphi_t, e_i\rangle|^2 = \|\varphi_t\|_1^2 \le c_0^2. \tag{2.31}$$

Integrating this inequality over $\mathbb{R}_+$ with respect to $d\tilde{\mu}$ we infer

$$\sum_{i=0}^{\infty} \int_0^{\infty} |e_i(t)|^2 d\tilde{\mu} \le c_0^2 \tag{2.32}$$

completing the proof of the lemma. □

2.3. **Lemma.** *Let w_i be the eigenfunctions of H_0, then there exist positive constants c and p such that*

$$\|w_i\|_1 \le c|\lambda_i|^p \|w_i\|_0 \qquad \forall\, i \in \mathbb{N}, \tag{2.33}$$

where $\|\cdot\|_0$ is the norm in $\mathcal{H}_0$.

Proof. We have

$$\|w_i\|_1^2 = \lambda_i \int_0^{\infty} t^{2-\frac{4}{n}} |w_i|^2. \tag{2.34}$$

Let $\epsilon > 0$ be arbitrary and define

$$q = \frac{2}{2-\frac{2}{n}} = \frac{n}{n-1} \tag{2.35}$$

and the conjugate exponent

$$q' = \frac{q}{q-1} = n, \tag{2.36}$$

then the integral on the right-hand side of (2.34) can be estimated from above by

$$\frac{1}{q}\epsilon^q \int_0^{\infty} \{t^{2-\frac{4}{n}}(1+t)^{\frac{2}{n}}\}^q |w_i|^2 + \frac{1}{q'}\epsilon^{-q'} \cdot \int_0^{\infty} (1+t)^{-\frac{2}{n}q'} |w_i|^2 \tag{2.37}$$

We note that by definition

$$\{t^{2-\frac{4}{n}}(1+t)^{\frac{2}{n}}\}^q \le (1+t)^2 \tag{2.38}$$

and that in view of (2.24)

$$\int_0^{\infty} (1+t)^2 |w_i|^2 \le c\|w_i\|_1^2. \tag{2.39}$$

Combining (2.34), (2.37) and (2.39) we then infer

$$\|w_i\|_1^2 \le c\frac{1}{q}\epsilon^q \lambda_i \|w_i\|_1^2 + c\frac{1}{q'}\epsilon^{-q'}\lambda_i\|w_i\|_0^2 \tag{2.40}$$

and deduce further, by choosing ϵ appropriately, that the result is valid with a different constant c. □

We are now ready to prove:

2.4. **Theorem.** *Let $\beta > 0$, then the operator*

$$e^{-\beta H_0} \tag{2.41}$$

is of trace class in $\mathcal{H}$, i.e.,

$$\operatorname{tr}(e^{-\beta H_0}) = \sum_{i=0}^{\infty} e^{-\beta\lambda_i} = c(\beta) < \infty. \tag{2.42}$$

Proof. In view of Lemma 2.2 the embedding

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \tag{2.43}$$

is Hilbert-Schmidt. Let

$$w_i \in \mathcal{H} \tag{2.44}$$

be an ONB of eigenfunctions, then

$$\begin{aligned} e^{-\beta\lambda_i} &= e^{-\beta\lambda_i}\|w_i\|^2 = e^{-\beta\lambda_i}\lambda_i^{-1}\|w_i\|_1^2 \\ &\le e^{-\beta\lambda_i}\lambda_i^{-1}c|\lambda_i|^{2p}\|w_i\|_0^2, \end{aligned} \tag{2.45}$$

in view of (2.33), but

$$\|w_i\|_0^2 = \|w_i\|_1^2\, \|\tilde w_i\|_0^2 = \lambda_i\|\tilde w_i\|_0^2, \tag{2.46}$$

where

$$\tilde w_i = w_i\|w_i\|_1^{-1} \tag{2.47}$$

is an ONB in $\mathcal{H}_1$, yielding

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} \le \tilde c \sum_{i=0}^{\infty}\|\tilde w_i\|_0^2 < \infty, \tag{2.48}$$

since j is Hilbert-Schmidt. □

There is also a spatial Hamiltonian H_1, which, in the case of the black holes considered, is a direct product of a classical harmonic oscillator in $\mathbb{R}$ and an elliptic operator A on a compact, smooth Riemannian manifold $M = M^{n-1}$, $n \ge 3$, with metric σ_{ij}, where A has the form

$$A\varphi = -(n-1)\Delta\varphi - \frac{n}{2}R\varphi \tag{2.49}$$

and the Laplacian is the Laplacian in M and R the scalar curvature of the metric. A is self-adjoint with domain

$$D(A) = H^{2,2}(M) \subset L^2(M), \tag{2.50}$$

where

$$H^{m,2}(M), \qquad m \in M, \tag{2.51}$$

are the usual Sobolev spaces with norm

$$\|\varphi\|_{m,2}^2 = \sum_{|\alpha|\le m} \int_M |D^\alpha \varphi|^2. \tag{2.52}$$

A has a pure point spectrum with countable many eigenvalues $\tilde\mu_j$ with finite multiplicities and mutually orthogonal eigenfunctions φ_j such that

$$\tilde\mu_0 < \tilde\mu_1 \le \cdots \tag{2.53}$$

and

$$\lim_j \tilde\mu_j = \infty. \tag{2.54}$$

We want to prove that

$$e^{-\beta A}, \qquad \beta > 0, \tag{2.55}$$

is of trace class in $L^2(M)$.

The proof of this result will follow the previous arguments very closely.

2.5. **Lemma.** *Let $m > \frac{n-1}{2}$, then the embedding*

$$j : H^{m,2}(M) \hookrightarrow L^2(M) \tag{2.56}$$

is Hilbert-Schmidt.

Proof. This result is due to Maurin and its proof is identical with the proof of Lemma 2.2 apart from some obvious modifications. □

We also need the lemma:

2.6. **Lemma.** *Let $m \in \mathbb{N}$, then there exists $c_m > 0$ such that*

$$\|\varphi\|_{2m,2}^2 \le c_m(\|A^m\varphi\|^2 + \|\varphi\|^2) \tag{2.57}$$

and the bilinear form

$$\langle A^m\varphi, A^m\psi\rangle_0 + \langle\varphi,\psi\rangle_0 \tag{2.58}$$

defines an equivalent scalar product in $H^{2m,2}(M)$, where

$$\langle\varphi,\psi\rangle_0 = \int_M \bar\varphi\psi. \tag{2.59}$$

Proof. Let

$$f \in H^{m,2}(M) \tag{2.60}$$

and

$$\varphi \in H^{2,2}(M) \tag{2.61}$$

a solution of

$$A\varphi = f, \tag{2.62}$$

then it is well-known that

$$\varphi \in H^{m+2,2}(M) \tag{2.63}$$

and there exists $\tilde{c}_m$ such that

$$\|\varphi\|_{m+2,2} \le \tilde{c}_m(\|f\|_{m,2} + \|\varphi\|_0). \tag{2.64}$$

The constant $\tilde{c}_m$ also depends on A and M. Using this estimate the relation (2.57) can be easily proved by induction. □

Now, we are ready to prove:

2.7. **Theorem.** *Let A be the self-adjoint operator in (2.49), then*

$$e^{-\beta A} \tag{2.65}$$

is of trace class in $L^2(M)$ for any $\beta > 0$.

Proof. Let $m > \frac{n-1}{4}$ and equip $H^{2m,2}(M)$ with the scalar product (2.58) such that

$$\|\varphi\|^2_{2m,2} = \langle A^m\varphi, A^m\varphi\rangle_0 + \langle\varphi,\varphi\rangle_0, \tag{2.66}$$

then any eigenfunctions φ_i, φ_j of A satisfy

$$\langle\varphi_i,\varphi_j\rangle_0 = 0 \implies \langle\varphi_i,\varphi_j\rangle_{2m,2} = 0. \tag{2.67}$$

Let (φ_j) be an ONB of eigenfunctions of A in $L^2(M)$ and define

$$\tilde{\varphi}_j = \varphi_i\|\varphi_j\|^{-1}_{2m,2}, \tag{2.68}$$

then the $\tilde{\varphi}_j$ form an ONB in $H^{2m,2}(M)$ and we conclude

$$\begin{aligned} e^{-\beta\tilde{\mu}_j} &= e^{-\beta\tilde{\mu}_j}\|\varphi_j\|^2_0 = e^{-\beta\tilde{\mu}_j}\|\varphi_j\|^2_{2m,2}\|\tilde{\varphi}_j\|^2_0 \\ &= e^{-\beta\tilde{\mu}_j}(1+|\tilde{\mu}_j|^{2m})\|\tilde{\varphi}_j\|^2_0 \le c_\beta\|\tilde{\varphi}_j\|^2_0 \end{aligned} \tag{2.69}$$

yielding

$$\sum_{j=0}^\infty e^{-\beta\tilde{\mu}_j} \le c_\beta\sum_{j=0}^\infty\|\tilde{\varphi}_j\|^2_0 < \infty \tag{2.70}$$

in view of Lemma 2.5. □

With the help of the preceding lemma we can now prove that, in case of the black holes, the spatial Hamiltonian H_1 has the property that

$$e^{-\beta H_1} \tag{2.71}$$

is of trace class for all $\beta > 0$, where we still have to define an appropriate Hilbert space.

We have

$$H_1 v = -\ddot{v} - Av, \tag{2.72}$$

where we write v as product

$$v(\tau,x) = \zeta(\tau)\varphi(x) \tag{2.73}$$

with

$$\tau \in \mathbb{R} \quad \wedge \quad x \in M = M^{n-1}, \tag{2.74}$$

where A is the differential operator in (2.49). Let φ_j be the eigenfunctions of A with eigenvalues $\tilde{\mu}_j$, then, for any eigenvalue λ_i we define

$$N_i = \{j \in \mathbb{N} : \tilde{\mu}_j \le \lambda_i\} \tag{2.75}$$

and $\omega_{ij} \ge 0$ such that

$$\omega_{ij}^2 + \tilde{\mu}_j = \lambda_i. \tag{2.76}$$

Note that

$$0 \in N_i \qquad \forall\, i \in \mathbb{N}, \tag{2.77}$$

since

$$\tilde{\mu}_0 \le 0. \tag{2.78}$$

Let

$$\zeta_{ijk}, \qquad k = 1, 2, \tag{2.79}$$

be the tempered distributions

$$\zeta_{ij1} = \frac{1}{\sqrt{2\pi}} e^{i\omega_{ij}\tau} \tag{2.80}$$

and

$$\zeta_{ij2} = \frac{1}{\sqrt{2\pi}} e^{-i\omega_{ij}\tau}, \tag{2.81}$$

where this distinction only occurs for

$$\omega_{ij} > 0. \tag{2.82}$$

Let $\hat{\zeta}_{ijk}$ be the Fourier transform of ζ_{ijk}, then

$$\hat{\zeta}_{ij1} = \delta_{\omega_{ij}} \quad \wedge \quad \hat{\zeta}_{ij2} = \delta_{-\omega_{ij}} \tag{2.83}$$

such that these tempered distributions are considered to be mutually „orthogonal“. The smooth functions

$$u_{ijk} = \zeta_{ijk}\varphi_j \tag{2.84}$$

satisfy

$$H_1 u_{ijk} = \lambda_i u_{ijk}. \tag{2.85}$$

Label the eigenvalues of H_1 including their multiplicities and denote them by $\tilde{\lambda}_i$. Then

$$\sum_{i=0}^{\infty} e^{-\beta\tilde{\lambda}_i} \le 2\sum_{i=0}^{\infty} e^{-\beta\lambda_i} n(\lambda_i) = 2\sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\lambda_i} e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i), \tag{2.86}$$

where

$$n(\lambda_i) = \#N_i. \tag{2.87}$$

2.8. **Lemma.** *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta_0 \le \beta \tag{2.88}$$

and for any $i \in \mathbb{N}$, the estimate

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta) \le c(\beta_0), \tag{2.89}$$

where $c(\beta_0)$ also depends on A but is independent of $i \in \mathbb{N}$.

Proof. Each N_i is the disjoint union

$$N_i' \,\dot\cup\, N_i'', \tag{2.90}$$

where

$$N_i' = \{j \in \mathbb{N}_i : \tilde\mu_j \le 0\} \tag{2.91}$$

and N_i'' is its complement. The operator A has only finitely many eigenvalues which are non-positive, i.e.,

$$\#N_i' \le n_0 \qquad \forall\, i \in \mathbb{N}, \tag{2.92}$$

hence

$$\begin{aligned} e^{-\frac{\beta}{2}\lambda_i} n_i(\lambda_i) &\le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\lambda_i} \le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\tilde\mu_j} \\ &\le n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde\mu_j} \\ &= n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde\mu_j}(1+|\tilde\mu|_j^{2m})\,\|\tilde\varphi_j\|_0^2 \\ &\le n_0 + c(\beta)\sum_{j=0}^{\infty}\|\tilde\varphi_j\|_0^2 < \infty, \end{aligned} \tag{2.93}$$

where we used (2.69). The estimate for the Hilbert-Schmidt norm of the embedding

$$j : H^{m,2}(M) \to L^2(M) \tag{2.94}$$

depends on A, since we used the equivalent norm given in (2.66), and

$$c(\beta) = \sup_{t>0} e^{-\frac{\beta}{2}t}(1+t^{2m}). \tag{2.95}$$

□

2.9. **Corollary.** *The sum on the left-hand side of (2.86) is finite and hence*

$$e^{-\beta H_1}, \qquad \beta > 0, \tag{2.96}$$

is of trace class provided we can define a Hilbert space $\mathcal{H}$ such that such that the eigendistributions form complete set of eigenvectors in $\mathcal{H}$ and H_1 is essentially self-adjoint in $\mathcal{H}$.

Proof. The first claim follows immediately by combining (2.93) and Theorem 2.4. In Lemma 3.1 on page 19 we shall define the Hilbert space $\mathcal{H}$ and shall prove that H_1 is essentially self-adjoint in $\mathcal{H}$ and that the eigendistributions form a complete set of eigenvectors in $\mathcal{H}$. ☐

The elliptic operator A also depend on Λ, since the underlying Riemannian metric depends on it. The estimates in the preceding lemma remain valid provided $|\Lambda|$ remains in a compact subset of $\mathbb{R}$, since the operator A is then still uniformly elliptic and smooth. However, when

$$|\Lambda| \to \infty, \tag{2.97}$$

then the relation (2.57) is no longer valid and a more sophisticated analysis is necessary to achieve a corresponding estimate. Let us treat the cases Schwarzschild-AdS and Kerr-AdS black holes separately.

For a Schwarzschild-AdS black hole the operator A can be written in the form

$$A = r_0^{-2}\tilde{A}, \tag{2.98}$$

where r_0 is the black hole radius and

$$\tilde{A}\varphi = -(n-1)\tilde{\Delta}\varphi - \frac{n}{2}\tilde{R}\varphi. \tag{2.99}$$

Here, the Laplacian and the scalar curvature $\tilde{R}$ refer to the corresponding quantities of $\mathbb{S}^{n-1}$ with the standard metric, cf. [4, equ. (2.12) and (2.14)]. The eigenfunctions of A are the eigenfunctions of $\tilde{A}$. Let μ_j be the eigenvalues of $\tilde{A}$ and $\tilde{\mu}_j$ the eigenvalues of A, then

$$\tilde{\mu}_j = r_0^{-2}\mu_j. \tag{2.100}$$

From the definition of the black hole radius

$$m r_0^{-(n-2)} = 1 + \frac{2}{n(n-1)}|\Lambda| r_0^2 \tag{2.101}$$

it is evident that

$$\lim_{|\Lambda|\to\infty} r_0 = 0 \tag{2.102}$$

and also

$$\lim_{|\Lambda|\to\infty} |\Lambda| r_0^2 = \infty, \tag{2.103}$$

though the latter result is only needed when we shall treat the Kerr-AdS case.

We can now prove:

2.10. **Lemma.** *Let β_0 be arbitrary and $|\Lambda_0|$ so large that*

$$r_0 < 1 \qquad \forall\, |\Lambda| > |\Lambda_0|, \tag{2.104}$$

then for any $i \in \mathbb{N}$, any $\beta \ge \beta_0$ and any $|\Lambda| > |\Lambda_0|$

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta) \le c(\beta_0), \tag{2.105}$$

where $c(\beta_0)$ also depends on $\tilde{A}$ but is independent of $|\Lambda|$ and $i \in \mathbb{N}$.

Proof. We follow the proof of Lemma 2.8 but use $\tilde{A}$ instead of A to define an equivalent norm in $H^{m,2}(M)$,

$$M = \mathbb{S}^{n-1}. \tag{2.106}$$

Then, we infer, cf. (2.93),

$$\begin{aligned} e^{-\frac{\beta}{2}\lambda_i} n_i(\lambda_i) &\le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\lambda_i} \le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &\le n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &= n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j} (1+|\mu|_j^{2m}) \,\|\tilde{\varphi}_j\|_0^2 \\ &\le n_0 + c(\beta)\sum_{j=0}^{\infty} \|\tilde{\varphi}_j\|_0^2 < \infty. \end{aligned} \tag{2.107}$$

Here, we used

$$\tilde{\mu}_j = r_0^{-2}\mu_j > \mu_j > 0. \tag{2.108}$$

$\square$

Let us now look at Kerr-AdS black holes. In [6, equ. (2.50)] we described the metric σ_{ij} on $M = \mathbb{S}^{n-1}$

$$\begin{aligned} ds_M^2 = &\frac{r^2+a^2}{1-a^2l^2}\big(\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j\big) \\ &+ a^2\frac{(1+l^2r^2)(r^2+a^2)}{r^2(1-a^2l^2)^2}\mu_i^2\mu_j^2 d\varphi^i d\varphi^j. \end{aligned} \tag{2.109}$$

Here

$$n = 2m, \qquad m \ge 2, \tag{2.110}$$

and the coordinates μ_i, $1 \le i \le m$ are subject to the constraint

$$\sum_{i=1}^m \mu_i^2 = 1. \tag{2.111}$$

They are the latitudinal coordinates of $\mathbb{S}^{n-1}$ and the φ_i, $1 \le i \le m$ are the azimuthal coordinates. The metric

$$\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j \tag{2.112}$$

is the standard metric of $\mathbb{S}^{n-1}$. The constant r is the radius of the event horizon, $a \ne 0$ the rotational parameter and

$$l^2 = -\frac{1}{m(2m-1)}\Lambda. \tag{2.113}$$

The relation

$$a^2l^2 < 1 \tag{2.114}$$

is assumed. We also require that a is small enough such that the scalar curvature R of the metric σ_{ij} is positive. We can write the metric as a conformal metric

$$\sigma_{ij} = \frac{r^2 + a^2}{1 - a^2l^2}\tilde{\sigma}_{ij}. \tag{2.115}$$

We note that the Schwarzschild-AdS black hole is obtained by setting $a = 0$ and that

$$\lim_{a\to 0} r = r_0, \tag{2.116}$$

the Schwarzschild black hole radius.

In order to prove the analogue of Lemma 2.10 we assume that, when

$$|\Lambda| \to \infty, \tag{2.117}$$

a is supposed so small that

$$\lim_{|\Lambda|\to\infty} |\Lambda|a^2 = 0 \tag{2.118}$$

and

$$\lim_{|\Lambda|\to\infty} |\Lambda|r^2 = \infty, \tag{2.119}$$

and we emphasize that these assumptions are always satisfied if $a = 0$, cf. (2.103). If these are satisfied, then the operator A can be expressed in the form

$$A = \frac{1 - a^2l^2}{r^2 + a^2}\tilde{A}, \tag{2.120}$$

where $\tilde{A}$ converges uniformly in $C^\infty(M)$ to the operator $\tilde{A}$ in (2.99), i.e., for large $|\Lambda|$ $\tilde{A}$ is uniformly elliptic and smooth such that the number of non-positive eigenvalues $n_0(\tilde{A})$ is bounded from above by the n_0 of the limit operator

$$n_0 \ge \limsup_{|\Lambda|\to\infty} n_0(\tilde{A}), \tag{2.121}$$

since n_0 is upper semi-continuous as it is well-known.

2.11. **Lemma.** *Under the assumptions* (2.118) *and* (2.119) *the results of Lemma* 2.10 *are also valid for the Kerr-AdS black hole, i.e., there exists* $|\Lambda_0| > 0$ *such that for all*

$$|\Lambda| > |\Lambda_0| \tag{2.122}$$

and for any β *satisfying*

$$0 < \beta_0 \le \beta, \tag{2.123}$$

where β_0 is arbitrary,

$$e^{-\frac{\beta}{2}\lambda_i}n(\lambda_i) \le c(\beta_0) \tag{2.124}$$

uniformly in $i \in \mathbb{N}$, $|\Lambda|$ and β.

Proof. The proof is identical to the proof of Lemma 2.10 by using the fact that the special $H^{m,2}(M)$ norm

$$\langle \tilde{A}^m\varphi, \tilde{A}^m\varphi\rangle_0 + \langle\varphi,\varphi\rangle_0, \tag{2.125}$$

with different m than used to express the dimension of M, is uniformly equivalent to the standard $H^{m,2}(M)$ norm, hence the Hilbert-Schmidt norm of the embedding

$$j : H^{m,2}(M) \hookrightarrow L^2(M) \tag{2.126}$$

is uniformly bounded. We also relied on

$$\tilde{\mu}_j = \frac{1-a^2l^2}{r^2+a^2}\mu_j > \mu_j > 0 \tag{2.127}$$

for $j \in N_i''$. ☐

Finally, let us derive the last result in this section.

2.12. **Lemma.** *Let λ_i be the temporal eigenvalues depending on Λ and let $\bar{\lambda}_i$ be the corresponding eigenvalues for*

$$|\Lambda| = 1, \tag{2.128}$$

then

$$\lambda_i = \bar{\lambda}_i|\Lambda|^{\frac{n-1}{n}}. \tag{2.129}$$

Proof. Let B and K be the bilinear forms defined in (2.3) resp. (2.4), where B corresponds to the cosmological constant Λ and let B_1 be the form with respect to the value

$$|\Lambda| = 1. \tag{2.130}$$

Moreover, let us denote the corresponding quadratic forms by the same symbols, then we have

$$\frac{B(\varphi)}{K(\varphi)} = |\Lambda|^{\frac{n-1}{n}}\frac{B_1(\varphi)}{K(\varphi)} \qquad \forall\, 0 \ne \varphi \in C_c^\infty(\mathbb{R}_+). \tag{2.131}$$

To prove (2.131) we introduce a new integration variable τ on the left-hand side

$$t = \mu\tau, \qquad \mu > 0, \tag{2.132}$$

to conclude

$$\frac{B(\varphi)}{K(\varphi)} = \mu^{-4\frac{n-1}{n}}\frac{B_1(\varphi)}{K(\varphi)} \qquad \forall\, 0 \ne \varphi \in C_c^\infty(\mathbb{R}_+). \tag{2.133}$$

provided

$$\mu = |\Lambda|^{-\frac{1}{4}}. \tag{2.134}$$

The relation (2.131) immediately implies (2.129). □

3. The partition function

We first define the partition function for the black holes and shall later show that the definitions and results are also applicable in case of the quantized globally hyperbolic spacetimes with a negative cosmological constant and asymptotically Euclidean Cauchy hypersurfaces.

We define the partition function by using the spatial Hamiltonian H_1 of the quantized black holes, Kerr or Schwarzschild, which is now defined in the separable Hilbert space $\mathcal{H}$ generated by the eigendistributions

$$u_{ijk} = w_i \zeta_{ijk} \varphi_j \tag{3.1}$$

which are smooth functions satisfying the eigenvalue equations

$$H_1 u_{ijk} = \lambda_i u_{ijk} \tag{3.2}$$

as well as

$$H_0 u_{ijk} = \lambda_i u_{ijk}, \tag{3.3}$$

where H_0 is the temporal Hamiltonian.

In order to explain how the eigendistributions can generate a Hilbert space let us relabel the eigenfunctions and the eigenvalues by $(u_i, \tilde{\lambda}_i)$ such that

$$H_1 u_i = \tilde{\lambda}_i u_i \tag{3.4}$$

and

$$H_0 u_i = \tilde{\lambda}_i u_i, \tag{3.5}$$

i.e., the multiplicities of the eigenvalues are now included in the labelling and the ordering is no longer strict

$$\tilde{\lambda}_0 \le \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots . \tag{3.6}$$

To define the Hilbert space $\mathcal{H}$ we simply declare that the eigendistributions are mutually orthogonal unit eigenvectors, hence defining a scalar product in the complex vector space $\mathcal{H}'$ spanned by these eigenvectors. We define the Hilbert space $\mathcal{H}$ to be its completion.

3.1. **Lemma.** *The linear operator H_1 with domain $\mathcal{H}'$ is essentially self-adjoint in $\mathcal{H}$. Let $\bar{H}_1$ be its closure, then the only eigenvectors of $\bar{H}_1$ are those of H_1.*

Proof. H_1 is obviously densely defined, symmetric and bounded from below

$$H_1 \ge \tilde{\lambda}_0 I > 0. \tag{3.7}$$

Since $\tilde{\lambda}_0 > 0$, the eigenvectors also span $R(H_1)$, i.e., $R(H_1)$ is dense. Let

$$w \in \mathcal{H} \tag{3.8}$$

be arbitrary, and let

$$H_1 v_i \in R(H_1) \tag{3.9}$$

be a sequence converging to w, then v_i is a Cauchy sequence, because

$$\tilde{\lambda}_0 \|v_i - v_j\|^2 \le \langle H_1 v_i - H_1 v_j, v_i - v_j\rangle \le \|H_1 v_i - H_1 v_j\| \, \|v_i - v_j\|, \tag{3.10}$$

hence

$$R(\bar{H}_1) = \mathcal{H} \tag{3.11}$$

and $\bar{H}_1$ is the unique s.a. extension of H_1.

It remains to prove that $\bar{H}_1$ has no additional eigenvectors. Thus, let u be an eigenvector of $\bar{H}_1$ with eigenvalue λ

$$\bar{H}_1 u = \lambda u, \tag{3.12}$$

and let

$$E(\tilde{\lambda}_i) \subset \mathcal{H}', \qquad i \in \mathbb{N}, \tag{3.13}$$

be the eigenspaces of H_1. Let us first assume that there exists j such that

$$\lambda = \tilde{\lambda}_j, \tag{3.14}$$

but

$$u \notin E(\tilde{\lambda}_j). \tag{3.15}$$

Without loss of generality we may assume

$$u \in E(\tilde{\lambda}_j)^\perp. \tag{3.16}$$

However, this leads to a contradiction, since then

$$u \in E(\tilde{\lambda}_i)^\perp \qquad \forall\, i \in \mathbb{N}, \tag{3.17}$$

and hence

$$u \in \mathcal{H}'^\perp \tag{3.18}$$

which implies $u = 0$.

Thus, let us assume

$$\lambda \ne \tilde{\lambda}_i \qquad \forall\, i \in \mathbb{N}, \tag{3.19}$$

but then (3.17) is again valid leading to the known contradiction. □

3.2. **Remark.** In the following we shall write H_1 instead of $\bar{H}_1$.

3.3. **Lemma.** *For any $\beta > 0$ the operator*

$$e^{-\beta H_1} \tag{3.20}$$

is of trace class in $\mathcal{H}$. Let

$$\mathscr{F} \equiv \mathscr{F}_+(\mathcal{H}) \tag{3.21}$$

be the symmetric Fock space generated by $\mathcal{H}$ and let

$$H = d\Gamma(H_1) \tag{3.22}$$

be the canonical extension of H_1 to $\mathscr{F}$. Then

$$e^{-\beta H} \tag{3.23}$$

is also of trace class in $\mathscr{F}$

$$\operatorname{tr}(e^{-\beta H}) = \prod_{i=0}^{\infty}(1-e^{-\beta\tilde{\lambda}_i})^{-1} < \infty. \tag{3.24}$$

Proof. The first part of the lemma has already been proved in Corollary 2.9 on page 14. This property can now be rephrased as

$$\operatorname{tr}(e^{-\beta H_1}) = \sum_{i=0}^{\infty} e^{-\beta\tilde{\lambda}_i} < \infty. \tag{3.25}$$

The second assertion is well known, since

$$H_1 \geq \tilde{\lambda}_0 I > 0, \tag{3.26}$$

and the properties (3.25) and (3.26) imply (3.24), cf. [1, Proposition 5.2.7] and [7, Volume II, p. 868], where the equation (3.24) is also proved. ☐

We then define the partition function Z by

$$Z = \operatorname{tr}(e^{-\beta H}) = \prod_{i=0}^{\infty}(1-e^{-\beta\tilde{\lambda}_i})^{-1} \tag{3.27}$$

and the density operator ρ in $\mathscr{F}$ by

$$\rho = Z^{-1}e^{-\beta H} \tag{3.28}$$

such that

$$\operatorname{tr}\rho = 1. \tag{3.29}$$

The von Neumann entropy S is then defined by

$$\begin{aligned} S &= -\operatorname{tr}(\rho\log\rho) \\ &= \log Z + \beta Z^{-1}\operatorname{tr}(He^{-\beta H}) \\ &= \log Z - \beta\frac{\partial \log Z}{\partial\beta} \\ &\equiv \log Z + \beta E, \end{aligned} \tag{3.30}$$

where E is the average energy

$$E = \operatorname{tr}(H\rho). \tag{3.31}$$

E can be expressed in the form

$$E = \sum_{i=0}^{\infty}\frac{\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i}-1}. \tag{3.32}$$

Here, we also set the Boltzmann constant

$$K_B = 1. \tag{3.33}$$

The parameter β is supposed to be the inverse of the absolute temperature T

$$\beta = T^{-1}. \tag{3.34}$$

In view of Lemma 2.12 on page 18 we can write the eigenvalues λ_i in the form

$$\lambda_i = \bar\lambda_i |\Lambda|^{\frac{n-1}{n}}, \tag{3.35}$$

where $\bar\lambda_i$ are the eigenvalues corresponding to $|\Lambda| = 1$. Hence, Z, S, and E can also be looked at as functions depending on β and Λ, or more conveniently, on (β, τ), where

$$\tau = |\Lambda|^{\frac{n-1}{n}}, \tag{3.36}$$

since the $\tilde\lambda_i$ can also be expressed as

$$\tilde\lambda_i = \lambda_j = \bar\lambda_j |\Lambda|^{\frac{n-1}{n}}, \tag{3.37}$$

where j is different from i

$$j \le i, \tag{3.38}$$

because of the multiplicities of $\tilde\lambda_i$. Let emphasize that the multiplicities also depend on Λ, hence it is best to simply note that

$$\tilde\lambda_0 = \lambda_0 = \bar\lambda_0 |\Lambda|^{\frac{n-1}{n}} \tag{3.39}$$

and that the $\tilde\lambda_i$ are ordered. We shall never use the relation (3.37) explicitly in the proofs of the subsequent theorems and lemmata referring to (3.35) instead.

3.4. **Theorem.** (i) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{3.40}$$

we have

$$\lim_{\Lambda \to 0} E = \infty \tag{3.41}$$

as well as

$$\lim_{\Lambda \to 0} S = \infty, \tag{3.42}$$

where the limites are uniform in β.

(ii) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{3.43}$$

we have

$$\lim_{|\Lambda| \to \infty} E = 0 \tag{3.44}$$

as well as

$$\lim_{|\Lambda| \to 0} S = 0, \tag{3.45}$$

where the limites are uniform in β.

Proof. „(i)“ We first observe that

$$E = \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i} - 1} \geq \sum_{i=0}^{\infty} \frac{\lambda_i}{e^{\beta\lambda_i} - 1} \tag{3.46}$$

Now, let $m \in \mathbb{N}$ be arbitrary, then

$$E \geq \sum_{i=0}^{m} \frac{\lambda_i}{e^{\beta\lambda_i} - 1} = \sum_{i=0}^{m} \frac{\bar{\lambda}_i\tau}{e^{\beta\lambda_i\tau} - 1} \tag{3.47}$$

and

$$\begin{aligned} \liminf_{\tau\to 0} E &\geq \lim_{\tau\to 0} \sum_{i=0}^{m} \frac{\bar{\lambda}_i\tau}{e^{\beta\lambda_i\tau} - 1} \\ &= (m+1)\beta^{-1} \geq (m+1)\beta_0^{-1} \end{aligned} \tag{3.48}$$

yielding

$$\lim_{\Lambda\to 0} E = \infty \tag{3.49}$$

uniformly in β.

Since $Z \geq 1$, the relation (3.42) follows as well.

„(ii)“ We estimate E from above by

$$\begin{aligned} E &= \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i e^{-\beta\tilde{\lambda}_i}}{1 - e^{-\beta\tilde{\lambda}_i}} = \sum_{i=0}^{\infty} \tilde{\lambda}_i e^{-\frac{\beta}{2}\tilde{\lambda}_i} e^{-\frac{\beta}{2}\tilde{\lambda}_i} (1 - e^{-\beta\tilde{\lambda}_i})^{-1} \\ &\leq (1 - e^{-\beta_0\tilde{\lambda}_0})^{-1} c(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\tilde{\lambda}_i}, \end{aligned} \tag{3.50}$$

where we used (3.43) and

$$\tilde{\lambda}_i e^{-\frac{\beta}{2}\tilde{\lambda}_i} \leq \sup_{t>0} t e^{-\frac{\beta}{2}t} = c(\beta) \leq c(\beta_0). \tag{3.51}$$

Furthermore, we know that

$$\begin{aligned} \sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\tilde{\lambda}_i} &\leq \tilde{c}(\beta) \sum_{i=0}^{\infty} e^{-\frac{\beta}{4}\lambda_i} \\ &\leq \tilde{c}(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta_0}{4}\lambda_i}, \end{aligned} \tag{3.52}$$

cf. Lemma 2.10 on page 15 and Lemma 2.11 on page 17, hence we obtain

$$E \leq (1 - e^{-\beta_0\bar{\lambda}_0\tau})^{-1} c(\beta_0)\tilde{c}(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta}{4}\bar{\lambda}_i\tau} \tag{3.53}$$

deducing further

$$\limsup_{\tau\to\infty} E \le c(\beta_0)\tilde c(\beta_0)\lim_{\tau\to\infty}\sum_{i=0}^\infty e^{-\frac{\beta}{4}\tilde\lambda_i\tau}=0 \tag{3.54}$$

uniformly in β and hence

$$\lim_{\tau\to\infty} E = 0. \tag{3.55}$$

It remains to prove that S vanishes in the limit. We have

$$\begin{aligned} Z&=\prod_{i=0}^\infty(1-e^{-\beta\tilde\lambda_i})^{-1}=\prod_{i=0}^\infty(1+e^{-\beta\tilde\lambda_i}(1-e^{-\beta\tilde\lambda_i})^{-1})\\ &\le \exp\{(1-e^{\beta_0\tilde\lambda_0})^{-1}\sum_{i=0}^\infty e^{-\beta\tilde\lambda_i}\}, \end{aligned} \tag{3.56}$$

where we used the inequality

$$\log(1+t)\le t\qquad\forall\, t\ge 0 \tag{3.57}$$

in the last step.

Applying then the arguments preceding the inequality (3.54) we conclude

$$\lim_{\tau\to\infty} Z = 1 \tag{3.58}$$

uniformly in β. □

3.5. **Remark.** The first part of the preceding theorem reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density we conjecture that the dark energy density should be proportional to the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho\eta = Z^{-1}\eta, \tag{3.59}$$

which is Z^{-1}.

The behaviour of Z with respect to Λ is described in the theorem:

3.6. **Theorem.** *Let $\beta_0>0$ be arbitrary, then, for any*

$$0<\beta\le\beta_0, \tag{3.60}$$

we have

$$\lim_{\Lambda\to 0} Z=\infty \tag{3.61}$$

and for any

$$\beta_0\le\beta \tag{3.62}$$

the relation

$$\lim_{|\Lambda|\to\infty} Z = 1 \tag{3.63}$$

is valid. The convergence in both limites is uniform in β.

Proof. „(3.60)“ Let $m \in \mathbb{N}$ be arbitrary, then

$$\begin{aligned} Z &\ge \prod_{i=0}^{\infty}(1-e^{-\beta\lambda_i})^{-1} = \prod_{i=0}^{\infty}(1-e^{-\beta\bar{\lambda}_i\tau})^{-1} \\ &\ge \prod_{i=0}^{m}(1-e^{-\beta_0\bar{\lambda}_i\tau})^{-1} \end{aligned} \tag{3.64}$$

and we infer

$$\lim_{\tau\to 0} Z = \liminf_{\tau\to 0} Z = \infty. \tag{3.65}$$

„(3.63)“ This limit relation has already been proved in (3.58). □

Let us now consider the quantized globally hyperbolic spacetimes with an asymptotically Euclidean Cauchy hypersurface. The eigenspaces

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{3.66}$$

of H_1 are separable but they are in general not finite dimensional as can be seen by the following counterexample

$$H_1 = -\Delta \tag{3.67}$$

in $\mathbb{R}^n$. The eigenspaces

$$\mathscr{E}_{\lambda_i}, \qquad \lambda_i > 0, \tag{3.68}$$

contain the tempered distributions

$$e^{i\langle k,x\rangle}, \qquad k \in \mathbb{S}^{n-1}_{\lambda_i}. \tag{3.69}$$

As a Hamel basis they generate a vector space the dimension of which is equal to the cardinality of $\mathbb{S}^{n-1}$. Of course, as a Schauder basis the functions with

$$k \in D \subset \mathbb{S}^{n-1}_{\lambda_i}, \tag{3.70}$$

where D is countable and dense, generate a dense subspace.

This example indicates that not all eigendistributions of H_1 might be physically relevant. Contrary to the cases of the black holes, where the selection of eigenvectors and eigendistributions was a natural process, only the temporal eigenvectors are naturally selected in the present situation and of course at least one matching spatial eigendistribution to obtain a solution of the wave equation. Hence, we could use H_0 to define the partition function. However, we believe this choice would be too restrictive, and we shall instead stipulate that we only pick at most

$$c|\lambda_i|^p \tag{3.71}$$

spatial eigendistributions in $\mathscr{E}_{\lambda_i}$, where c and p are arbitrary but fixed constants, i.e., we assume that

$$n(\lambda_i) \le c|\lambda_i|^p \qquad \forall\, i \in \mathbb{N}. \tag{3.72}$$

With this assumption it becomes evident that the results and conjectures of Theorem 3.4, Remark 3.5 and Theorem 3.6 are also valid in case of globally hyperbolic spacetimes with asymptotically Euclidean hypersurfaces.

4. The Friedmann universes with negative cosmological constants

In [3, Remark 6.11] we observed that, if the Cauchy hypersurface $\mathcal{S}_0$ is a space of constant curvature and if the wave equation (1.1) on page 2 is only considered for functions u which do not depend on x, then this equation is identical to the equation obtained by quantizing the Hamilton constraint in a Friedman universe without matter but including a cosmological constant. The equation is then the ODE

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - \frac{n}{2}Rt^{2-\frac{4}{n}}u + nt^2\Lambda u = 0, \qquad 0 < t < \infty, \tag{4.1}$$

where R is the scalar curvature of $\mathcal{S}_0$. We cannot apply our previous arguments to the solutions of this ODE. However, if we consider instead the more general equation (1.1), where u is also allowed to depend on x, which certainly is more general and accurate, then the previous arguments can be applied if the curvature $\tilde{\kappa}$ of $\mathcal{S}_0$ vanishes

$$\tilde{\kappa} = 0. \tag{4.2}$$

The scalar curvature, which is equal to

$$R = n(n-1)\tilde{\kappa}, \tag{4.3}$$

then vanishes too and

$$\mathcal{S}_0 = \mathbb{R}^n. \tag{4.4}$$

We are now in the situation which we analyzed at the end of the previous section, where now the spatial Hamiltonian is

$$H_1 = -(n-1)\Delta \tag{4.5}$$

and some spatial eigendistributions are shown in (3.69) on page 25. However, since we consider the quantized version of a Friedmann universe we shall look for radially symmetric eigendistributions, i.e., we look for smooth functions $v = v(x)$ satisfying

$$v(x) = \varphi(r) \tag{4.6}$$

such that

$$\Delta v = \ddot{\varphi} + (n-1)r^{-1}\dot{\varphi} = -\mu^2\varphi \quad \text{in} \quad r > 0, \tag{4.7}$$

where $\mu > 0$. Obviously, it is sufficient to assume $\mu = 1$, because, if φ is an eigenfunction for $\mu = 1$, then

$$\tilde{\varphi}(r) = \varphi(\mu r) \tag{4.8}$$

is an eigenfunction for the eigenvalue μ^2. Therefore, let us choose $\mu = 1$.

We shall express the solution φ with the help of a Bessel function J_ν. Let ψ be a solution of the Bessel equation

$$\ddot{\psi} + r^{-1}\dot{\psi} + (1 - r^{-2}\nu^2)\psi = 0, \tag{4.9}$$

where

$$\nu = \frac{n-2}{2}, \tag{4.10}$$

then the function

$$\varphi(r) = r^{-\nu}\psi \tag{4.11}$$

satisfies

$$r\ddot{\varphi} + (2\nu + 1)\dot{\varphi} + r\varphi = 0, \tag{4.12}$$

which is equivalent to (4.7) with $\mu = 1$. The Bessel equation (4.9) has the two independent solutions J_ν and Y_ν, the Bessel functions of first kind resp. of second kind. It is well known that the functions

$$r^{-\nu}J_\nu \tag{4.13}$$

can be expressed as power series in the variable r^2, cf. [2, equ. (21), p. 420], i.e., the function

$$v(x) = \varphi(r) = r^{-\nu}J_\nu \tag{4.14}$$

is smooth in $\mathbb{R}^n$, while the functions

$$r^{-\nu}Y_\nu \tag{4.15}$$

have a singularity in $r = 0$. Hence, there exists exactly one smooth radially symmetric solution v of the eigenvalue equation

$$-\Delta v = \lambda^2 v, \qquad \lambda > 0, \tag{4.16}$$

which is given by

$$v = (\lambda r)^{-\nu}J_\nu(\lambda r). \tag{4.17}$$

This solution also vanishes at infinity, hence it is uniformly bounded and a tempered distribution.

A solution of the wave equation (1.1) on page 2, in case of a quantized Friedmann universe, is therefore given by a sequence

$$u_i = w_i(t)v_i(x), \qquad i \in \mathbb{N}, \tag{4.18}$$

where w_i is a temporal eigenfunction and v_i a spatial eigenfunction. The u_i are also eigenfunctions for the temporal Hamiltonian as well as for the spatial Hamiltonian. Each eigenvalue has multiplicity one. We have therefore proved:

4.1. **Theorem.** *The results in Theorem 3.4, Remark 3.5 and Theorem 3.4 are also valid, if the quantized spacetime $N = N^{n+1}$, $n \geq 3$, is a Friedmann universe without matter but with a negative cosmological constant Λ and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian H_1 all have multiplicity one.*

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