THE QUANTUM DEVELOPMENT OF AN ASYMPTOTICALLY EUCLIDEAN CAUCHY HYPERSURFACE

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Abstract. In our model of quantum gravity the quantum development of a Cauchy hypersurface is governed by a wave equation derived as the result of a canonical quantization process. To find physically interesting solutions of the wave equation we employ the separation of variables by considering a temporal eigenvalue problem which has a complete countable set of eigenfunctions with positive eigenvalues and also a spatial eigenvalue problem which has a complete set of eigendistributions. Assuming that the Cauchy hypersurface is asymtotically Euclidean we prove that the temporal eigenvalues are also spatial eigenvalues and the product of corresponding eigenfunctions and eigendistributions, which will be smooth functions with polynomial growth, are the physically interesting solutions of the wave equation. We consider these solutions to describe the quantum development of the Cauchy hypersurface.

Contents

1. Introduction

In general relativity the Cauchy development of a Cauchy hypersurface $\mathcal{S}_0$ is governed by the Einstein equations, where of course the second fundamental form of $\mathcal{S}_0$ has also to be specified.

In the model of quantum gravity we developed in a series of papers [6, 7, 4, 8, 9, 10] we pick a Cauchy hypersurface, which is then only considered to

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be a complete Riemannian manifold $(\mathcal{S}_0, g_{ij})$ of dimension $n \ge 3$, and define its quantum development to be described by special solutions of the wave equation

$$\begin{aligned}\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + \alpha_1\frac{n}{8}t^{2-\frac{4}{n}}F_{ij}F^{ij}u\\ + \alpha_2\frac{n}{4}t^{2-\frac{4}{n}}\gamma_{ab}\sigma^{ij}\Phi^a_i\Phi^b_i u + \alpha_2\frac{n}{2}mt^{2-\frac{4}{n}}V(\Phi)u + nt^2\Lambda u = 0,\end{aligned}\tag{1.1}$$

in a globally hyperbolic spacetime

$$Q = (0,\infty)\times\mathcal{S}_0,\tag{1.2}$$

cf. [9]. The preceding wave equation describes the interaction of a given complete Riemannian metric g_{ij} in $\mathcal{S}_0$ with a given Yang-Mills and Higgs field; R is the scalar curvature of g_{ij}, V is the potential of the Higgs field, Λ a negative cosmological constant, m a positive constant, α_1, α_2 are positive coupling constants and the other symbols should be self-evident. The existence of the time variable, and its range, is due to the quantization process.

1.1. **Remark.** For the results and arguments in [9] it was completely irrelevant that the values of the Higgs field Φ lie in a Lie algebra, i.e., Φ could also be just an arbitrary scalar field, or we could consider a Higgs field as well as an another arbitrary scalar field. Hence, let us stipulate that the Higgs field could also be just an arbitrary scalar field.

If $\mathcal{S}_0$ is compact we also proved a spectral resolution of equation (1.1) by first considering a stationary version of the hyperbolic equation, namely, the elliptic eigenvalue equation

$$\begin{aligned}&-(n-1)\Delta v - \frac{n}{2}Rv + \alpha_1\frac{n}{8}F_{ij}F^{ij}v\\ &+\alpha_2\frac{n}{4}\gamma_{ab}\sigma^{ij}\Phi^a_i\Phi^b_i v + \alpha_2\frac{n}{2}mV(\Phi)v = \mu v.\end{aligned}\tag{1.3}$$

It has countably many solutions (v_i, μ_i) such that

$$\mu_0 < \mu_1 \le \mu_2 \le \cdots,\tag{1.4}$$

$$\lim \mu_i = \infty.\tag{1.5}$$

Let v be an eigenfunction with eigenvalue $\mu > 0$, then we looked at solutions of (1.1) of the form

$$u(x,t) = w(t)v(x).\tag{1.6}$$

u is then a solution of (1.1) provided w satisfies the implicit eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} - \mu t^{2-\frac{4}{n}}w - nt^2\Lambda w = 0,\tag{1.7}$$

where Λ is the eigenvalue.

We proved in [9] that for any stationary eigenfunction v_j with positive eigenvalue μ_j there is complete sequence of eigenfunctions w_{ij} of the temporal implicit eigenvalue problem such that the functions

$$u_{ij}(t,x) = w_{ij}(t)v_j(x) \tag{1.8}$$

are solutions of the wave equation, cf. also [8, Section 6].

However, for non-compact Cauchy hypersurfaces one has to use a different approach in order to quantize the wave equation (1.1). Let us first consider the temporal eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w \tag{1.9}$$

in the Sobolev space

$$H_0^{1,2}(\mathbb{R}_+^*). \tag{1.10}$$

Here,

$$\Lambda < 0 \tag{1.11}$$

is the cosmological constant.

The eigenvalue problem (1.9) can be solved by considering the generalized eigenvalue problem for the bilinear forms

$$B(w,\tilde{w}) = \int_{\mathbb{R}_+^*}\{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}' + n|\Lambda|t^2\bar{w}\tilde{w}\} \tag{1.12}$$

and

$$K(w,\tilde{w}) = \int_{\mathbb{R}_+^*} t^{2-\frac{4}{n}}\bar{w}\tilde{w} \tag{1.13}$$

in the Sobolev space $\mathcal{H}$ which is the completion of

$$C_c^\infty(\mathbb{R}_+^*,\mathbb{C}) \tag{1.14}$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$B(w,\varphi) = \lambda K(w,\varphi) \quad \forall\, \varphi \in \mathcal{H} \tag{1.15}$$

which is equivalent to (1.9).

1.2. **Theorem.** *The eigenvalue problem (1.15) has countably many solutions (w_i,λ_i) such that*

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \tag{1.16}$$

$$\lim \lambda_i = \infty, \tag{1.17}$$

and

$$K(w_i,w_j) = \delta_{ij}. \tag{1.18}$$

The w_i are complete in $\mathcal{H}$ as well as in $L^2(\mathbb{R}_+^*)$.

Secondly, let A be the elliptic operator on the left-hand side of (1.3), assuming that its coefficients are smooth and bounded in any

$$C^m(\mathcal{S}_0), \qquad m \in \mathbb{N}, \tag{1.19}$$

then A is self-adjoint in $L^2(\mathcal{S}_0, \mathbb{C})$ and, if $\mathcal{S}_0$ is asymptotically Euclidean, i.e., if it satisfies the very mild conditions in Assumption 3.1 on page 14, then the Schwartz space $\mathscr{S}$ of rapidly decreasing functions can also be defined in $\mathcal{S}_0$,

$$\mathscr{S} = \mathscr{S}(\mathcal{S}_0), \tag{1.20}$$

such that

$$\mathscr{S} \subset L^2(\mathcal{S}_0) \subset \mathscr{S}' \tag{1.21}$$

is a Gelfand triple and the eigenvalue problem in $\mathscr{S}'$

$$Af = \lambda f \tag{1.22}$$

has a solution for any $\lambda \in \sigma(A)$, cf. Theorem 2.5 on page 9. Let

$$(\mathscr{E}_\lambda)_{\lambda \in \sigma(A)} \tag{1.23}$$

be the set of eigendistributions in $\mathscr{S}'$ satisfying

$$Af(\lambda) = \lambda f(\lambda), \qquad f(\lambda) \in \mathscr{E}_\lambda, \tag{1.24}$$

then the $f(\lambda)$ are actually smooth functions in $\mathcal{S}_0$ with polynomial growth, cf. [10, Theorem 3]. Moreover, due to a result of Donnelly [1], we know that

$$[0, \infty) \subset \sigma_{\text{ess}}(A), \tag{1.25}$$

hence, any temporal eigenvalue λ_i in Theorem 1.2 is also a spatial eigenvalue of A in $\mathscr{S}'$

$$Af(\lambda_i) = \lambda_i f(\lambda_i). \tag{1.26}$$

Since the eigenspaces $\mathscr{E}_{\lambda_i}$ are separable we deduce that for each i there is an at most countable basis of eigendistributions in $\mathscr{E}_{\lambda_i}$

$$v_{ij} \equiv f_j(\lambda_i), \qquad 1 \le j \le n(i) \le \infty, \tag{1.27}$$

satisfying

$$Av_{ij} = \lambda_i v_{ij}, \tag{1.28}$$

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{1.29}$$

The functions

$$u_{ij} = w_i v_{ij} \tag{1.30}$$

are then smooth solutions of the wave equations. They are considered to describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

Let us summarize this result as a theorem:

1.3. **Theorem.** *Let A and $\mathcal{S}_0$ satisfy the conditions in (1.19) and Assumption 3.1, and let w_i resp. v_{ij} be the countably many solutions of the temporal resp. spatial eigenvalue problems, then*

$$u_{ij} = w_i v_{ij} \tag{1.31}$$

are smooth solutions of the wave equation (1.1). They describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

1.4. **Remark.** We used a similar approach to describe the quantum development of the event horizon of an AdS blackhole, see [11].

2. Existence of a complete set of eigendistributions

Let H be a separable Hilbert space, $\mathcal{S}$ a complete nuclear space and

$$j : \mathcal{S} \hookrightarrow H \tag{2.1}$$

an embedding such that $j(\mathcal{S})$ is dense in H. The triple

$$\mathcal{S} \subset H \subset \mathcal{S}' \tag{2.2}$$

is then called a Gelfand triple and H a *rigged* Hilbert space. Moreover, we require that the semi-norms $\|\cdot\|_p$ defining the topology of $\mathcal{S}$ are a countable family. In view of the assumption (2.1) at least of one the semi-norms is already a norm, since there exist a constant c and a semi-norm $\|\cdot\|_p$ such that

$$\|j(\varphi)\| \le c\|\varphi\|_p \qquad \forall\, \varphi \in \mathcal{S}, \tag{2.3}$$

hence $\|\cdot\|_p$ is a norm since j is injective. But then there exists an equivalent sequence of norms generating the topology of $\mathcal{S}$. Since $\mathcal{S}$ is nuclear we may also assume that the norms are derived from a scalar product, cf. [12, Theorem 2, p. 292].

Let $\mathcal{S}_p$ be the completion of $\mathcal{S}$ with respect to $\|\cdot\|_p$, then

$$\mathcal{S} = \bigcap_{p=1}^{\infty} \mathcal{S}_p \tag{2.4}$$

and

$$\mathcal{S}' = \bigcup_{p=1}^{\infty} \mathcal{S}'_p. \tag{2.5}$$

A nuclear space $\mathcal{S}$ having these properties is called a nuclear countably Hilbert space or a nuclear Fréchet Hilbert space.

Let A be a self-adjoint operator in H with spectrum

$$\Lambda = \sigma(A). \tag{2.6}$$

Identifying $\mathcal{S}$ with $j(\mathcal{S})$ we assume

$$A(\mathcal{S}) \subset \mathcal{S} \tag{2.7}$$

and we want to prove that for any $\lambda \in \Lambda$ there exists

$$0 \neq f(\lambda) \in \mathcal{S}' \tag{2.8}$$

satisfying

$$\langle f(\lambda), A\varphi \rangle = \lambda \langle f(\lambda), \varphi \rangle \quad \forall \varphi \in \mathcal{S}. \tag{2.9}$$

$f(\lambda)$ is then called a generalized eigenvector, or an eigendistribution, if $\mathcal{S}'$ is a space of distributions. The crucial point is that we need to prove the existence of a generalized eigenvector for any $\lambda \in \Lambda$.

2.1. **Definition.** We define

$$\mathscr{E}_\lambda = \{\, f \in \mathscr{S}' : Af = \lambda f \,\} \tag{2.10}$$

to be the generalized eigenspace of A with eigenvalue $\lambda \in \Lambda$ provided

$$\mathscr{E}_\lambda \neq \{0\}. \tag{2.11}$$

If (2.11) is valid for all $\lambda \in \Lambda$, then we call

$$(\mathscr{E}_\lambda)_{\lambda \in \Lambda} \tag{2.12}$$

a complete system of generalized eigenvectors of A in $\mathcal{S}'$.

2.2. **Lemma.** *If $\mathcal{S}$ is separable, then each $\mathscr{E}_\lambda \neq \{0\}$ is also separable in the inherited strong topology of $\mathcal{S}'$.*

Proof. The Hilbert spaces $\mathcal{S}_p$ are all separable by assumption, so are their duals $\mathcal{S}'_p$. Let $\mathcal{B}_p$ be a countable dense subset of $\mathcal{S}'_p$ and set

$$\mathcal{B} = \bigcup_{p=1}^{\infty} \mathcal{B}_p, \tag{2.13}$$

Then $\mathcal{B}$ is dense in $\mathcal{S}'$ in the strong topology. Indeed, consider $f \in \mathcal{S}'$ and a bounded subset $B \subset \mathcal{S}$, then there exists p such that $f \in \mathcal{S}'_p$, in view of (2.5), and for any $g \in \mathcal{B}_p$ we obtain

$$\sup_{\varphi \in B} |\langle f - g, \varphi \rangle| \le \|f - g\|_{-p} \sup_{\varphi \in B} \|\varphi\|_p \le c_B \|f - g\|_{-p} \tag{2.14}$$

proving the claim. □

Let E be the spectral measure of A mapping Borel sets of Λ to projections in H, then we can find an at most countable family of mutually orthogonal unit vectors

$$v_i \in H, \qquad 1 \le i \le m \le \infty, \tag{2.15}$$

and mutually orthogonal subspaces

$$H_i \in H \tag{2.16}$$

which are generated by the vectors

$$E(\Omega) v_i, \qquad \Omega \in \mathscr{B}(\Lambda), \tag{2.17}$$

where Ω is an arbitrary Borel set in Λ, such that

$$H = \bigoplus_{i=1}^{m} H_i. \tag{2.18}$$

Each subspace H_i is isomorphic to the function space

$$\hat{H}_i = L^2(\Lambda, \mathbb{C}, \mu_i) \equiv L^2(\Lambda, \mu_i), \tag{2.19}$$

where μ_i is the positive Borel measure

$$\mu_i = \langle E v_i, v_i \rangle. \tag{2.20}$$

We have

$$\mu_i(\Lambda) = 1 \tag{2.21}$$

and there exists a unitary U from H_i onto $\hat{H}_i$ such that

$$\langle u, v \rangle = \int_\Lambda \bar{\hat{u}}(\lambda)\hat{v}(\lambda) d\mu_i \qquad \forall\, u, v \in H_i \tag{2.22}$$

where we have set

$$\hat{u} = Uu \qquad \forall\, u \in H_i. \tag{2.23}$$

Hence, there exists a unitary surjective operator, also denoted by U,

$$U : H \to \hat{H} = \bigoplus_{i=1}^{m} \hat{H}_i \tag{2.24}$$

such that $u = (u^i)$ is mapped to

$$\hat{u} = Uu = (Uu^i) = (\hat{u}^i) \tag{2.25}$$

and

$$\hat{u}^i = \hat{u}^i(\lambda) \in L^2(\Lambda, \mu_i). \tag{2.26}$$

Moreover, if $u \in D(A)$, then

$$\widehat{Au} = (\widehat{Au}^i(\lambda)) = (\lambda \hat{u}^i) = \lambda \hat{u}. \tag{2.27}$$

For a proof of these well-known results see e.g. [2, Chap. I, Appendix, p. 127].

2.3. **Remark.** We define the positive measure

$$\mu = \sum_{i=1}^{m} 2^{-i} \mu_i \tag{2.28}$$

in Λ, and we shall always have this measure in mind when referring to null sets in Λ. Moreover, applying the Radon-Nikodym theorem, we conclude that there are non-negative Borel functions, which we express in the form $h_i^2, 0 \le h_i$, such that

$$h_i^2 \in L^1(\Lambda, \mu) \tag{2.29}$$

and

$$d\mu_i = h_i^2 d\mu. \tag{2.30}$$

The map

$$v \in L^2(\Lambda, \mu_i) \to h_i v \in L^2(\Lambda, \mu) \tag{2.31}$$

is a unitary embedding.

2.4. **Lemma.** *The functions h_i satisfy the following relations*

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 < \infty \qquad \mu \text{ a.e.} \tag{2.32}$$

and

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 \neq 0 \qquad \mu \text{ a.e.} \tag{2.33}$$

Replacing the values of h_i on the exceptional null sets by 2^{-i} the two previous relations are valid everywhere in Λ.

Proof. (i) We first prove that, for a fixed i, h_i cannot vanish on a Borel set G with positive μ_i measure, $\mu_i(G) > 0$. We argue by contradiction assuming that h_i would vanish on a Borel set G with $\mu_i(G) > 0$. Let $v \in H$ be arbitrary and let v^i be the component belonging to H_i, then

$$\begin{aligned} \int_G |\hat{v}^i|^2 d\mu_i &= \int_\Lambda \chi_G |\hat{v}^i|^2 d\mu_i \\ &= \int_\Lambda \chi_G h_i^2 |\hat{v}^i|^2 d\mu = 0, \end{aligned} \tag{2.34}$$

and we deduce

$$\hat{v}^i = 0 \qquad \mu_i \text{ a.e. in } G \qquad \forall\, v \in H, \tag{2.35}$$

a contradiction, since the $\hat{v}^i$ generate $L^2(\Lambda, \mu_i)$.

(ii) Now, let $G \subset \Lambda$ be an arbitrary Borel set satisfying $\mu(G) > 0$ and define $\hat{\psi} = (\hat{\psi}^i)$ by setting

$$\hat{\psi}^i = \chi_G 2^{-i}, \tag{2.36}$$

then we obtain

$$\begin{aligned} \|\hat{\psi}\|^2 &= \sum_{i=1}^{m} \int_\Lambda \chi_G 2^{-2i} d\mu_i \\ &= \sum_{i=1}^{m} \int_\Lambda \chi_G 2^{-2i} h_i^2 d\mu < \infty \end{aligned} \tag{2.37}$$

concluding

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 < \infty \qquad \mu \text{ a.e.} \tag{2.38}$$

as well as

$$\sum_{i=1}^{m} 2^{-2i} h_i^2 \neq 0 \qquad \mu \text{ a.e.}, \tag{2.39}$$

where the last conclusion is due to the result proved in (i), since there must existst an i such that $\mu_i(G) > 0$. □

Now we can prove:

2.5. **Theorem.** *Let H be a separable rigged Hilbert space as above assuming that the nuclear space $\mathcal{S}$ is a Fréchet Hilbert space, and let A be a self-adjoint operator in H satisfying (2.7). Then there exists a complete system of generalized eigenvectors $(\mathscr{E}_\lambda)_{\lambda \in \varLambda}$. If $\mathcal{S}$ is separable, then each eigenspace $\mathscr{E}_\lambda$ is separable.*

Proof. Since $\mathcal{S}$ is nuclear there exists a norm $\|\cdot\|_p$ such that the embedding

$$j : \mathcal{S}_p \hookrightarrow H \tag{2.40}$$

is nuclear, i.e., we can write

$$j(\varphi) = \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle u_k \qquad \forall\, \varphi \in \mathcal{S}, \tag{2.41}$$

where

$$0 \le \lambda_k \quad \wedge \quad \sum_{k=1}^{\infty} \lambda_k < \infty, \tag{2.42}$$

$$f_k \in \mathcal{S}'_p \quad \wedge \quad \|f_k\| = 1, \tag{2.43}$$

and $u_k \in H$ is an orthonormal sequence. We may, and shall, also assume

$$u_k \in D(A), \tag{2.44}$$

since $D(A)$ is dense in H: Let

$$v_k \in D(A) \tag{2.45}$$

be a sequence of linearly independent vectors generating a dense subspace in H, then we can define an orthonormal basis $(\tilde{v}_k)$ in H which spans the same subspace. Hence, there exists a unitary map T such that

$$\tilde{v}_k = T u_k \qquad \forall\, k \in \mathbb{N}. \tag{2.46}$$

Instead of the embedding j we can then consider the embedding

$$T \circ j \tag{2.47}$$

proving our claim. Thus, we shall assume (2.44) which is convenient but not necessary.

We immediately infer from the assumption that $j(\mathcal{S})$ is dense in H the following conclusions:

(2.48) The (u_k) are complete in H,

$$0 < \lambda_k \qquad \forall\, k, \tag{2.49}$$

and

$$\text{for all } k \text{ there exists } \varphi \in \mathcal{S} \text{ such that } \langle f_k, \varphi\rangle \neq 0. \tag{2.50}$$

Let U be the unitary operator in (2.24), then we define

$$\hat{\varphi} = U \circ j(\varphi) = \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi\rangle \hat{u}_k \tag{2.51}$$

such that

$$\hat{u}_k = (\hat{u}_k^i(\lambda))_{1\le i\le m} \tag{2.52}$$

$$\hat{u}_k^i \in L^2(\varLambda, \mu_i). \tag{2.53}$$

Applying the embedding in (2.31) we can also express $\hat{u}_k$ in the form

$$\hat{u}_k = (h_i \hat{u}_k^i(\lambda))_{1\le i\le m} \tag{2.54}$$

$$h_i \hat{u}_k^i \in L^2(\varLambda, \mu). \tag{2.55}$$

Similarly we have

$$\hat{\varphi} = (h_i \hat{\varphi}^i) \tag{2.56}$$

and

$$\widehat{A\varphi} = (\lambda h_i \hat{\varphi}^i), \tag{2.57}$$

in view of (2.27). Here, we identify φ and $j\varphi$, i.e.,

$$A\varphi \equiv A(j\varphi). \tag{2.58}$$

We want to prove that

$$\begin{aligned} \widehat{A(j\varphi)} &= \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi\rangle \widehat{Au_k} \\ &= \lambda \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi\rangle \hat{u}_k. \end{aligned} \tag{2.59}$$

Indeed, for any bounded Borel set $\varOmega \subset \varLambda$

$$AE(\varOmega) \tag{2.60}$$

is a self-adjoint bounded operator in H such that

$$\|AE(\varOmega)\| \le \sup_{\lambda\in\varOmega} |\lambda|. \tag{2.61}$$

Hence, we deduce

$$AE(\varOmega)(j\varphi) = \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi\rangle AE(\varOmega) u_k \tag{2.62}$$

and

$$\widehat{AE(\varOmega)u_k} = \lambda \chi_{\varOmega} \hat{u}_k \tag{2.63}$$

and we infer

$$\chi_\Omega \widehat{A(j\varphi)} = \chi_\Omega \lambda \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle \hat{u}_k \\ = \chi_\Omega \lambda \hat{\varphi}. \tag{2.64}$$

Since $\Omega \subset \Lambda$ is an arbitrary bounded Borel set we conclude

$$\widehat{A(j\varphi)} = \lambda \sum_{k=1}^{\infty} \lambda_k \langle f_k, \varphi \rangle \hat{u}_k \\ = \lambda \hat{\varphi}. \tag{2.65}$$

The right-hand side of the second equation is square integrable and therefore the right-hand side of the first equation too.

Let us set

$$\hat{\varphi}(\lambda) = (h_i \hat{\varphi}^i(\lambda)). \tag{2.66}$$

$h_i \hat{\varphi}^i$ is an equivalence class and to define $h_i \hat{\varphi}^i(\lambda)$ as a complex number for a fixed $\lambda \in \Lambda$ requires to pick a representative of the equivalence class in order to define $h_i \hat{\varphi}^i(\lambda)$. It is well-known that for a given representative $h_i \hat{\varphi}^i(\lambda)$ is well defined for almost every $\lambda \in \Lambda$, i.e., apart from a null set. We shall show that $\hat{\varphi}(\lambda)$ can be well defined for *any* $\lambda \in \Lambda$ and any $\varphi \in \mathcal{S}$. The choices we shall have to make will be independent of φ.

Firstly, let us define the product

$$h_i \hat{u}_k^i \tag{2.67}$$

unambiguously. In view of Lemma 2.4 h_i is everywhere finite, i.e., we only have to consider the case when $h_i = 0$ and $|\hat{u}_k^i| = \infty$. In this case we stipulate that

$$h_i \hat{u}_k^i = 0. \tag{2.68}$$

This definition insures that the integrals, e.g.,

$$\int_\Lambda |h_i \hat{u}_k^i|^2 d\mu \tag{2.69}$$

will give the correct values, because of Lebesgue's monotone convergence theorem: approximate $|\hat{u}_k^i|$ by

$$\min(|\hat{u}_k^i|, r), \qquad r \in \mathbb{N}. \tag{2.70}$$

Secondly, we observe that

$$1 = \|\hat{u}_k\|^2 = \sum_{i=1}^{m} \int_\Lambda |h_i \hat{u}_k^i(\lambda)|^2, \tag{2.71}$$

and hence

$$\sum_{i=1}^{m} |h_i \hat{u}_i^k(\lambda)|^2 < \infty \qquad \text{a.e. in } \Lambda. \tag{2.72}$$

Thirdly, we have

$$\sum_{k=1}^{\infty}\sum_{i=1}^{m}|h_i\hat u_k^i(\lambda)|^2\ne 0\qquad\text{a.e. in } \varLambda. \tag{2.73}$$

Indeed, suppose there were a Borel set

$$G\subset\varLambda \tag{2.74}$$

such that

$$0<\mu(G)=\sum_i 2^{-i}\mu_i(G) \tag{2.75}$$

and

$$\sum_{k=1}^{\infty}\sum_{i=1}^{m}|h_i\hat u_k^i(\lambda)|^2=0\qquad\text{in } G, \tag{2.76}$$

then there would exist j such that

$$\mu_j(G)>0 \tag{2.77}$$

and we would deduce

$$0=\sum_{k=1}^{\infty}\int_G|h_j\hat u_k^j|^2d\mu=\sum_{k=1}^{\infty}\int_G|\hat u_k^j|^2d\mu_j, \tag{2.78}$$

contradicting the fact that the $(\hat u_k^j)$ are a basis for $L^2(\varLambda,\mu_j)$.

Fourthly, we have

$$\sum_k\sum_i\int_{\varLambda}\lambda_k|h_i\hat u_k^i(\lambda)|^2d\mu=\sum_k\lambda_k\|\hat u_k\|^2=\sum_k\lambda_k<\infty, \tag{2.79}$$

hence we deduce

$$\sum_k\sum_i\lambda_k|h_i\hat u_k^i(\lambda)|^2<\infty\qquad\text{a.e. in } \varLambda. \tag{2.80}$$

Now, for any (i,k) we choose a particular representative of $h_i\hat u_k^i$ by first picking the representative of h_i we defined in Lemma 2.4 and a representative of $\hat u_k^i$ satisfying the relations in (2.72), (2.73) and (2.80) and then defining the values of these particular representatives in the exceptional null sets occurring in the just mentioned relations by

$$h_i=2^{-i}\quad\wedge\quad\hat u_k^i=2^{-i}2^{-k}. \tag{2.81}$$

Then $h_i\hat u_k^i(\lambda)$ is well defined for any $\lambda\in\varLambda$ and the relations in (2.72), (2.73) and (2.80) are valid for any $\lambda\in\varLambda$.

Moreover, the series

$$h_i\hat\varphi^i(\lambda)=\sum_k\lambda_k\langle f_k,\varphi\rangle h_i\hat u_k^i(\lambda) \tag{2.82}$$

converges absolutely, since

$$\begin{aligned}\sum_k \lambda_k|\langle f_k,\varphi\rangle||h_i\hat{u}_k^i(\lambda)| &\le \|\varphi\|_p\sum_k \lambda_k|h_i\hat{u}_k^i(\lambda)|\\ &\le \|\varphi\|_p(\sum_k \lambda_k)^{\frac12}(\sum_k \lambda_k|h_i\hat{u}_k^i(\lambda)|^2)^{\frac12}<\infty,\end{aligned}\tag{2.83}$$

in view of (2.80).

2.6. **Definition.** Let us define the sequence space

$$l_2=\{\,(a_k^i)\colon\ \sum_k\sum_i|a_k^i|^2<\infty\,\}\tag{2.84}$$

with scalar product

$$\langle (a_k^i),(b_k^i)\rangle=\sum_k(\sum_i \bar{a}_k^i b_k^i).\tag{2.85}$$

Thus, we have

$$(\lambda_k\langle f_k,\varphi\rangle h_i\hat{u}_k^i(\lambda))\in l_2,\tag{2.86}$$

since

$$\lambda_k^2<\lambda_k\tag{2.87}$$

for k large. By a slight abuse of language we shall also call this sequence $\hat{\varphi}(\lambda)$,

$$\hat{\varphi}(\lambda)=(\lambda_k\langle f_k,\varphi\rangle h_i\hat{u}_k^i(\lambda)).\tag{2.88}$$

We are now ready to complete the proof of the theorem. Let $\lambda\in\mathit{\Lambda}$ be arbitrary, then there exists a pair (i_0,k_0) such that

$$h_{i_0}\hat{u}_{k_0}^{i_0}(\lambda)\ne 0,\tag{2.89}$$

in view of (2.73), which is now valid for any $\lambda\in\mathit{\Lambda}$. Define

$$f(\lambda)=(h_{i_0}\hat{u}_{k_0}^{i_0}(\lambda))\in l_2\tag{2.90}$$

to be the sequence with just one non-trivial term. We may consider

$$f(\lambda)\in\mathcal{S}_p'\subset\mathcal{S}'\tag{2.91}$$

by defining

$$\langle f(\lambda),\varphi\rangle=\langle f(\lambda),\hat{\varphi}(\lambda)\rangle\qquad\forall\,\varphi\in\mathcal{S},\tag{2.92}$$

where the right-hand side is the scalar product in l_2. Indeed, we obtain

$$\begin{aligned}|\langle f(\lambda),\varphi\rangle|&=\lambda_{k_0}|\langle f_{k_0},\varphi\rangle||h_{i_0}\hat{u}_{k_0}^{i_0}(\lambda)|^2\\ &\le\lambda_{k_0}|h_{i_0}\hat{u}_{k_0}^{i_0}(\lambda)|^2\|\varphi\|_p^2\quad\forall\,\varphi\in\mathcal{S}\end{aligned}\tag{2.93}$$

yielding

$$f(\lambda)\in\mathcal{S}_p'.\tag{2.94}$$

Furthermore,

$$f(\lambda) \neq 0, \tag{2.95}$$

since there exists $\varphi \in \mathcal{S}$ such that

$$\langle f_{k_0}, \varphi \rangle \neq 0, \tag{2.96}$$

in view of (2.50).

$f(\lambda)$ is also a generalized eigenvector of A with eigenvalue λ, since

$$\langle f(\lambda), A\varphi \rangle = \langle f(\lambda), \widehat{A\varphi}(\lambda) \rangle = \langle f(\lambda), \lambda \hat{\varphi}(\lambda) \rangle = \lambda \langle f(\lambda), \varphi \rangle \tag{2.97}$$

because of (2.57) and (2.59). The final conclusions are derived from Lemma 2.2. □

3. Properties of $\sigma(A)$ in the asymptotically Euclidean case

Let A be the elliptic operator

$$\begin{aligned} &-(n-1)\Delta v - \frac{n}{2} R v + \alpha_1 \frac{n}{8} F_{ij} F^{ij} v \\ &+ \alpha_2 \frac{n}{4} \gamma_{ab} g^{ij} \Phi^a_i \Phi^b_i v + \alpha_2 \frac{n}{2} m V(\Phi) v. \end{aligned} \tag{3.1}$$

We want to prove that

$$[0, \infty) \subset \sigma(A), \tag{3.2}$$

in order to be able to quantize the wave equation (1.1) on page 2. Using the results in [1] we shall show that (3.2) or even the stronger result

$$[0, \infty) \subset \sigma_{\text{ess}}(A), \tag{3.3}$$

where $\sigma_{\text{ess}}(A)$ is the essential spectrum, is valid provided the following assumptions are satisfied:

3.1. **Assumption.** We assume there exists a compact $K \subset \mathcal{S}_0$ and a coordinate system (x^i) covering $\mathcal{S}_0 \backslash K$ such that $\mathcal{S}_0 \backslash K$ is diffeomorphic with an exterior region

$$\Omega \subset \mathbb{R}^n \tag{3.4}$$

and

$$x = (x^i) \in \Omega. \tag{3.5}$$

The metric (g_{ij}) then has to satisfy

$$\lim_{|x| \to \infty} g_{ij}(x) = \delta_{ij}, \tag{3.6}$$

$$\lim_{|x| \to \infty} g_{ij,k}(x) = 0, \tag{3.7}$$

where a comma indicates partial differentiation, and there is a constant c such that

$$cr \leq |x| \leq c^{-1} r \qquad \forall x \in \Omega, \tag{3.8}$$

where r is the geometric distance to a base point $p \in K$.

Furthermore, we require that the lower order terms of A vanish at infinity, i.e.,

$$\lim_{|x|\to\infty} \{|R| + |F_{ij}F^{ij}| + |\gamma_{ab}g^{ij}\Phi_i^a\Phi_i^b| + |V(\Phi)|\} = 0. \tag{3.9}$$

Let us refer the lower order terms with the symbol $V = V(x)$ such that

$$A = (n-1)\{-\Delta + V\}, \tag{3.10}$$

then we shall prove

3.2. **Theorem.** *The operator A in (3.10) has the property*

$$[0,\infty) \subset \sigma_{\text{ess}}(A). \tag{3.11}$$

Proof. We first prove the result for the operator $(-\Delta + V)$. Let us define a positive function

$$b \in C^\infty(\mathcal{S}_0), \tag{3.12}$$

such that

$$b(x) = |x| \qquad \forall\, x \notin B_R(p), \tag{3.13}$$

where $B_R(p)$ is a large geodesic ball containing the compact set K. In view of the assumptions (3.6), (3.7) and (3.8) b satisfies the conditions (i), (ii) and (iii) in [1, Properties 2.1]. Moreover, the assumption (3.9), which implies

$$\lim_{|x|\to\infty} |V| = 0, \tag{3.14}$$

insures that the condition (iv) in [1, Theorem 2.4] can be applied yielding

$$[0,\infty) = \sigma_{\text{ess}}(-\Delta + V). \tag{3.15}$$

However, since only the inclusion

$$[0,\infty) \subset \sigma_{\text{ess}}(-\Delta + V). \tag{3.16}$$

is proved while the reverse inclusion is merely referred to, and we could not look at the given references, we shall only use (3.16). This relation is proved by constructing, for each $\epsilon > 0$ and $\lambda > 0$, an infinite dimensional subspace G_ϵ of $C_c^2(\mathcal{S}_0)$ such that

$$\int_M |(-\Delta + V - \lambda^2)v|^2 \le \epsilon^2 \int_M |v|^2 \qquad \forall\, v \in G_\epsilon. \tag{3.17}$$

Multiplying this inequality by $(n-1)^2$ we infer that (3.16) is also valid when the operator $(-\Delta + V)$ is replaced by

$$A = (n-1)(-\Delta + V) \tag{3.18}$$

proving the theorem. ∎

4. The quantization of the wave equation

The quantization of the hyperbolic equation (1.1) on page 2 will be achieved by splitting the equation into two equations: A temporal eigenvalue equation, an ODE, and a spatial elliptic eigenvalue equation.

Let us first consider the temporal eigenvalue equation

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w}+n|\Lambda|t^2w=\lambda t^{2-\frac{4}{n}}w \tag{4.1}$$

in the Sobolev space

$$H_0^{1,2}(\mathbb{R}_+^*). \tag{4.2}$$

Here,

$$\Lambda<0 \tag{4.3}$$

is a cosmological constant.

The eigenvalue problem (4.1) can be solved by considering the generalized eigenvalue problem for the bilinear forms

$$B(w,\tilde{w})=\int_{\mathbb{R}_+^*}\{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}'+n|\Lambda|t^2\bar{w}\tilde{w}\} \tag{4.4}$$

and

$$K(w,\tilde{w})=\int_{\mathbb{R}_+^*}t^{2-\frac{4}{n}}\bar{w}\tilde{w} \tag{4.5}$$

in the Sobolev space $\mathcal{H}$ which is the completion of

$$C_c^\infty(\mathbb{R}_+^*,\mathbb{C}) \tag{4.6}$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$B(w,\varphi)=\lambda K(w,\varphi)\quad\forall\,\varphi\in\mathcal{H} \tag{4.7}$$

which is equivalent to (4.1).

4.1. **Theorem.** *The eigenvalue problem (4.7) has countably many solutions (w_i,λ_i) such that*

$$0<\lambda_0<\lambda_1<\lambda_2<\cdots, \tag{4.8}$$

$$\lim\lambda_i=\infty, \tag{4.9}$$

and

$$K(w_i,w_j)=\delta_{ij}. \tag{4.10}$$

The w_i are complete in $\mathcal{H}$ as well as in $L^2(\mathbb{R}_+^*)$.

Proof. The quadratic form K is compact with respect to the quadratic form B as one can easily prove, cf. [3, Lemma 6.8], and hence a proof of the result, except for the strict inequalities in (4.8), can be found in [5, Theorem 1.6.3, p. 37]. Each eigenvalue has multiplicity one since we have a linear ODE of order two and all solutions satisfy the boundary condition

$$w_i(0) = 0. \tag{4.11}$$

The kernel is two-dimensional and the condition (4.11) defines a one-dimensional subspace. Note, that we considered only real valued solutions to apply this argument. □

The elliptic eigenvalue equation has the form

$$Av = \lambda v, \tag{4.12}$$

where A is the elliptic operator in (3.1) on page 14 and $v \in C^\infty(\mathcal{S}_0)$. A is a self-adjoint operator in $L^2(\mathcal{S}_0, \mathbb{C})$. Let

$$\mathscr{S} = \mathscr{S}(\mathcal{S}_0) \tag{4.13}$$

be the Schwartz space of rapidly decreasing smooth functions, then $\mathscr{S}$ is a separable nuclear Fréchet Hilbert space and

$$\mathscr{S} \subset L^2(\mathcal{S}_0, \mathbb{C}) \subset \mathscr{S}' \tag{4.14}$$

a Gelfand triple. Applying the results of Theorem 2.5 on page 9 we infer that there exists a complete system of eigendistributions

$$(\mathscr{E}_\lambda)_{\lambda \in \sigma(A))} \tag{4.15}$$

in $\mathscr{S}'$, i.e.,

$$Af(\lambda) = \lambda f(\lambda) \qquad \forall\, f(\lambda) \in \mathscr{E}_\lambda. \tag{4.16}$$

These eigendistributions are actually smooth functions in $\mathcal{S}_0$ with polynomial growth as we proved in [10, Theorem 3]. Assuming, furthermore, that the conditions in Assumption 3.1 on page 14 are satisfied we conclude that

$$[0, \infty) \subset \sigma_{\text{ess}}(A), \tag{4.17}$$

in view of Theorem 3.2 on page 15, i.e., the equation (4.12) is valid for all $\lambda \in \mathbb{R}_+$, and we conclude further that each temporal eigenvalue λ_i of the equation (4.1) can also be looked at as a spatial eigenvalue of the equation (4.12). Since the eigenspaces $\mathscr{E}_{\lambda_i}$ are separable we deduce that for each i there is an at most countable basis of eigendistributions in $\mathscr{E}_{\lambda_i}$

$$v_{ij} \equiv f_j(\lambda_i), \qquad 1 \le j \le n(i) \le \infty, \tag{4.18}$$

satisfying

$$Av_{ij} = \lambda_i v_{ij}, \tag{4.19}$$

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{4.20}$$

The functions

$$u_{ij} = w_i v_{ij} \tag{4.21}$$

are then smooth solutions of the wave equations. They are considered to describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

Let us summarize this result as a theorem:

4.2. **Theorem.** *Let $\mathcal{S}_0$ satisfy the conditions in Assumption 3.1 and let w_i resp. v_{ij} be the countably many solutions of the temporal resp. spatial eigenvalue problems, then*

$$u_{ij} = w_i v_{ij} \tag{4.22}$$

are smooth solutions of the wave equation. They describe the quantum development of the Cauchy hypersurface $\mathcal{S}_0$.

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