TRACE CLASS ESTIMATES AND APPLICATIONS

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Abstract. We prove trace class estimates for self-adjoint elliptic operators defined in $\mathbb{R}^n$ or $\mathbb{R}_+$. These estimates are also applicable when a physical system is governed by a wave equation by employing separation of variables to obtain corresponding temporal and spatial Hamiltonians. It is shown, in one important example, that the resulting Hamiltonians are of trace class such that quantum statistics can be applied to the system.

Contents

1. Introduction

Consider a physical system that can be described by a separable Hilbert space $\mathcal{H}$ and a self-adjoint operator H assuming that H has a pure point spectrum. If one wants to apply quantum statistics to this system, then, for any $\beta > 0$, the operator

$$e^{-\beta H} \tag{1.1}$$

has to be of trace class in $\mathcal{H}$, or, if H is extended to the corresponding symmetric Fock space $\mathcal{F}_+(\mathcal{H})$, the extended operator in (1.1) has to be of trace class in $\mathcal{F}_+(\mathcal{H})$. In case H is a Schrödinger operator or, more generally, a self-adjoint elliptic operator in a bounded domain of $\mathbb{R}^n$ with homogenous

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boundary conditions, it is well-known that the operator in (1.1) is of trace class because of Weyl's asymptotic behaviour formula for the eigenvalues λ_j,

$$\lambda_j \sim C_n(\frac{j}{V})^{\frac{2}{n}}, \tag{1.2}$$

where C_n is the so-called Weyl constant, V the Euclidean volume of the domain and the λ_j are labelled such that

$$\lambda_1 \le \lambda_2 \le \cdots \tag{1.3}$$

We prefer to start the numbering with $j = 0$ instead of $j = 1$, though this is of course irrelevant as far as the asymptotic formulas are concerned, but it might become relevant if more precise estimates are considered. Hence, when citing estimates the labelling in (1.3) will always be assumed.

Weyl used variational methods and properties of the Green's function to obtain the asymptotic estimates, cf. [12] and also [2, Kap. VI.4]. Li and Yau proved a lower bound

$$\lambda_j \ge \frac{nC_n}{n+2}(\frac{j}{V})^{\frac{2}{n}} \tag{1.4}$$

assuming the eigenvalues to be positive; they used the heat kernel for this estimate, cf. [10].

In case of unbounded domains we do not know of any asymptotic or lower estimates which would imply the operator in (1.1) to be of trace class—apart from special cases, when the eigenvalues are explicitly known.

In this paper we shall consider self-adjoint elliptic differential operators defined in $\mathbb{R}_+$ or $\mathbb{R}^n$, $n \ge 2$, and shall prove, by imposing reasonable assumptions, that the operator in (1.1) is of trace class. The proof will not rely on showing either asymptotic or explicit lower estimates but we shall instead construct explicit majorants the existence of which will infer

$$\operatorname{tr}(e^{-\beta H}) < \infty. \tag{1.5}$$

One crucial ingredient in the proof is a generalization of Maurin's Hilbert-Schmidt type embedding theorem, cf. [11, Theorem 1, p. 336], to unbounded domains with special weighted measures combined with an interpolation inequality involving the norm of the target space of the Hilbert-Schmidt embedding.

These new trace class estimates can especially be applied when the physical system is defined by a wave equation, which is either obtained by a classical description or is the result of a (first) quantization process. In either case it is worthwhile to use, if possible, a separation of variables to split a solution u of the wave equation into a product

$$u(t,x) = w(t)v(x) \tag{1.6}$$

and then finding temporal and spatial self-adjoint operators H_0 resp. H_1 such that one of them has a pure point spectrum with eigenvalues λ_i while, for the other operator, it is possible to find corresponding eigendistributions for each of the eigenvalues λ_i. Assuming, e.g., that H_0 has a pure point spectrum with corresponding mutually orthogonal eigenfunctions w_i and H_1 has smooth eigendistributions v_{ij} satisfying

$$H_1 v_{ij} = \lambda_i v_{ij} \qquad \forall\, j \tag{1.7}$$

then

$$u_{ij} = w_i v_{ij} \tag{1.8}$$

would be solutions of the wave equation. Weyl used this approach to analyze the radiation of a black body, cf. [12, Kap. 6], though in this case the spatial Hamiltonian H_1 had a pure point spectrum and the temporal Hamiltonian H_0, which was just the classical harmonic oscillator,

$$H_0 w = -\ddot{w}, \tag{1.9}$$

had only a continuous spectrum.

We are especially interested in a wave equation which we obtained, in our model of quantum gravity, as the result of a canonical quantization process applied to a globally hyperbolic spacetime with a cosmological constant. This wave equation has the form

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{1.10}$$

and is defined in a quantum spacetime

$$N = \mathbb{R}_+ \times \mathcal{S}_0, \tag{1.11}$$

where $\mathcal{S}_0$ is a n-dimensional, $n \ge 3$, Cauchy hypersurface of the original spacetime, or, in case of black holes, the smooth limit of Cauchy hypersurfaces. The Laplacian and the scalar curvature correspond to the metric σ_{ij} in $\mathcal{S}_0$, cf. [5, Theorem 6.9], where we derived this wave equation after a canonical quantization process, see also [4]. The cosmological constant Λ is supposed to be negative. We applied this model to a Schwarzschild-AdS resp. Kerr-AdS black hole and to a globally hyperbolic spacetime with an asymptotic Euclidean Cauchy hypersurface. In all three cases we obtained a sequence of smooth functions as solutions of the wave equation which are a product of temporal eigenfunctions and spatial eigendistributions.

In case of the globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface the solutions to the wave equation can be expressed in the form

$$u_{ij} = w_i v_{ij}, \qquad i \in \mathbb{N},\ 1 \le j \le m \le \infty, \tag{1.12}$$

where the w_i are the eigenfunctions of a temporal Hamilton operator H_0

$$H_0 w_i = \lambda_i w_i \tag{1.13}$$

and the λ_i have multiplicity one such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{1.14}$$

and for each fixed i the, at most countably many, v_{ij} generate an eigenspace

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{1.15}$$

of a spatial Hamiltonian H_1, i.e.,

$$H_1 v_{ij} = \lambda_i v_{ij}. \tag{1.16}$$

We have

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{1.17}$$

In case of the black holes the description is a bit more complicated and we refer the reader to Section 4, where it is also proved that the trace class estimates can be applied to both the temporal as well as to the spatial Hamiltonian.

Let us now give a more detailed summary of our results. First, for the general trace class estimates. We consider eigenvalue problems in $\mathbb{R}^n$, $n \geq 2$. Let A be the linear elliptic operator

$$Au = -D_i(a^{ij} D_j u) + b(x)u, \tag{1.18}$$

where

$$a^{ij}, b \in L^\infty_{\text{loc}}(\mathbb{R}^n), \tag{1.19}$$

a^{ij} is symmetric and we assume there exists $a_0 > 0$ such that

$$a_0 |\xi|^2 \leq a^{ij} \xi_i \xi_j \qquad \forall\, \xi \in \mathbb{R}^n \tag{1.20}$$

and that there exists $R_0 > 1$ and positive p, c_1 such that

$$c_1 |x|^p \leq b(x) \qquad \forall\, |x| \geq R_0. \tag{1.21}$$

Then, we proved:

1.1. **Theorem.** *The operator A is essentially self-adjoint in $\mathcal{H} = L^2(\mathbb{R}^n)$ with a pure point spectrum*

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \tag{1.22}$$

Let H be its self-adjoint extension then, for any $\beta > 0$,

$$e^{-\beta H} \tag{1.23}$$

is of trace class in $\mathcal{H}$.

Next, let us consider a Sturm-Liouville operator A in $\mathbb{R}_+$ of the form

$$Au = -(au')' + bu, \tag{1.24}$$

where a dot or a prime indicates differentiation, and corresponding eigenvalue problems

$$Au = \lambda \varphi_0 u, \tag{1.25}$$

where the coefficients a, b and the function φ_0 are all measurable and locally bounded in $\mathbb{R}_+$, and b is even locally bounded in $[0, \infty)$, and they satisfy

$$a(t) \geq a_0 > 0 \qquad \forall\, t \in \mathbb{R}_+, \tag{1.26}$$

and there exist positive constants c_1, c_2, p, r and $t_0 > 1$ such that

$$b(t) \geq c_1 t^p \qquad \forall\, t \geq t_0, \tag{1.27}$$

$$\varphi_0(t) \leq c_2 t^r \qquad \forall\, t \geq t_0, \tag{1.28}$$

and

$$0 < r < p, \tag{1.29}$$

where the function φ_0 is also positive almost everywhere. Then we proved:

1.2. **Theorem.** *The eigenvalue problem*

$$Au = \lambda \varphi_0 u \tag{1.30}$$

has countably many solutions (λ_i, w_i) *such that*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1.31}$$

and the w_i *form an ONB in*

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \tag{1.32}$$

$$d\mu = \varphi_0 dt. \tag{1.33}$$

The operator

$$\varphi_0^{-1} A \tag{1.34}$$

is essentially self-adjoint in $\mathcal{H}$*. Let* H_0 *be its self-adjoint extension then, for any* $\beta > 0$*,*

$$e^{-\beta H_0} \tag{1.35}$$

is of trace class in $\mathcal{H}$*.*

Finally, let us describe the results with respect to the wave equation (1.10). In Section 4 we shall prove that the wave equation can be expressed in the form

$$\varphi_0(H_0 u - H_1 u) = 0, \tag{1.36}$$

where $u = u(t,x)$ is a smooth function, $x \in \mathcal{S}_0$ and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{1.37}$$

H_0 is an operator which satisfies the assumptions in the previous theorem and in Section 5 we shall define an abstract Hilbert space $\mathcal{H}$, where the eigendistributions of H_1 form an ONB, such that H_0 and H_1 have the same eigenvalues but with different multiplicities. H_1 is also essentially self-adjoint in $\mathcal{H}$. Let $\tilde{H}_1$ be the unique self-adjoint extension of H_1, namely its closure, then we shall prove that for any $\beta > 0$

$$e^{-\beta \tilde{H}_1} \tag{1.38}$$

is of trace class in $\mathcal{H}$. In addition $\tilde{H}_1$ satisfies

$$\tilde{H}_1 \geq \lambda_0 I, \qquad \lambda_0 > 0. \tag{1.39}$$

Let

$$H \equiv d\varGamma(\tilde{H}_1) \tag{1.40}$$

be the canonical extension of $\tilde{H}_1$ to the symmetric Fock space

$$\mathscr{F} = \mathscr{F}_+(\mathcal{H}), \tag{1.41}$$

then

$$e^{-\beta H} \tag{1.42}$$

is of trace class in $\mathscr{F}$ because of (1.38) and (1.39), cf. [1, Prop. 5.2.27]. Hence we can define the partition function

$$Z = \operatorname{tr}(e^{-\beta H}), \tag{1.43}$$

the density operator

$$\rho = Z^{-1}e^{-\beta H} \tag{1.44}$$

and the von Neumann entropy

$$S = -\operatorname{tr}(\rho \log \rho) = \log Z + \beta E, \tag{1.45}$$

where E is the average energy and $\beta > 0$ the inverse temperature

$$\beta = T^{-1}. \tag{1.46}$$

Here is a summary of the results derived in Section 5:

1.3. **Theorem.** (i) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{1.47}$$

we have

$$\lim_{\Lambda \to 0} E = \infty \tag{1.48}$$

as well as

$$\lim_{\Lambda \to 0} S = \infty, \tag{1.49}$$

where the limites are uniform in β.

(ii) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{1.50}$$

we have

$$\lim_{|\Lambda| \to \infty} E = 0 \tag{1.51}$$

as well as

$$\lim_{|\Lambda| \to \infty} S = 0, \tag{1.52}$$

where the limites are uniform in β.

The behaviour of Z with respect to Λ is described in the theorem:

1.4. **Theorem.** *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{1.53}$$

we have

$$\lim_{\Lambda \to 0} Z = \infty \tag{1.54}$$

and for any

$$\beta_0 \le \beta \tag{1.55}$$

the relation

$$\lim_{|\Lambda| \to \infty} Z = 1 \tag{1.56}$$

is valid. The convergence in both limites is uniform in β.

1.5. **Remark.** The first part of Theorem 1.3 reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density we consider the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho\eta = Z^{-1}\eta, \tag{1.57}$$

i.e., the dark energy density should be proportional to Z^{-1}.

In Section 6 we also applied quantum statistics to the quantized version of a Friedmann universe and proved:

1.6. **Theorem.** *The results in the last two theorems and the conjectures in the remark above are also valid, if the quantized spacetime $N = N^{n+1}$, $n \ge 3$, is a Friedmann universe without matter but with a negative cosmological constant Λ and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian H_1 all have multiplicity one.*

1.7. **Remark.** Let us also mention that we use Planck units in this paper, i.e.,

$$c = G = \hbar = K_B = 1. \tag{1.58}$$

2. Trace class estimates in $\mathbb{R}_+$

Let us first consider a Sturm-Liouville operator A in $\mathbb{R}_+$ of the form

$$Au = -(au')' + bu, \tag{2.1}$$

where a dot or a prime indicates differentiation, and corresponding eigenvalue problems

$$Au = \lambda\varphi_0 u, \tag{2.2}$$

where the coefficients a, b and the function φ_0 are all measurable and locally bounded in $\mathbb{R}_+$, and b is even locally bounded in $[0,\infty)$, and they satisfy

$$a(t) \ge a_0 > 0 \qquad \forall\, t \in \mathbb{R}_+, \tag{2.3}$$

and there exist positive constants c_1, c_2, p, r and $t_0 > 1$ such that

$$b(t) \ge c_1 t^p \qquad \forall\, t \ge t_0, \tag{2.4}$$

$$\varphi_0(t) \le c_2 t^r \qquad \forall\, t \ge t_0, \tag{2.5}$$

and

$$0 < r < p, \tag{2.6}$$

where φ_0 is also assumed to be positive almost everywhere (a.e.), and where the specification

$$\forall\, t \ge t_0 \tag{2.7}$$

means

$$\text{a.e. in } \{t \ge t_0\} \tag{2.8}$$

when used in connection with measurable functions which are not assumed to be continuous.

We define the bilinear forms

$$B(u,v) = \langle Au, v\rangle = \int_{\mathbb{R}_+} \{a\bar{u}'v' + b\bar{u}v\} \tag{2.9}$$

and

$$K(u,v) = \int_{\mathbb{R}_+} \varphi_0 \bar{u}v \tag{2.10}$$

for

$$u, v \in C_c^\infty(\mathbb{R}_+, \mathbb{C}), \tag{2.11}$$

and we denote the corresponding quadratic forms by $B(u)$ resp. $K(u)$.

2.1. **Lemma.** *Define*

$$b_0(t) = \begin{cases} 0, & 0 \le t < t_0, \\ b(t), & t_0 \le t, \end{cases} \tag{2.12}$$

and

$$B_0(u) = \int_{\mathbb{R}_+} \{a|u'|^2 + b_0|u|^2\}, \tag{2.13}$$

then, for any $\epsilon > 0$*, there exists* c_ϵ *such that*

$$\|u\|_2^2 = \int_{\mathbb{R}_+} |u|^2 \le \epsilon B_0(u) + c_\epsilon K(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+). \tag{2.14}$$

Proof. This compactness lemma is well-known. However, we give a short proof for the convenience of the reader. We argue by contradiction and assume there would exist $\epsilon > 0$ and a sequence

$$u_k \in C_c^\infty(\mathbb{R}_+) \tag{2.15}$$

such that

$$\|u\|_2^2 > \epsilon B_0(u_k) + kK(u_k). \tag{2.16}$$

Without loss of generality we may assume that

$$\|u_k\|_2^2 = 1. \tag{2.17}$$

Hence the u_k would be uniformly bounded in the Sobolev space

$$H^{1,2}(J) \tag{2.18}$$

with norm

$$\|u\|_{1,2}^2 = \int_J (|u'|^2 + |u|^2), \tag{2.19}$$

for any bounded interval

$$J \Subset [0, \infty), \tag{2.20}$$

and we would deduce

$$\lim_{k\to\infty} K(u_k) = 0. \tag{2.21}$$

Moreover, by applying the Sobolev embedding theorem, we would know that a subsequence, not relabelled, would converge strongly in any

$$L^2(J, \mathbb{C}) \tag{2.22}$$

to a function u. In view of Fatou's lemma, we would also infer

$$K(u) \le \lim K(u_k) = 0 \tag{2.23}$$

and thus

$$u \equiv 0. \tag{2.24}$$

But this would lead to a contradiction, since, for any $m > t_0$, we would have

$$\begin{aligned} 1 &= \int_0^m |u_k|^2 + \int_m^\infty |u_k|^2 \\ &\le \int_0^m |u_k|^2 + c_1^{-1} m^{-p} \int_m^\infty b_0 |u_k|^2 \\ &\le \int_0^m |u_k|^2 + c_1^{-1} m^{-p} \limsup B_0(u_k) \end{aligned} \tag{2.25}$$

yielding

$$1 \le c_1^{-1} m^{-p} \limsup B_0(u_k) \le c_1^{-1} m^{-p} \epsilon^{-1} \qquad \forall\, m \ge t_0, \tag{2.26}$$

in view of (2.16) and (2.17). □

As an immediate corollary we obtain

2.2. **Corollary.** *There exists a positive constant c_0 such that*

$$\|u\|^2 \equiv \|u\|_2^2 \le B(u) + c_0 K(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+) \tag{2.27}$$

and

$$\frac{1}{2} B_0(u) \le B(u) + c_0 K(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+). \tag{2.28}$$

Proof. Since b is bounded in $I = [0, t_0]$ we conclude, in view of (2.14),

$$\begin{aligned} B(u) &\ge B_0(u) - c\|u\|_2^2 \\ &\ge B_0(u) - c\epsilon B_0(u) - cc_\epsilon K(u) \\ &= (1 - c\epsilon) B_0(u) - cc_\epsilon K(u) \\ &\ge \|u\|_2^2 - c_0 K(u), \end{aligned} \tag{2.29}$$

if we choose

$$\epsilon = \frac{1}{2c} \tag{2.30}$$

and c_0 appropriately, proving both estimates. □

In view of the Poincaré inequality on bounded intervals, we also conclude that there exists $c > 0$ such that

$$\|u\|_{1,2}^2 \le c B_0(u) \qquad \forall\, u \in C_c^\infty(\mathbb{R}_+). \tag{2.31}$$

2.3. **Definition.** We define the Hilbert space $\mathcal{H}_1$ as the completion of $C_c^\infty(\mathbb{R}_+)$ with respect to the scalar product defined by the bilinear form

$$B + c_0 K, \tag{2.32}$$

cf. Corollary 2.2, and we denote this scalar product by the symbol

$$\langle \cdot, \cdot \rangle_1 \tag{2.33}$$

and corresponding norm

$$\|\cdot\|_1. \tag{2.34}$$

The Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \tag{2.35}$$

where

$$d\mu = \varphi_0(t)dt. \tag{2.36}$$

The corresponding scalar product is K and it is also characterized by the symbol

$$\langle \cdot, \cdot \rangle \tag{2.37}$$

and corresponding norm

$$\|\cdot\|. \tag{2.38}$$

Using the arguments in the proof of Lemma 2.1, the results of Corollary 2.2 and the assumptions (2.5) and (2.6) we immediately obtain:

2.4. **Lemma.** *The embedding*

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H} \tag{2.39}$$

is compact, i.e., if $u_k \in \mathcal{H}_1$ converges weakly to u

$$u_k \rightharpoonup u, \tag{2.40}$$

then

$$j(u_k) \to j(u). \tag{2.41}$$

We conclude further that the generalized eigenvalue problem

$$B(u, v) = \lambda K(u, v) \qquad \forall\, v \in \mathcal{H}_1 \tag{2.42}$$

can be solved by a variational process which goes back to Courant-Hilbert [2, Kap. 6]. We describe it in the following theorem:

2.5. **Theorem.** *Let $\mathcal{H}$ be a complex, separable Hilbert space, B and K sesquilinear, symmetric forms on $\mathcal{H}$ and assume there exists a positive constant c_0 such that*

$$B + c_0 K \tag{2.43}$$

is an equivalent scalar product in $\mathcal{H}$. K is also supposed to be a compact form in $\mathcal{H}$, i.e.,

$$u_k \rightharpoonup u \implies K(u_k) \to K(u). \tag{2.44}$$

Then the eigenvalue problem

$$B(u, v) = \lambda K(u, v) \qquad \forall\, v \in \mathcal{H}_1 \tag{2.45}$$

has countably many eigenvalues with finite multiplicities. If we label the eigenvectors such that

$$\lambda_0 \le \lambda_1 \le \cdots \tag{2.46}$$

then

$$\lim_{i\to\infty} \lambda_i = \infty, \tag{2.47}$$

and

$$-c_0 < \lambda_0. \tag{2.48}$$

There exists a sequence of corresponding eigenvectors u_i which are complete in $\mathcal{H}$ satisfying

$$K(u_i, u_j) = \delta_{ij} \tag{2.49}$$

and

$$B(u_i, u_j) = \lambda_i K(u_i, u_j) \tag{2.50}$$

as well as the expansion

$$B(u, v) = \sum_i \lambda_i K(u, u_i) K(u_i, v) \tag{2.51}$$

and

$$K(u, v) = \sum_i K(u, u_i) K(u_i, v). \tag{2.52}$$

The pairs (λ_i, u_i) are defined by the variational problems

$$\begin{aligned} \lambda_i &= \inf\{ \frac{B(u)}{K(u)} : 0 \ne u \in \mathcal{H},\, K(u, u_j) = 0 \quad \forall\, 0 \le j \le i-1 \} \\ &= B(u_i, u_i). \end{aligned} \tag{2.53}$$

This theorem is well-known. A proof can be found in [3, Theorem 1.6.3].

We apply this theorem to the previously defined Hilbert space $\mathcal{H}_1$ and the bilinear (sesquilinear) forms B and K. Let (λ_i, w_i) be the corresponding pairs of eigenvalues and eigenvectors, then the w_i satisfy the ODE

$$A w_i = \lambda_i \varphi_0 w_i \tag{2.54}$$

in the weak sense. The operator

$$\tilde{A} = \varphi_0^{-1} A \tag{2.55}$$

is symmetric in

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \qquad d\mu = \varphi_0 dt, \tag{2.56}$$

and the w_i are eigenfunctions of $\tilde{A}$

$$\tilde{A} w_i = \lambda_i w_i. \tag{2.57}$$

The equation (2.54) is equivalent to

$$\varphi_0 \tilde{A} w_i = \lambda_i \varphi_0 w_i \tag{2.58}$$

and $\tilde{A}$, with domain

$$D(\tilde{A}) = \langle w_i : i \in \mathbb{N} \rangle \subset \mathcal{H}, \tag{2.59}$$

is essentially self-adjoint as will be proved later, Lemma 5.1 on page 37, in a more general setting. We denote its unique self-adjoint extension by H_0.

We shall now prove that

$$e^{-\beta H_0}, \qquad \beta > 0, \tag{2.60}$$

is of trace class in $\mathcal{H}$.

First, we need two lemmata:

2.6. **Lemma.** *The embedding*

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 = L^2(\mathbb{R}_+, d\tilde{\mu}), \tag{2.61}$$

where

$$d\tilde{\mu} = (1+t)^{-2} dt, \tag{2.62}$$

is Hilbert-Schmidt.

Proof. Maurin was the first to prove that the embedding

$$H^{m,2}(\Omega) \hookrightarrow L^2(\Omega), \tag{2.63}$$

where

$$\Omega \subset \mathbb{R}^n \tag{2.64}$$

is a bounded domain, is Hilbert-Schmidt provided

$$m > \frac{n}{2}, \tag{2.65}$$

cf. [11, Theorem 1, p. 336]. We adapt his proof to the present situation.

Let $w \in \mathcal{H}_1$, then, assuming w is real valued,

$$\begin{aligned} |w(t)|^2 = 2\int_0^t \dot{w}w &\le 2\int_o^\infty |\dot{w}|^2 + \frac{1}{2}\int_0^\infty |w|^2 \\ &\le c\|w\|_1^2 \end{aligned} \tag{2.66}$$

for all $t > 0$, where $\|\cdot\|_1$ is the norm in $\mathcal{H}_1$. To derive the last inequality in (2.66) we used Corollary 2.2. The estimate

$$|w(t)| \le c\|w\|_1 \qquad \forall\, t > 0 \tag{2.67}$$

is of course also valid for complex valued functions from which infer that, for any $t > 0$, the linear form

$$w \to w(t), \qquad w \in \mathcal{H}_1, \tag{2.68}$$

is continuous, hence it can be expressed as

$$w(t) = \langle \varphi_t, w \rangle, \tag{2.69}$$

where

$$\varphi_t \in \mathcal{H}_1 \tag{2.70}$$

and

$$\|\varphi_t\|_1 \le c. \tag{2.71}$$

Now, let

$$e_i \in \mathcal{H}_1 \tag{2.72}$$

be an ONB, then

$$\sum_{i=0}^{\infty} |e_i(t)|^2 = \sum_{i=0}^{\infty} |\langle \varphi_t, e_i\rangle|^2 = \|\varphi_t\|_1^2 \le c^2. \tag{2.73}$$

Integrating this inequality over $\mathbb{R}_+$ with respect to $d\tilde{\mu}$ we infer

$$\sum_{i=0}^{\infty} \int_0^\infty |e_i(t)|^2 d\tilde{\mu} \le c^2 \tag{2.74}$$

completing the proof of the lemma. □

2.7. **Lemma.** *Let w_i be the eigenfunctions of H_0, then there exist positive constants c and γ such that*

$$\|w_i\|_1 \le c|\lambda_i + c_0|^\gamma \|w_i\|_0 \qquad \forall\, i \in \mathbb{N}, \tag{2.75}$$

where $\|\cdot\|_0$ is the norm in $\mathcal{H}_0$.

Proof. We have, in view of (2.32) and (2.5),

$$\begin{aligned} \|w_i\|_1^2 &= (\lambda_i + c_0) \int_0^\infty \varphi_0(t)|w_i|^2 \\ &\le (\lambda_i + c_0)\Big\{\int_0^{t_0} \varphi_0(t)|w_i|^2 + c_2 \int_{t_0}^\infty t^r |w_i|^2\Big\}. \end{aligned} \tag{2.76}$$

To estimate the second integral in the braces we exploit the assumptions (2.4) and (2.6) and choose m so large that

$$r \le p - \frac{p}{m}, \tag{2.77}$$

and hence,

$$t^r \le t^{p-\frac{p}{m}} \qquad \forall\, t \ge t_0 > 1. \tag{2.78}$$

Then, choosing small positive constants δ and ϵ, we apply Young's inequality, with

$$q = \frac{p}{p - p\delta} = \frac{1}{1-\delta} \tag{2.79}$$

and

$$q' = \delta^{-1} \tag{2.80}$$

to estimate the integral from above by

$$\begin{aligned} &\frac{1}{q}\epsilon^q \int_{t_0}^\infty \{t^{p-\frac{p}{m}}(1+t)^{\frac{p}{m}-p\delta}\}^q |w_i|^2 \\ &\qquad + \frac{1}{q'}\epsilon^{-q'} \int_{t_0}^\infty (1+t)^{-(\frac{p}{m}-p\delta)q'} |w_i|^2. \end{aligned} \tag{2.81}$$

Choosing, now, δ so small such that

$$(\frac{p}{m} - p\delta)\delta^{-1} > 2 \tag{2.82}$$

the preceding integrals can be estimated from above by

$$\frac{1}{q}\epsilon^q \int_{t_0}^\infty (1+t)^p |w_i|^2 + \frac{1}{q'}\epsilon^{-q'} \int_0^\infty (1+t)^{-2}|w_i|^2 \tag{2.83}$$

which in turn can be estimated by

$$\frac{1}{q}\epsilon^q c\|w_i\|_1^2 + \frac{1}{q'}\epsilon^{-q'}\|w_i\|_0^2, \tag{2.84}$$

in view of (2.27).

The first integral in the braces on the right-hand side of (2.76) can be estimated by

$$\begin{aligned} \int_0^{t_0} \varphi_0(t)|w_i|^2 &\le \frac{1}{2}c(1+t_0)^2\epsilon^2 \int_0^\infty |w_i|^2 \\ &\quad + \frac{1}{2}\epsilon^{-2}\int_0^\infty (1+t)^{-2}|w_i|^2 \\ &\le \tilde{c}\epsilon^2\|w_i\|_1^2 + \frac{1}{2}\epsilon^{-2}\|w_i\|_0^2, \end{aligned} \tag{2.85}$$

because of (2.27).

Choosing now ϵ, γ and c appropriately the result follows. □

We are now ready to prove:

2.8. **Theorem.** *Let $\beta > 0$, then the operator*

$$e^{-\beta H_0} \tag{2.86}$$

is of trace class in $\mathcal{H}$, i.e.,

$$\operatorname{tr}(e^{-\beta H_0}) = \sum_{i=0}^\infty e^{-\beta\lambda_i} = c(\beta) < \infty. \tag{2.87}$$

Proof. In view of Lemma 2.6 the embedding

$$j : \mathcal{H}_1 \hookrightarrow \mathcal{H}_0 \tag{2.88}$$

is Hilbert-Schmidt. Let

$$w_i \in \mathcal{H} \tag{2.89}$$

be an ONB of eigenfunctions, then

$$\begin{aligned} e^{-\beta\lambda_i} &= e^{-\beta\lambda_i}\|w_i\|^2 = e^{-\beta\lambda_i}|\lambda_i + c_0|^{-1}\|w_i\|_1^2 \\ &\le e^{\beta c_0}e^{-\beta(\lambda_i + c_0)}|\lambda_i + c_0|^{-1}c|\lambda_i + c_0|^{2\gamma}\|w_i\|_0^2, \end{aligned} \tag{2.90}$$

in view of (2.75), but

$$\|w_i\|_0^2 = \|w_i\|_1^2\,\|\tilde{w}_i\|_0^2 = (\lambda_i + c_0)\|\tilde{w}_i\|_0^2, \tag{2.91}$$

where

$$\tilde{w}_i = w_i \|w_i\|_1^{-1} \tag{2.92}$$

is an ONB in $\mathcal{H}_1$, yielding

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} \le c_\beta \sum_{i=0}^{\infty} \|\tilde{w}_i\|_0^2 < \infty, \tag{2.93}$$

since j is Hilbert-Schmidt. □

3. Trace class estimates in $\mathbb{R}^n$

Let us now consider eigenvalue problems in $\mathbb{R}^n$, $n \ge 2$, and let A be the linear elliptic operator

$$Au = -D_i(a^{ij} D_j u) + b(x)u, \tag{3.1}$$

where

$$a^{ij}, b \in L^\infty_{\text{loc}}(\mathbb{R}^n), \tag{3.2}$$

a^{ij} is symmetric and there exists $a_0 > 0$ such that

$$a_0 |\xi|^2 \le a^{ij} \xi_i \xi_j \qquad \forall\, \xi \in \mathbb{R}^n \tag{3.3}$$

and there exists $R_0 > 1$ and positive p, c_1 such that

$$c_1 |x|^p \le b(x) \qquad \forall\, |x| \ge R_0. \tag{3.4}$$

Then, we look at the eigenvalue problem

$$Au = \lambda u. \tag{3.5}$$

This eigenvalue problem can be solved by similar, if not identical, arguments as in the case of the Sturm-Liouville operator.

We define the bilinear forms

$$B(u,v) = \int_{\mathbb{R}^n} a^{ij} D_i \bar{u} D_j v \tag{3.6}$$

and

$$K(u,v) = \int_{\mathbb{R}^n} \bar{u} v \tag{3.7}$$

in $C_c^\infty(\mathbb{R}^n, \mathbb{C})$, and one can easily prove the analogues of Corollary 2.2 on page 10 and Theorem 2.5 on page 11, i.e., there exists $c_0 > 0$ such that

$$B + c_0 K \ge K, \tag{3.8}$$

K is compact relative to $B + c_0 K$, and there exists countably many pairs (λ_i, u_i) of eigenvalues with corresponding eigenfunctions satisfying the properties specified in Theorem 2.5, and we shall now prove that

$$e^{-\beta H}, \qquad \beta > 0, \tag{3.9}$$

is of trace class, where

$$H = \bar{A} \tag{3.10}$$

is the unique self-adjoint extension of A. We recall that A satisfies the estimate (2.28) on page 10 which can be rephrased as

$$A + c_0 \geq \frac{1}{2}\{-D_i(a^{ij}D_j) + b_0\}, \tag{3.11}$$

where

$$b_0(x) = \begin{cases} 0, & |x| \leq R_0, \\ b(x), & |x| > R_0. \end{cases} \tag{3.12}$$

The right-hand side of (3.11) is a strictly positive operator. Since eigenvalues, obtained by the variational process described in Theorem 2.5, also satisfy a minimax principle, cf. e.g., [3, Theorem 1.6.4], we conclude that

$$\mu_i \leq \tilde{\lambda}_i \qquad \forall\, i \in \mathbb{N}, \tag{3.13}$$

where μ_i are the ordered eigenvalues of the operator on the right-hand side of (3.11) and $\tilde{\lambda}_i$ the ordered eigenvalues of $A + c_0$. Hence, it suffices to prove that

$$\sum_{i=0}^{\infty} e^{-\beta\mu_i} < \infty. \tag{3.14}$$

For reasons that will become apparent later, we shall derive trace class estimates for the operator

$$\tilde{A}u = -\alpha_0 \Delta u + \Theta u, \tag{3.15}$$

where

$$\alpha_0 = \frac{a_0}{2}, \tag{3.16}$$

$$\Theta(x) = \frac{c_1}{2}\eta_0 |x|^{p_0}, \tag{3.17}$$

$$p_0 = \min(p, 1) \tag{3.18}$$

and η_0 is a cut-off function such that

$$\eta_0(x) = \begin{cases} 0, & |x| \leq R_0, \\ 1, & |x| \geq 2R_0. \end{cases} \tag{3.19}$$

We emphasize that

$$\Theta \leq \frac{1}{2} b_0 \tag{3.20}$$

and hence, due to the inequalities (3.3) and (3.11),

$$A + c_0 \geq \tilde{A}. \tag{3.21}$$

Therefore, it will suffice to prove that $\tilde{A}$ is a trace class operator. To simplify notations let us also drop the tilde and let us write A for the operator in (3.15), i.e.,

$$Au = -\alpha_0 \Delta u + \Theta u. \tag{3.22}$$

Furthermore, the previous definitions of the bilinear form B and the Hilbert space $\mathcal{H}_1$ are also adopted while the Hilbert space $\mathcal{H}$ is now $L^2(\mathbb{R}^n)$. A is essentially self-adjoint in $\mathcal{H}$ with domain

$$D(A) = \langle u_i : i \in \mathbb{N} \rangle, \tag{3.23}$$

where u_i are a sequence of mutually orthogonal eigenfunctions of A

$$Au_i = \lambda_i u_i. \tag{3.24}$$

Note that

$$0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \tag{3.25}$$

We shall first prove that the eigenfunctions of A are smooth with uniformly bounded norms

$$\|u_i\|^2_{m,2} = \sum_{|\alpha| \le m} \int_{\mathbb{R}^n} |D^\alpha u|^2 \tag{3.26}$$

in the usual Sobolev spaces $H^{m,2}(\mathbb{R}^n)$.

3.1. **Theorem.** *Let $u \in H^{m-1,2}(\mathbb{R}^n) \cap \mathcal{H}_1$ be a weak solution of the equation*

$$-\alpha_0 \Delta u + \Theta u = f, \tag{3.27}$$

where $f \in H^{m-2,2}(\mathbb{R}^n)$, $m \ge 2$, and assume that

$$\|u\|^2_{m-1,2} + \sum_{|\alpha| \le m-2} \int_{\mathbb{R}^n} \Theta |D^\alpha u|^2 \le c \|f\|^2_{m-3,2}, \tag{3.28}$$

then $u \in H^{m,2}(\mathbb{R}^n)$ and

$$\|u\|^2_{m,2} + \sum_{|\alpha| \le m-1} \int_{\mathbb{R}^n} \Theta |D^\alpha u|^2 \le c \|f\|^2_{m-2,2}, \tag{3.29}$$

where the constants c depend on m, Θ, p_0, n and α_0.

Proof. We shall prove the theorem by induction. First, in the lemma below we shall prove that the theorem is valid for $m = 2$. Thus, let us assume that the theorem is correct for $m = q \ge 2$ and show that it is then also valid for $m = q + 1$.

Fix $1 \le k \le n$ and define

$$v = D_k u. \tag{3.30}$$

Differentiating (3.27) we obtain

$$-\alpha_0 \Delta v + \Theta v = D_k f - D_k \Theta u \equiv \tilde{f}. \tag{3.31}$$

We observe that

$$\tilde{f} \in H^{q-2,2}(\mathbb{R}^n) \tag{3.32}$$

and that

$$\|\tilde{f}\|^2_{q-2,2} \le c \|f\|^2_{q-1,2}, \tag{3.33}$$

because

$$
\begin{aligned}
\|D_k\Theta u\|^2_{q-2,2} &\le c\{\|u\|_{q-2,2} + \sum_{|\alpha|\le q-2}\Theta|D^\alpha u|^2\} \\
&\le c\{\|u\|_{q,2} + \sum_{|\alpha|\le q-1}\int_{\mathbb{R}^n}\Theta|D^\alpha u|^2\} \qquad (3.34)\\
&\le c\|f\|^2_{q-2,2}
\end{aligned}
$$

in view of the definition of Θ and (3.29). Applying then the induction hypothesis for $m = q$ we conclude that the theorem is also valid for $m = q+1$. □

3.2. **Lemma.** *The preceding theorem is valid for $m = 2$, i.e., any weak solution $u \in \mathcal{H}_1$ of*

$$-\alpha_0\Delta u + \Theta u = f \tag{3.35}$$

satisfies the estimates (3.28) *and* (3.29)*, where we note that*

$$H^{-1,2}(\mathbb{R}^n) = \{\, D_i g^i + g_0 \colon g_0, g^i \in L^2(\mathbb{R}^n)\,\} \tag{3.36}$$

is the dual space of $H^{1,2}(\mathbb{R}^n)$ and

$$L^2(\mathbb{R}^n) \hookrightarrow H^{-1,2}(\mathbb{R}^n) \subset \mathcal{H}_1'. \tag{3.37}$$

The equation (3.35) *has also a unique solution which can be found by minimizing a functional if we consider f and u to be real valued. Of course we then also obtain a solution for complex valued f.*

Proof. First, the existence of a solution $u \in \mathcal{H}_1$ of (3.35) satisfying

$$B(u) = \langle Au, u\rangle \le c\|f\|^2 \tag{3.38}$$

is obvious, since

$$K(v) = \|v\|^2 \tag{3.39}$$

is compact relative to B, and for real valued f and v and $\epsilon > 0$ we have

$$
\begin{aligned}
|\langle f, v\rangle| &\le \frac{1}{2}\epsilon\|v\|^2 + \frac{1}{2}\epsilon^{-1}\|f\|^2 \\
&\le \frac{1}{2}\epsilon\lambda_0^{-1}B(v) + \frac{1}{2}\epsilon^{-1}\|f\|^2,
\end{aligned}
\tag{3.40}
$$

where $0 < \lambda_0$ is the smallest eigenvalue of A. It then immediately follows that the variational problem

$$J(v) = B(v) - 2\langle f, v\rangle \to \min \qquad \forall\, v \in \mathcal{H}_1 \tag{3.41}$$

has a unique solution u, which is also a weak solution of the corresponding Euler-Lagrange equation, and that u satisfies (3.38) which is equivalent to (3.28) for $m = 2$.

Secondly, to prove (3.29) for $m = 2$ we note that

$$u \in C^\infty(\mathbb{R}^n), \tag{3.42}$$

in view of the interior L^2-estimates, since A is uniformly elliptic with smooth coefficients. Hence, choosing a cut-off function η

$$0 \le \eta \in C_c^\infty(\mathbb{R}^n) \tag{3.43}$$

such that

$$|D\eta| \le 2 \tag{3.44}$$

and $1 \le k \le n$ we have

$$D_k u\eta^2 \in H^{1,2}(\mathbb{R}^n). \tag{3.45}$$

Multiplying (3.35) by

$$-D_k(D^k u\eta^2), \tag{3.46}$$

where we use summation convention, integrating by parts and employing some trivial estimates, we deduce

$$\begin{aligned} &\frac{\alpha_0}{2}\int_{\mathbb{R}^n} |D^2u|^2\eta^2 + \frac{1}{2}\int_{\mathbb{R}^n} \Theta|Du|^2\eta^2 \\ &\le c\{\|f\|^2 + \|u\|_{1,2}^2 + \int_{\mathbb{R}^n} \Theta|u|^2\} \le c\|f\|^2, \end{aligned} \tag{3.47}$$

where we also used (3.38), (3.44) and where the symbol c may represent different constants. Since η is an arbitrary cut-off function, only subject to (3.44), the result follows. $\square$

As a corollary to Theorem 3.1 and Lemma 3.2 we obtain

3.3. **Theorem.** *Let $f \in H^{m-2,2}(\mathbb{R}^n)$, $m \ge 2$, then the equation*

$$Au = -\alpha_0 \Delta u + \Theta u = f \tag{3.48}$$

has a unique solution $u \in H^{m,2}(\mathbb{R}^n) \cap \mathcal{H}_1$ satisfying

$$\|u\|_{m,2}^2 + \sum_{|\alpha| \le m-1} \int_{\mathbb{R}^n} \Theta|D^\alpha u|^2 \le c\|f\|_{m-2}^2, \tag{3.49}$$

where c depends on m, n, Θ, p_0 and α_0.

Moreover, the eigenfunctions u satisfying

$$Au = \lambda u \tag{3.50}$$

are smooth and the $H^{m,2}$-norm can be estimated by

$$\|u\|_{m,2}^2 \le c_m \lambda^m \|u\|^2 \qquad \forall\, m \ge 1, \tag{3.51}$$

where c_m also depends on the smallest eigenvalue λ_0 of A.

Proof. It suffices to prove the last estimate, which can be deduced from (3.49) by induction

$$\|u\|_{m,2}^2 \le c\lambda^2 \|u\|_{m-2}^2 \le c\lambda^2 \lambda^{m-2} \|u\|^2 = c\lambda^m \|u\|^2. \tag{3.52}$$

The proof for $m = 1$ follows from

$$\|u\|_{1,2}^2 \le c(1 + \lambda_0^{-1})B(u) = c(1 + \lambda_0^{-1})\lambda\|u\|^2. \tag{3.53}$$

□

3.4. **Lemma.** *Let $\mathcal{H}_{2m}(\mathbb{R}^n)$, $m \geq 1$, be the completion of $C_c^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the scalar product*

$$\langle A^m u, A^m v\rangle = \int_{\mathbb{R}^n} A^m \bar{u} A^m v, \tag{3.54}$$

then

$$\|u\|^2_{2m,2} \leq c\|A^m u\|^2 \qquad \forall\, u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{3.55}$$

$$\|A^{m-1}u\|^2 \leq c\|A^m u\|^2 \qquad \forall\, u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{3.56}$$

and the eigenfunctions of A are complete in $\mathcal{H}_{2m}(\mathbb{R}^n)$ for any $m \geq 1$. Furthermore, if the eigenfunctions are mutually orthogonal in $L^2(\mathbb{R}^n)$ then they are also mutually orthogonal in $\mathcal{H}_{2m}(\mathbb{R}^n)$ and vice versa.

Proof. We prove the first estimate by induction.

„(3.55)“ The estimate is valid for $m = 1$, in view of Theorem 3.3.

Suppose the estimate is valid for $q \geq 1$ and let u be test function, then

$$\begin{aligned} \|u\|^2_{2(q+1),2} &\leq c\|Au\|^2_{2q,2} \\ &\leq c\|A^q(Au)\|^2 \\ &= c\|A^{q+1}u\|^2, \end{aligned} \tag{3.57}$$

where we used Theorem 3.3 in the first inequality and the induction hypothesis in the second.

„(3.56)“ Let $m \geq 1$, then

$$\|A^{m-1}u\|^2 \leq \lambda_0^{-1}\langle AA^{m-1}u, A^{m-1}u\rangle \leq \lambda_0^{-1}\|A^m u\|\,\|A^{m-1}u\|. \tag{3.58}$$

It remains to prove the completeness of the eigenfunctions u_i obtained in Theorem 2.5 on page 11. They are complete in $\mathcal{H}_1$ but also in $L^2(\mathbb{R}^n)$ because of the Parseval's identity (2.52).

If they were not complete in $H_{2m}(\mathbb{R}^n)$ for some m, then there would exist $0 \neq u \in H_{2m}(\mathbb{R}^n)$ such that

$$0 = \langle A^m u, A^m u_i\rangle = \langle u, A^{2m}u_i\rangle = \lambda_i^{2m}\langle u, u_i\rangle \qquad \forall\, i \in \mathbb{N}, \tag{3.59}$$

hence we would infer

$$u = 0; \tag{3.60}$$

a contradiction. □

The elliptic operator A with

$$D(A) = C_c^\infty(\mathbb{R}^n) \subset \mathcal{H} = L^2(\mathbb{R}^n) \tag{3.61}$$

is essentially self-adjoint, for a proof see Lemma 5.1 on page 37. Let us denote its unique self-adjoint extension by the same symbol since the domain of the extension is $\mathcal{H}_2(\mathbb{R}^n)$. We are almost ready to prove the trace class estimates for A but we need to additional lemmata.

3.5. **Lemma.** *Let $\mathcal{H}_0$ be the Hilbert space*

$$\mathcal{H}_0 = L^2(\mathbb{R}^n, d\mu) \tag{3.62}$$

where

$$d\mu = (1+|x|)^{-(n+1)}, \tag{3.63}$$

then the embedding

$$j : \mathcal{H}_{2m}(\mathbb{R}^n) \hookrightarrow \mathcal{H}_0 \tag{3.64}$$

is Hilbert-Schmidt provided $m > \frac{n}{2}$.

Proof. As in the proof of Lemma 2.6 on page 13 we adapt Maurin's original proof for bounded subsets of $\mathbb{R}^n$ to the present situation. Let φ be a real valued test function

$$\varphi \in C_c^\infty(\mathbb{R}^n) \tag{3.65}$$

and S the differential operator

$$S = D_1 \circ D_2 \circ \cdots \circ D_n, \tag{3.66}$$

then

$$\varphi^2(x) = \int_{-\infty}^{x^1} \cdots \int_{-\infty}^{x^n} S(\varphi^2). \tag{3.67}$$

The integrand can be expressed in the form

$$S(\varphi^2) = \sum_{|\alpha|+|\beta|=n} c_{\alpha\beta} D^\alpha \varphi S^\beta \varphi \tag{3.68}$$

with multiindices α, β and constants $c_{\alpha\beta}$, where some constants may be zero. Hence, we deduce

$$|\varphi|^2 \le c\|\varphi\|^2_{n,2} \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^n). \tag{3.69}$$

This estimate is of course also valid for complex valued $u \in \mathcal{H}_{2m}(\mathbb{R}^n)$.

Now, let $m > \frac{n}{2}$ and let e_i be an ONB in $\mathcal{H}_{2m}(\mathbb{R}^n)$ consisting of eigenfunctions of A, then, for any $x \in \mathbb{R}^n$, the map

$$u \to u(x), \qquad u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{3.70}$$

is continuous, because of (3.69) and (3.55), hence it can be expressed in the form

$$u(x) = \langle A^m \varphi_x, A^m u\rangle \qquad \forall\, u \in \mathcal{H}_{2m}(\mathbb{R}^n), \tag{3.71}$$

where

$$\varphi_x \in \mathcal{H}_{2m}(\mathbb{R}^n) \tag{3.72}$$

and

$$\|A^m \varphi_x\| \le c \tag{3.73}$$

is uniformly bounded independent of x. If we choose especially $u = e_i$ then, for any $x \in \mathbb{R}^n$,

$$\sum_{i=0}^{\infty} |e_i(x)|^2 = \sum_{i=0}^{\infty} |\langle A^m \varphi_x, A^m e_i \rangle|^2 = \|A^m \varphi_x\|^2 \le c^2. \tag{3.74}$$

Integrating now with respect to measure in (3.63) completes the proof of the lemma. □

The next lemma is analogous to Lemma 2.7 on page 14.

3.6. **Lemma.** *Let u_i be an eigenfunction of A with eigenvalue λ_i, then there exist positive constants c and γ such that*

$$\|u_i\|_1^2 = B(u_i) =\le c\lambda_i^{\gamma} \|u_i\|_0^2, \tag{3.75}$$

where c, γ are independent of u_i and $\|\cdot\|_0$ is the norm in $\mathcal{H}_0$.

Proof. We have

$$B(u_i) = \int_{\mathbb{R}^n} \{\alpha_0 |Du_i|^2 + \Theta |u_i|^2\} = \lambda_i \|u_i\|^2. \tag{3.76}$$

Moreover, we know, in view of (3.17) and (3.19), that

$$\Theta(x) \ge \frac{1}{2} c_1 |x|^{p_0} \qquad \forall\, |x| \ge 2R_0 > 1, \tag{3.77}$$

where $p_0 > 0$. Choosing small positive δ, ϵ and applying Young's inequality with

$$q = \frac{p_0}{p_0 - p_0\delta} = \frac{1}{1-\delta} \tag{3.78}$$

and

$$q' = \delta^{-1} \tag{3.79}$$

to estimate the L^2-norm on the right-hand side of (3.76) from above by

$$\frac{1}{q}\epsilon^q \int_{\mathbb{R}^n} (1+|x|)^{p_0} |u_i|^2 + \frac{1}{q'}\epsilon^{-q'} \int_{\mathbb{R}^n} (1+|x|)^{-p_0(1-\delta)\delta^{-1}} |u_i|^2. \tag{3.80}$$

Choosing δ so small that

$$p_0 \delta^{-1} > n + 2 \tag{3.81}$$

we deduce

$$\|u_i\|^2 \le c\frac{1}{q}\epsilon^q B(u_i) + c\frac{1}{q'}\epsilon^{-q'} \|u_i\|_0^2 \tag{3.82}$$

leading immediately to the desired estimate by choosing ϵ appropriately. □

Now we can prove:

3.7. **Theorem.** *Let A be the elliptic differential operator*

$$Au = -\alpha_0 \Delta u + \Theta u, \tag{3.83}$$

then

$$e^{-\beta A}, \qquad \beta > 0, \tag{3.84}$$

is of trace class in $L^2(\mathbb{R}^n)$, i.e.,

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} < \infty. \tag{3.85}$$

Proof. Let (u_i) be an ONB of eigenfunctions of A in $\mathcal{H} = L^2(\mathbb{R}^n)$ and let $m > \frac{n}{2}$, then

$$\begin{aligned} e^{-\beta\lambda_i} &= e^{-\beta\lambda_i}\|u_i\|^2 = e^{-\beta\lambda_i}\lambda_i^{-1}B(u_i) \\ &\le e^{-\beta\lambda_i}\lambda_i^{-1}c\lambda_i^{\gamma}\|u_i\|_0^2 \\ &\le e^{-\beta\lambda_i}\lambda_i^{-1}c\lambda_i^{\gamma}\|A^m u_i\|^2\|\tilde{u}_i\|_0^2 \\ &= ce^{-\beta\lambda_i}\lambda_i^{2m+\gamma-1}\|\tilde{u}_i\|_0^2, \end{aligned} \tag{3.86}$$

$$\tilde{u}_i = \frac{u_i}{\|A^m u_i\|} \tag{3.87}$$

and where we also used the estimate (3.75) to derive the first inequality in (3.86).

Hence, we infer

$$e^{-\beta\lambda_i} \le c_\beta \|\tilde{u}_i\|_0^2, \tag{3.88}$$

where

$$c_\beta = c \sup_{t>0} e^{-\beta t} t^{2m+\gamma-1}, \tag{3.89}$$

and we finally conclude

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} \le c_\beta \sum_{i=0}^{\infty} \|\tilde{u}_i\|_0^2 < \infty, \tag{3.90}$$

because the embedding

$$j : \mathcal{H}_{2m}(\mathbb{R}^n) \hookrightarrow \mathcal{H}_0 \tag{3.91}$$

is Hilbert-Schmidt, in view of Lemma 3.5. □

4. The Hamiltonians governing quantum gravity

In three recent papers we applied our model of quantum gravity to a globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface [7] and to a Schwarzschild-AdS [6] resp. Kerr-AdS black hole [8]. In all three cases the quantized model had the same structure, namely, it consisted of special solutions to a wave equation

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0, \tag{4.1}$$

in a quantum spacetime

$$N = \mathbb{R}_+ \times \mathcal{S}_0, \tag{4.2}$$

where $\mathcal{S}_0$ is a n-dimensional, $n \geq 3$, Cauchy hypersurface of the original spacetime, or, in case of black holes, the smooth limit of Cauchy hypersurfaces. The Laplacian and the scalar curvature correspond to the metric σ_{ij} in $\mathcal{S}_0$, cf. [5, Theorem 6.9], where we derived this wave equation after a canonical quantization process. The special solutions are a sequence of smooth functions which are a product of temporal and spatial eigenfunctions of elliptic operators, where the spatial eigenfunctions are eigendistributions.

In case of the globally hyperbolic spacetime with an asymptotically Euclidean Cauchy hypersurface the solutions to the wave equation can be expressed in the form

$$u_{ij} = w_i v_{ij}, \qquad i \in \mathbb{N},\ 1 \leq j \leq m \leq \infty, \tag{4.3}$$

where the w_i are the eigenfunctions of a temporal Hamilton operator H_0

$$H_0 w_i = \lambda_i w_i \tag{4.4}$$

and the λ_i have multiplicity one such that

$$0 < \lambda_0 < \lambda_1 < \cdots \tag{4.5}$$

and for each fixed i the at most countably many v_{ij} generate an eigenspace

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{4.6}$$

of a spatial Hamiltonian H_1, i.e.,

$$H_1 v_{ij} = \lambda_i v_{ij}. \tag{4.7}$$

We have

$$v_{ij} \in C^\infty(\mathcal{S}_0) \cap \mathscr{S}'(\mathcal{S}_0). \tag{4.8}$$

In the two remaining cases of the black holes the special solutions are labelled by three indices

$$u_{ijk} = w_i \zeta_{ijk} \varphi_j, \tag{4.9}$$

where the w_i are the same temporal eigenfunctions as before, the φ_j are the eigenfunctions of an elliptic operator A on a smooth compact Riemannian manifold (M, σ_{ij}), where topologically

$$M \simeq \mathbb{S}^{n-1}, \tag{4.10}$$

at least in the physically interesting cases, i.e.,

$$A\varphi_j = \tilde{\mu}_j \varphi_j, \tag{4.11}$$

$$\tilde{\mu}_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots \tag{4.12}$$

The φ_j form a mutually orthogonal basis of $L^2(M)$. For a Schwarzschild-AdS black hole we know that

$$\tilde{\mu}_0 \le 0, \tag{4.13}$$

and for a Kerr-AdS black hole this condition can be assured by assuming that the rotational parameter a is small enough such that the scalar curvature of σ_{ij} is positive. Let us emphasize that we considered in [8] Kerr-AdS black holes of odd dimensions

$$\dim N = 2m + 1, \qquad m \ge 2, \tag{4.14}$$

and assumed that all rotational parameters a_i are equal

$$a_i = a \ne 0 \qquad \forall\, 1 \le i \le m. \tag{4.15}$$

The ζ_{ijk} are eigendistributions in $\mathscr{S}'(\mathbb{R})$ satisfying

$$-\zeta''_{ijk} = \omega_{ij}^2 \zeta_{ijk}, \qquad k = 1, 2, \tag{4.16}$$

where

$$\zeta_{ij1}(\tau) = \frac{1}{\sqrt{2\pi}} e^{i\omega_{ij}\tau} \tag{4.17}$$

and

$$\zeta_{ij2}(\tau) = \frac{1}{\sqrt{2\pi}} e^{-i\omega_{ij}\tau}, \tag{4.18}$$

where

$$\omega_{ij} \ge 0 \tag{4.19}$$

is defined by the relation

$$\lambda_i = \tilde{\mu}_j + \omega_{ij}^2, \tag{4.20}$$

i.e., for any $i \in \mathbb{N}$ we look for all j satisfying

$$\tilde{\mu}_j \le \lambda_i \tag{4.21}$$

and then choose $\omega_{ij} \ge 0$ satisfying (4.20). Let N_i be the set of integers such that the $\tilde{\mu}_j$ satisfy (4.21), then the smooth functions

$$\zeta_{ijk}\varphi_j \tag{4.22}$$

are mutually orthogonal in $L^2(M, \sigma_{ij})$—for fixed i and k; note that we only have two different eigendistributions ζ_{ijk}, if

$$\omega_{ij} > 0, \tag{4.23}$$

otherwise we have only one. The eigendistributions ζ_{ij1} and ζ_{ij2} are also considered to be „orthogonal“ since their Fourier transforms

$$\hat{\zeta}_{ijk} = \delta_{\pm\omega_{ij}} \tag{4.24}$$

have disjoint supports.

Finally, the smooth functions u_{ijk} in (4.9) can be considered to be mutually orthogonal since u_{ijk} and $u_{i'j'k'}$ are mutually orthogonal in

$$L^2(\mathbb{R}_+, d\mu) \otimes L^2(M), \tag{4.25}$$

where

$$d\mu = t^{2-\frac{4}{n}} dt, \tag{4.26}$$

if

$$\omega_{ij} = \omega_{i'j'} \quad \wedge \quad k = k' \tag{4.27}$$

and as tempered distributions otherwise.

The u_{ijk} are eigendistributions for both the temporal Hamiltonian H_0 as well as for the spatial Hamiltonian H_1 with the same eigenvalues λ_i, where now the eigenvalues have finite multiplicities different from 1 by definition of the eigendistributions and the u_{ijk} also solve the wave equation, since the wave equation can be expressed as

$$\varphi_0(H_0 u - H_1 u) = 0, \tag{4.28}$$

where $u = u(t, x)$ is a smooth function

$$x \in \mathcal{S}_0 = \mathbb{R} \times M \tag{4.29}$$

and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{4.30}$$

In Section 5 we shall prove that we can define an abstract Hilbert space $\mathcal{H}$, where the eigendistributions u_{ijk} resp. u_{ij} in (4.3) form a basis of mutually orthogonal unit vectors such that the Hamiltonian H_1 can be defined on the dense subspace, which is the algebraic span of the basis vectors, as an essentially self-adjoint operator. Let $\tilde{H}_1$ be its unique self-adjoint extension, namely its closure, then we shall prove that for any $\beta > 0$

$$e^{-\beta \tilde{H}_1} \tag{4.31}$$

is of trace class in $\mathcal{H}$. In addition $\tilde{H}_1$ satisfies

$$\tilde{H}_1 \ge \lambda_0 I, \qquad \lambda_0 > 0. \tag{4.32}$$

The temporal eigenfunctions w_i solve the equation

$$H_0 w_i = \lambda_i w_i, \tag{4.33}$$

where

$$H_0 w_i = \varphi_0^{-1}(-\frac{1}{32}\frac{n^2}{n-1}\ddot{w}_i + nt^2|\Lambda| w_i), \tag{4.34}$$

which is equivalent to

$$-\frac{1}{32}\frac{n^2}{n-1}\ddot{w}_i + nt^2|\Lambda| w_i = \lambda_i \varphi_0 w_i, \tag{4.35}$$

i.e., it is one of the Sturm-Liouville eigenvalue problems which we considered in (2.2) on page 8, where now

$$Au = -\frac{1}{32}\frac{n^2}{n-1}\ddot{u} + nt^2|\varLambda|u, \tag{4.36}$$

$$b(t) = nt^2|\varLambda| \tag{4.37}$$

and

$$\varphi_0(t) = t^{2-\frac{4}{n}}. \tag{4.38}$$

The eigenvalues are obtained by looking at the generalized eigenvalue problem

$$B(u,v) = \lambda K(u,v) \qquad \forall\, v \in \mathcal{H}_1, \tag{4.39}$$

where

$$B(u,v) = \langle Au, v\rangle \tag{4.40}$$

and

$$K(u,v) = \int_{\mathbb{R}_+} t^{2-\frac{n}{4}}\bar{u}v, \tag{4.41}$$

cf. Theorem 2.5 on page 11, where now

$$c_0 = 0. \tag{4.42}$$

Hence, the assumptions of Theorem 2.8 on page 15 are all satisfied and we conclude

4.1. **Theorem.** *Let $\beta > 0$ and let H_0 be the Hamiltonian in (4.34), then the operator*

$$e^{-\beta H_0} \tag{4.43}$$

is of trace class $L^2(\mathbb{R}_+, d\mu)$.

There is also a spatial Hamiltonian H_1, which, in the case of the black holes considered, is a direct product of a classical harmonic oscillator in $\mathbb{R}$ and an elliptic operator A on a compact, smooth Riemannian manifold $M = M^{n-1}$, $n \ge 3$, with metric σ_{ij}, where A has the form

$$A\varphi = -(n-1)\varDelta\varphi - \frac{n}{2}R\varphi \tag{4.44}$$

and the Laplacian is the Laplacian in M and R the scalar curvature of the metric. A is self-adjoint with domain

$$D(A) = H^{2,2}(M) \subset L^2(M), \tag{4.45}$$

where

$$H^{m,2}(M), \qquad m \in M, \tag{4.46}$$

are the usual Sobolev spaces with norm

$$\|\varphi\|_{m,2}^2 = \sum_{|\alpha|\le m} \int_M |D^\alpha \varphi|^2. \tag{4.47}$$

A has a pure point spectrum with countable many eigenvalues $\tilde{\mu}_j$ with finite multiplicities and mutually orthogonal eigenfunctions φ_j such that

$$\tilde{\mu}_0 < \tilde{\mu}_1 \le \cdots \tag{4.48}$$

and

$$\lim_j \tilde{\mu}_j = \infty. \tag{4.49}$$

We want to prove that

$$e^{-\beta A}, \qquad \beta > 0, \tag{4.50}$$

is of trace class in $L^2(M)$.

The proof of this result will follow the arguments in Section 3 very closely.

4.2. **Lemma.** *Let $m > \frac{n-1}{2}$, then the embedding*

$$j : H^{m,2}(M) \hookrightarrow L^2(M) \tag{4.51}$$

is Hilbert-Schmidt.

Proof. This result is due to Maurin and its proof is identical with the proof of Lemma 2.6 apart from some obvious modifications. □

We also need the lemma:

4.3. **Lemma.** *Let $m \in \mathbb{N}$, then there exists $c_m > 0$ such that*

$$\|\varphi\|_{2m,2}^2 \le c_m(\|A^m\varphi\|^2 + \|\varphi\|^2) \tag{4.52}$$

and the bilinear form

$$\langle A^m\varphi, A^m\psi\rangle_0 + \langle\varphi,\psi\rangle_0 \tag{4.53}$$

defines an equivalent scalar product in $H^{2m,2}(M)$, where

$$\langle\varphi,\psi\rangle_0 = \int_M \bar{\varphi}\psi. \tag{4.54}$$

Proof. Let

$$f \in H^{m,2}(M) \tag{4.55}$$

and

$$\varphi \in H^{2,2}(M) \tag{4.56}$$

a solution of

$$A\varphi = f, \tag{4.57}$$

then it is well-known that

$$\varphi \in H^{m+2,2}(M) \tag{4.58}$$

and there exists $\tilde{c}_m$ such that

$$\|\varphi\|_{m+2,2} \le \tilde{c}_m(\|f\|_{m,2} + \|\varphi\|_0). \tag{4.59}$$

The constant $\tilde{c}_m$ also depends on A and M. Using this estimate the relation (4.52) can be easily proved by induction. ☐

Now, we are ready to prove:

4.4. **Theorem.** *Let A be the self-adjoint operator in (4.44), then*

$$e^{-\beta A} \tag{4.60}$$

is of trace class in $L^2(M)$ for any $\beta > 0$.

Proof. Let $m > \frac{n-1}{4}$ and equip $H^{2m,2}(M)$ with the scalar product (4.53) such that

$$\|\varphi\|^2_{2m,2} = \langle A^m\varphi, A^m\varphi\rangle_0 + \langle\varphi,\varphi\rangle_0, \tag{4.61}$$

then any eigenfunctions φ_i, φ_j of A satisfy

$$\langle\varphi_i,\varphi_j\rangle_0 = 0 \implies \langle\varphi_i,\varphi_j\rangle_{2m,2} = 0. \tag{4.62}$$

Let (φ_j) be an ONB of eigenfunctions of A in $L^2(M)$ and define

$$\tilde{\varphi}_j = \varphi_i\|\varphi_j\|^{-1}_{2m,2}, \tag{4.63}$$

then the $\tilde{\varphi}_j$ form an ONB in $H^{2m,2}(M)$ and we conclude

$$\begin{aligned} e^{-\beta\tilde{\mu}_j} &= e^{-\beta\tilde{\mu}_j}\|\varphi_j\|^2_0 = e^{-\beta\tilde{\mu}_j}\|\varphi_j\|^2_{2m,2}\,\|\tilde{\varphi}_j\|^2_0 \\ &= e^{-\beta\tilde{\mu}_j}(1+|\tilde{\mu}_j|^{2m})\|\tilde{\varphi}_j\|^2_0 \le c_\beta\|\tilde{\varphi}_j\|^2_0 \end{aligned} \tag{4.64}$$

yielding

$$\sum_{j=0}^\infty e^{-\beta\tilde{\mu}_j} \le c_\beta\sum_{j=0}^\infty\|\tilde{\varphi}_j\|^2_0 < \infty \tag{4.65}$$

in view of Lemma 4.2. ☐

With the help of the preceding lemma we can now prove that, in case of the black holes, the spatial Hamiltonian H_1 has the property that

$$e^{-\beta H_1} \tag{4.66}$$

is of trace class for all $\beta > 0$, where we still have to define an appropriate Hilbert space.

We have

$$H_1 v = -\ddot{v} - Av, \tag{4.67}$$

where we write v as product

$$v(\tau,x) = \zeta(\tau)\varphi(x) \tag{4.68}$$

with

$$\tau\in\mathbb{R} \quad\wedge\quad x\in M = M^{n-1}, \tag{4.69}$$

where A is the differential operator in (4.44). Let φ_j be the eigenfunctions of A with eigenvalues $\tilde{\mu}_j$, then, for any eigenvalue λ_i we define

$$N_i = \{j \in \mathbb{N} : \tilde{\mu}_j \le \lambda_i\} \tag{4.70}$$

and $\omega_{ij} \ge 0$ such that

$$\omega_{ij}^2 + \tilde{\mu}_j = \lambda_i. \tag{4.71}$$

Note that

$$0 \in N_i \qquad \forall\, i \in \mathbb{N}, \tag{4.72}$$

since

$$\tilde{\mu}_0 \le 0. \tag{4.73}$$

Let

$$\zeta_{ijk}, \qquad k = 1, 2, \tag{4.74}$$

be the tempered distributions

$$\zeta_{ij1} = \frac{1}{\sqrt{2\pi}} e^{i\omega_{ij}\tau} \tag{4.75}$$

and

$$\zeta_{ij2} = \frac{1}{\sqrt{2\pi}} e^{-i\omega_{ij}\tau}, \tag{4.76}$$

where this distinction only occurs for

$$\omega_{ij} > 0. \tag{4.77}$$

Let $\hat{\zeta}_{ijk}$ be the Fourier transform of ζ_{ijk}, then

$$\hat{\zeta}_{ij1} = \delta_{\omega_{ij}} \quad \wedge \quad \hat{\zeta}_{ij2} = \delta_{-\omega_{ij}} \tag{4.78}$$

such that these tempered distributions are considered to be mutually „orthogonal“. The smooth functions

$$u_{ijk} = \zeta_{ijk}\varphi_j \tag{4.79}$$

satisfy

$$H_1 u_{ijk} = \lambda_i u_{ijk}. \tag{4.80}$$

Label the eigenvalues of H_1 including their multiplicities and denote them by $\tilde{\lambda}_i$. Then

$$\sum_{i=0}^{\infty} e^{-\beta\tilde{\lambda}_i} \le 2\sum_{i=0}^{\infty} e^{-\beta\lambda_i} n(\lambda_i) = 2\sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\lambda_i} e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i), \tag{4.81}$$

where

$$n(\lambda_i) = \#N_i. \tag{4.82}$$

4.5. **Lemma.** *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta_0 \le \beta \tag{4.83}$$

and for any $i \in \mathbb{N}$, the estimate

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta) \le c(\beta_0), \tag{4.84}$$

where $c(\beta_0)$ also depends on A but is independent of $i \in \mathbb{N}$.

Proof. Each N_i is the disjoint union

$$N_i' \,\dot\cup\, N_i'', \tag{4.85}$$

where

$$N_i' = \{j \in \mathbb{N}_i : \tilde\mu_j \le 0\} \tag{4.86}$$

and N_i'' is its complement. The operator A has only finitely many eigenvalues which are non-positive, i.e.,

$$\# N_i' \le n_0 \qquad \forall\, i \in \mathbb{N}, \tag{4.87}$$

hence

$$\begin{aligned} e^{-\frac{\beta}{2}\lambda_i} n_i(\lambda_i) &\le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\lambda_i} \le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\tilde\mu_j} \\ &\le n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde\mu_j} \\ &= n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde\mu_j}(1+|\tilde\mu|_j^{2m})\,\|\tilde\varphi_j\|_0^2 \\ &\le n_0 + c(\beta)\sum_{j=0}^{\infty}\|\tilde\varphi_j\|_0^2 < \infty, \end{aligned} \tag{4.88}$$

where we used (4.64). The estimate for the Hilbert-Schmidt norm of the embedding

$$j : H^{m,2}(M) \to L^2(M) \tag{4.89}$$

depends on A, since we used the equivalent norm given in (4.61), and

$$c(\beta) = \sup_{t>0} e^{-\frac{\beta}{2}t}(1+t^{2m}). \tag{4.90}$$

□

4.6. **Corollary.** *The sum on the left-hand side of (4.81) is finite and hence*

$$e^{-\beta H_1}, \qquad \beta > 0, \tag{4.91}$$

is of trace class provided we can define a Hilbert space $\mathcal{H}$ such that the eigendistributions form a complete set of eigenvectors in $\mathcal{H}$ and H_1 is essentially self-adjoint in $\mathcal{H}$.

Proof. The first claim follows immediately by combining (4.88) and Theorem 2.8. In Lemma 5.1 on page 37 we shall define the Hilbert space $\mathcal{H}$ and shall prove that H_1 is essentially self-adjoint in $\mathcal{H}$ and that the eigendistributions form a complete set of eigenvectors in $\mathcal{H}$. □

The elliptic operator A also depend on Λ, since the underlying Riemannian metric depends on it. The estimates in the preceding lemma remain valid provided $|\Lambda|$ remains in a compact subset of $\mathbb{R}$, since the operator A is then still uniformly elliptic and smooth. However, when

$$|\Lambda| \to \infty, \tag{4.92}$$

then the relation (4.52) is no longer valid and a more sophisticated analysis is necessary to achieve a corresponding estimate. Let us treat the cases Schwarzschild-AdS and Kerr-AdS black holes separately.

For a Schwarzschild-AdS black hole the operator A can be written in the form

$$A = r_0^{-2}\tilde{A}, \tag{4.93}$$

where r_0 is the black hole radius and

$$\tilde{A}\varphi = -(n-1)\tilde{\Delta}\varphi - \frac{n}{2}\tilde{R}\varphi. \tag{4.94}$$

Here, the Laplacian and the scalar curvature $\tilde{R}$ refer to the corresponding quantities of $\mathbb{S}^{n-1}$ with the standard metric, cf. [6, equ. (2.12) and (2.14)]. The eigenfunctions of A are the eigenfunctions of $\tilde{A}$. Let μ_j be the eigenvalues of $\tilde{A}$ and $\tilde{\mu}_j$ the eigenvalues of A, then

$$\tilde{\mu}_j = r_0^{-2}\mu_j. \tag{4.95}$$

From the definition of the black hole radius

$$mr_0^{-(n-2)} = 1 + \frac{2}{n(n-1)}|\Lambda| r_0^2 \tag{4.96}$$

it is evident that

$$\lim_{|\Lambda|\to\infty} r_0 = 0 \tag{4.97}$$

and also

$$\lim_{|\Lambda|\to\infty} |\Lambda| r_0^2 = \infty, \tag{4.98}$$

though the latter result is only needed when we shall treat the Kerr-AdS case.

We can now prove:

4.7. **Lemma.** *Let $\beta_0 > 0$ be arbitrary and $|\Lambda_0|$ so large that*

$$r_0 < 1 \qquad \forall\, |\Lambda| > |\Lambda_0|, \tag{4.99}$$

then for any $i \in \mathbb{N}$, any $\beta \geq \beta_0$ and any $|\Lambda| > |\Lambda_0|$

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \leq c(\beta) \leq c(\beta_0), \tag{4.100}$$

where $c(\beta_0)$ also depends on $\tilde{A}$ but is independent of $|\Lambda|$ and $i \in \mathbb{N}$.

Proof. We follow the proof of Lemma 4.5 but use $\tilde{A}$ instead of A to define an equivalent norm in $H^{m,2}(M)$,

$$M = \mathbb{S}^{n-1}. \tag{4.101}$$

Then, we infer, cf. (4.88),

$$\begin{aligned} e^{-\frac{\beta}{2}\lambda_i} n_i(\lambda_i) &\le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\lambda_i} \le n_0 + \sum_{j\in N_i''} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &\le n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j} \\ &= n_0 + \sum_{j\ge n_0} e^{-\frac{\beta}{2}\tilde{\mu}_j}(1+|\mu|_j^{2m})\,\|\tilde{\varphi}_j\|_0^2 \\ &\le n_0 + c(\beta)\sum_{j=0}^{\infty}\|\tilde{\varphi}_j\|_0^2 < \infty. \end{aligned} \tag{4.102}$$

Here, we used

$$\tilde{\mu}_j = r_0^{-2}\mu_j > \mu_j > 0. \tag{4.103}$$

□

Let us now look at Kerr-AdS black holes. In [8, equ. (2.50)] we described the metric σ_{ij} on $M = \mathbb{S}^{n-1}$

$$\begin{aligned} ds_M^2 = \frac{r^2+a^2}{1-a^2l^2}\left(\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j\right) \\ + a^2\frac{(1+l^2r^2)(r^2+a^2)}{r^2(1-a^2l^2)^2}\mu_i^2\mu_j^2 d\varphi^i d\varphi^j. \end{aligned} \tag{4.104}$$

Here

$$n = 2m, \qquad m \ge 2, \tag{4.105}$$

and the coordinates μ_i, $1 \le i \le m$ are subject to the constraint

$$\sum_{i=1}^m \mu_i^2 = 1. \tag{4.106}$$

They are the latitudinal coordinates of $\mathbb{S}^{n-1}$ and the φ_i, $1 \le i \le m$, are the azimuthal coordinates. The metric

$$\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j \tag{4.107}$$

is the standard metric of $\mathbb{S}^{n-1}$. The constant r is the radius of the event horizon, $a \ne 0$ the rotational parameter and

$$l^2 = -\frac{1}{m(2m-1)}\Lambda. \tag{4.108}$$

The relation

$$a^2l^2 < 1 \tag{4.109}$$

is assumed. We also require that a is small enough such that the scalar curvature R of the metric σ_{ij} is positive. We can write the metric as a conformal metric

$$\sigma_{ij} = \frac{r^2 + a^2}{1 - a^2l^2}\tilde{\sigma}_{ij}. \tag{4.110}$$

Let us also note that the Schwarzschild-AdS black hole is obtained by setting $a = 0$ and that

$$\lim_{a \to 0} r = r_0, \tag{4.111}$$

is the Schwarzschild black hole radius.

In order to prove the analogue of Lemma 4.7 we assume that, when

$$|\Lambda| \to \infty, \tag{4.112}$$

a is supposed to be so small that

$$\lim_{|\Lambda| \to \infty} |\Lambda| a^2 = 0 \tag{4.113}$$

and

$$\lim_{|\Lambda| \to \infty} |\Lambda| r^2 = \infty, \tag{4.114}$$

and we emphasize that these assumptions are always satisfied if $a = 0$, cf. (4.98). If these are satisfied, then the operator A can be expressed in the form

$$A = \frac{1 - a^2l^2}{r^2 + a^2}\tilde{A}, \tag{4.115}$$

where $\tilde{A}$ converges uniformly in $C^\infty(M)$ to the operator $\tilde{A}$ in (4.94), i.e., for large $|\Lambda|$, $\tilde{A}$ is uniformly elliptic and smooth such that the number of non-positive eigenvalues $n_0(\tilde{A})$ is bounded from above by the n_0 of the limit operator

$$n_0 \ge \limsup_{|\Lambda| \to \infty} n_0(\tilde{A}), \tag{4.116}$$

since n_0 is upper semi-continuous as it is well-known.

4.8. **Lemma.** *Under the assumptions* (4.113) *and* (4.114) *the results of Lemma* 4.7 *are also valid for the Kerr-AdS black hole, i.e., there exists* $|\Lambda_0| > 0$ *such that for all*

$$|\Lambda| > |\Lambda_0| \tag{4.117}$$

and for any β *satisfying*

$$0 < \beta_0 \le \beta, \tag{4.118}$$

where β_0 is arbitrary,

$$e^{-\frac{\beta}{2}\lambda_i} n(\lambda_i) \le c(\beta_0) \tag{4.119}$$

uniformly in $i \in \mathbb{N}$, $|\Lambda|$ and β.

Proof. The proof is identical to the proof of Lemma 4.7 by using the fact that the special $H^{m,2}(M)$ norm

$$\langle \tilde{A}^m \varphi, \tilde{A}^m \varphi\rangle_0 + \langle \varphi, \varphi\rangle_0, \tag{4.120}$$

with different m than used to express the dimension of M, is uniformly equivalent to the standard $H^{m,2}(M)$ norm, hence the Hilbert-Schmidt norm of the embedding

$$j : H^{m,2}(M) \hookrightarrow L^2(M) \tag{4.121}$$

is uniformly bounded. We also relied on

$$\tilde{\mu}_j = \frac{1 - a^2 l^2}{r^2 + a^2} \mu_j > \mu_j > 0 \tag{4.122}$$

for $j \in N_i''$. □

Finally, let us derive the last result in this section.

4.9. **Lemma.** *Let λ_i be the temporal eigenvalues depending on Λ and let $\bar{\lambda}_i$ be the corresponding eigenvalues for*

$$|\Lambda| = 1, \tag{4.123}$$

then

$$\lambda_i = \bar{\lambda}_i |\Lambda|^{\frac{n-1}{n}}. \tag{4.124}$$

Proof. Let B and K be the bilinear forms defined in (4.40) resp. (4.41), where B corresponds to the cosmological constant Λ, and let B_1 be the form with respect to the value

$$|\Lambda| = 1. \tag{4.125}$$

Moreover, let us denote the corresponding quadratic forms by the same symbols, then we have

$$\frac{B(\varphi)}{K(\varphi)} = |\Lambda|^{\frac{n-1}{n}} \frac{B_1(\varphi)}{K(\varphi)} \qquad \forall\, 0 \ne \varphi \in C_c^\infty(\mathbb{R}_+). \tag{4.126}$$

To prove (4.126) we introduce a new integration variable τ on the left-hand side

$$t = \mu\tau, \qquad \mu > 0, \tag{4.127}$$

to conclude

$$\frac{B(\varphi)}{K(\varphi)} = \mu^{-4\frac{n-1}{n}} \frac{B_1(\varphi)}{K(\varphi)} \qquad \forall\, 0 \ne \varphi \in C_c^\infty(\mathbb{R}_+). \tag{4.128}$$

provided

$$\mu = |\Lambda|^{-\frac{1}{4}}. \tag{4.129}$$

The relation (4.126) immediately implies (4.124). □

5. The partition function

We first define the partition function for the black holes and shall later show that the definitions and results are also applicable in case of the quantized globally hyperbolic spacetimes with a negative cosmological constant and asymptotically Euclidean Cauchy hypersurfaces.

We define the partition function by using the spatial Hamiltonian H_1 of the quantized black holes, Kerr or Schwarzschild, which is now defined in the separable Hilbert space $\mathcal{H}$ generated by the eigendistributions

$$u_{ijk} = w_i \zeta_{ijk} \varphi_j \tag{5.1}$$

which are smooth functions satisfying the eigenvalue equations

$$H_1 u_{ijk} = \lambda_i u_{ijk} \tag{5.2}$$

as well as

$$H_0 u_{ijk} = \lambda_i u_{ijk}, \tag{5.3}$$

where H_0 is the temporal Hamiltonian.

In order to explain how the eigendistributions can generate a Hilbert space let us relabel the eigenfunctions and the eigenvalues by $(u_i, \tilde{\lambda}_i)$ such that

$$H_1 u_i = \tilde{\lambda}_i u_i \tag{5.4}$$

and

$$H_0 u_i = \tilde{\lambda}_i u_i, \tag{5.5}$$

i.e., the multiplicities of the eigenvalues are now included in the labelling and the ordering is no longer strict

$$\tilde{\lambda}_0 \le \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots. \tag{5.6}$$

To define the Hilbert space $\mathcal{H}$ we simply declare that the eigendistributions are mutually orthogonal unit eigenvectors, hence defining a scalar product in the complex vector space $\mathcal{H}'$ spanned by these eigenvectors. We define the Hilbert space $\mathcal{H}$ to be its completion.

5.1. **Lemma.** *The linear operator H_1 with domain $\mathcal{H}'$ is essentially self-adjoint in $\mathcal{H}$. Let $\bar{H}_1$ be its closure, then the only eigenvectors of $\bar{H}_1$ are those of H_1.*

Proof. H_1 is obviously densely defined, symmetric and bounded from below

$$H_1 \ge \tilde{\lambda}_0 I > 0. \tag{5.7}$$

Since $\tilde{\lambda}_0 > 0$, the eigenvectors also span $R(H_1)$, i.e., $R(H_1)$ is dense. Let

$$w \in \mathcal{H} \tag{5.8}$$

be arbitrary, and let

$$H_1 v_i \in R(H_1) \tag{5.9}$$

be a sequence converging to w, then v_i is a Cauchy sequence, because

$$\tilde\lambda_0 \|v_i - v_j\|^2 \le \langle H_1 v_i - H_1 v_j, v_i - v_j\rangle \le \|H_1 v_i - H_1 v_j\| \, \|v_i - v_j\|, \tag{5.10}$$

hence

$$R(\bar H_1) = \mathcal{H} \tag{5.11}$$

and $\bar H_1$ is the unique s.a. extension of H_1.

It remains to prove that $\bar H_1$ has no additional eigenvectors. Thus, let u be an eigenvector of $\bar H_1$ with eigenvalue λ

$$\bar H_1 u = \lambda u, \tag{5.12}$$

and let

$$E(\tilde\lambda_i) \subset \mathcal{H}', \qquad i \in \mathbb{N}, \tag{5.13}$$

be the eigenspaces of H_1. Let us first assume that there exists j such that

$$\lambda = \tilde\lambda_j, \tag{5.14}$$

but

$$u \notin E(\tilde\lambda_j). \tag{5.15}$$

Without loss of generality we may assume

$$u \in E(\tilde\lambda_j)^\perp. \tag{5.16}$$

However, this leads to a contradiction, since then

$$u \in E(\tilde\lambda_i)^\perp \qquad \forall\, i \in \mathbb{N}, \tag{5.17}$$

and hence

$$u \in \mathcal{H}'^\perp \tag{5.18}$$

which implies $u = 0$.

Thus, let us assume

$$\lambda \ne \tilde\lambda_i \qquad \forall\, i \in \mathbb{N}, \tag{5.19}$$

but then (5.17) is again valid leading to the known contradiction. □

5.2. **Remark.** In the following we shall write H_1 instead of $\bar H_1$.

5.3. **Lemma.** *For any $\beta > 0$ the operator*

$$e^{-\beta H_1} \tag{5.20}$$

is of trace class in $\mathcal{H}$. Let

$$\mathscr{F} \equiv \mathscr{F}_+(\mathcal{H}) \tag{5.21}$$

be the symmetric Fock space generated by $\mathcal{H}$ and let

$$H = d\Gamma(H_1) \tag{5.22}$$

be the canonical extension of H_1 to $\mathscr{F}$. Then

$$e^{-\beta H} \tag{5.23}$$

is also of trace class in $\mathscr{F}$

$$\operatorname{tr}(e^{-\beta H}) = \prod_{i=0}^{\infty}(1-e^{-\beta\tilde{\lambda}_i})^{-1} < \infty. \tag{5.24}$$

Proof. The first part of the lemma has already been proved in Corollary 4.6 on page 32. This property can now be rephrased as

$$\operatorname{tr}(e^{-\beta H_1}) = \sum_{i=0}^{\infty} e^{-\beta\tilde{\lambda}_i} < \infty. \tag{5.25}$$

The second assertion is well known, since

$$H_1 \geq \tilde{\lambda}_0 I > 0, \tag{5.26}$$

and the properties (5.25) and (5.26) imply (5.24), cf. [1, Proposition 5.2.7] and [9, Volume II, p. 868], where the equation (5.24) is also proved. □

We then define the partition function Z by

$$Z = \operatorname{tr}(e^{-\beta H}) = \prod_{i=0}^{\infty}(1-e^{-\beta\tilde{\lambda}_i})^{-1} \tag{5.27}$$

and the density operator ρ in $\mathscr{F}$ by

$$\rho = Z^{-1}e^{-\beta H} \tag{5.28}$$

such that

$$\operatorname{tr}\rho = 1. \tag{5.29}$$

The von Neumann entropy S is then defined by

$$\begin{aligned} S &= -\operatorname{tr}(\rho\log\rho) \\ &= \log Z + \beta Z^{-1}\operatorname{tr}(He^{-\beta H}) \\ &= \log Z - \beta\frac{\partial \log Z}{\partial\beta} \\ &\equiv \log Z + \beta E, \end{aligned} \tag{5.30}$$

where E is the average energy

$$E = \operatorname{tr}(H\rho). \tag{5.31}$$

E can be expressed in the form

$$E = \sum_{i=0}^{\infty}\frac{\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i}-1}. \tag{5.32}$$

Here, we also set the Boltzmann constant

$$K_B = 1. \tag{5.33}$$

The parameter β is supposed to be the inverse of the absolute temperature T

$$\beta = T^{-1}. \tag{5.34}$$

In view of Lemma 4.9 on page 36 we can write the eigenvalues λ_i in the form

$$\lambda_i = \bar{\lambda}_i |\varLambda|^{\frac{n-1}{n}}, \tag{5.35}$$

where $\bar{\lambda}_i$ are the eigenvalues corresponding to $|\varLambda| = 1$. Hence, Z, S, and E can also be looked at as functions depending on β and $\varLambda$, or more conveniently, on (β, τ), where

$$\tau = |\varLambda|^{\frac{n-1}{n}}, \tag{5.36}$$

since the $\tilde{\lambda}_i$ can also be expressed as

$$\tilde{\lambda}_i = \lambda_j = \bar{\lambda}_j |\varLambda|^{\frac{n-1}{n}}, \tag{5.37}$$

where j is different from i

$$j \le i, \tag{5.38}$$

because of the multiplicities of $\tilde{\lambda}_i$. Let emphasize that the multiplicities also depend on $\varLambda$, hence it is best to simply note that

$$\tilde{\lambda}_0 = \lambda_0 = \bar{\lambda}_0 |\varLambda|^{\frac{n-1}{n}} \tag{5.39}$$

and that the $\tilde{\lambda}_i$ are ordered. We shall never use the relation (5.37) explicitly in the proofs of the subsequent theorems and lemmata referring to (5.35) instead.

5.4. **Theorem.** (i) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$0 < \beta \le \beta_0, \tag{5.40}$$

we have

$$\lim_{\varLambda \to 0} E = \infty \tag{5.41}$$

as well as

$$\lim_{\varLambda \to 0} S = \infty, \tag{5.42}$$

where the limites are uniform in β.

(ii) *Let $\beta_0 > 0$ be arbitrary, then, for any*

$$\beta \ge \beta_0, \tag{5.43}$$

we have

$$\lim_{|\varLambda| \to \infty} E = 0 \tag{5.44}$$

as well as

$$\lim_{|\varLambda| \to \infty} S = 0, \tag{5.45}$$

where the limites are uniform in β.

Proof. „(i)“ We first observe that

$$E = \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i}{e^{\beta\tilde{\lambda}_i}-1} \ge \sum_{i=0}^{\infty} \frac{\lambda_i}{e^{\beta\lambda_i}-1} \tag{5.46}$$

Now, let $m \in \mathbb{N}$ be arbitrary, then

$$E \ge \sum_{i=0}^{m} \frac{\lambda_i}{e^{\beta\lambda_i}-1} = \sum_{i=0}^{m} \frac{\bar{\lambda}_i\tau}{e^{\beta\bar{\lambda}_i\tau}-1} \tag{5.47}$$

and

$$\begin{aligned} \liminf_{\tau\to 0} E &\ge \lim_{\tau\to 0} \sum_{i=0}^{m} \frac{\bar{\lambda}_i\tau}{e^{\beta\bar{\lambda}_i\tau}-1} \\ &= (m+1)\beta^{-1} \ge (m+1)\beta_0^{-1} \end{aligned} \tag{5.48}$$

yielding

$$\lim_{\Lambda\to 0} E = \infty \tag{5.49}$$

uniformly in β.

Since $Z \ge 1$, the relation (5.42) follows as well.

„(ii)“ We estimate E from above by

$$\begin{aligned} E &= \sum_{i=0}^{\infty} \frac{\tilde{\lambda}_i e^{-\beta\tilde{\lambda}_i}}{1-e^{-\beta\tilde{\lambda}_i}} = \sum_{i=0}^{\infty} \tilde{\lambda}_i e^{-\frac{\beta}{2}\tilde{\lambda}_i} e^{-\frac{\beta}{2}\tilde{\lambda}_i}(1-e^{-\beta\tilde{\lambda}_i})^{-1} \\ &\le (1-e^{-\beta_0\tilde{\lambda}_0})^{-1} c(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\tilde{\lambda}_i}, \end{aligned} \tag{5.50}$$

where we used (5.43) and

$$\tilde{\lambda}_i e^{-\frac{\beta}{2}\tilde{\lambda}_i} \le \sup_{t>0} t e^{-\frac{\beta}{2}t} = c(\beta) \le c(\beta_0). \tag{5.51}$$

Furthermore, we know that

$$\begin{aligned} \sum_{i=0}^{\infty} e^{-\frac{\beta}{2}\tilde{\lambda}_i} &\le \tilde{c}(\beta) \sum_{i=0}^{\infty} e^{-\frac{\beta}{4}\lambda_i} \\ &\le \tilde{c}(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta_0}{4}\lambda_i}, \end{aligned} \tag{5.52}$$

cf. Lemma 4.7 on page 33 and Lemma 4.8 on page 35, hence we obtain

$$E \le (1-e^{-\beta_0\bar{\lambda}_0\tau})^{-1} c(\beta_0)\tilde{c}(\beta_0) \sum_{i=0}^{\infty} e^{-\frac{\beta}{4}\bar{\lambda}_i\tau} \tag{5.53}$$

deducing further

$$\limsup_{\tau\to\infty} E \le c(\beta_0)\tilde{c}(\beta_0) \lim_{\tau\to\infty} \sum_{i=0}^{\infty} e^{-\frac{\beta}{4}\bar{\lambda}_i\tau} = 0 \tag{5.54}$$

uniformly in β and hence

$$\lim_{\tau\to\infty} E = 0. \tag{5.55}$$

It remains to prove that S vanishes in the limit. We have

$$\begin{aligned} Z &= \prod_{i=0}^{\infty}(1-e^{-\beta\tilde\lambda_i})^{-1} = \prod_{i=0}^{\infty}(1+e^{-\beta\tilde\lambda_i}(1-e^{-\beta\tilde\lambda_i})^{-1}) \\ &\le \exp\{(1-e^{\beta_0\tilde\lambda_0})^{-1}\sum_{i=0}^{\infty}e^{-\beta\tilde\lambda_i}\}, \end{aligned} \tag{5.56}$$

where we used the inequality

$$\log(1+t)\le t \qquad \forall\, t\ge 0 \tag{5.57}$$

in the last step.

Applying then the arguments preceding the inequality (5.54) we conclude

$$\lim_{\tau\to\infty} Z = 1 \tag{5.58}$$

uniformly in β. □

5.5. **Remark.** The first part of the preceding theorem reveals that the energy becomes very large for small values of $|\Lambda|$. Since this is the energy obtained by applying quantum statistics to the quantized version of a black hole or of a globally hyperbolic spacetime—assuming its Cauchy hypersurfaces are asymptotically Euclidean—a small negative cosmological constant might be responsible for the dark matter, where we equate the energy of the quantized universe with matter. As source for the dark energy density we conjecture that the dark energy density should be proportional to the eigenvalue of the density operator ρ with respect to the vacuum vector η

$$\rho\eta = Z^{-1}\eta, \tag{5.59}$$

which is Z^{-1}.

The behaviour of Z with respect to Λ is described in the theorem:

5.6. **Theorem.** *Let $\beta_0>0$ be arbitrary, then, for any*

$$0<\beta\le\beta_0, \tag{5.60}$$

we have

$$\lim_{\Lambda\to 0} Z = \infty \tag{5.61}$$

and for any

$$\beta_0\le\beta \tag{5.62}$$

the relation

$$\lim_{|\Lambda|\to\infty} Z = 1 \tag{5.63}$$

is valid. The convergence in both limites is uniform in β.

Proof. „(5.60)“ Let $m \in \mathbb{N}$ be arbitrary, then

$$\begin{aligned} Z &\ge \prod_{i=0}^{\infty}(1-e^{-\beta\lambda_i})^{-1} = \prod_{i=0}^{\infty}(1-e^{-\beta\bar{\lambda}_i\tau})^{-1} \\ &\ge \prod_{i=0}^{m}(1-e^{-\beta_0\bar{\lambda}_i\tau})^{-1} \end{aligned} \tag{5.64}$$

and we infer

$$\lim_{\tau\to 0} Z = \liminf_{\tau\to 0} Z = \infty. \tag{5.65}$$

„(5.63)“ This limit relation has already been proved in (5.58). □

Let us now consider the quantized globally hyperbolic spacetimes with an asymptotically Euclidean Cauchy hypersurface. The eigenspaces

$$\mathscr{E}_{\lambda_i} \subset \mathscr{S}'(\mathcal{S}_0) \tag{5.66}$$

of H_1 are separable but they are in general not finite dimensional as can be seen by the following counterexample

$$H_1 = -\Delta \tag{5.67}$$

in $\mathbb{R}^n$. The eigenspaces

$$\mathscr{E}_{\lambda_i}, \qquad \lambda_i > 0, \tag{5.68}$$

contain the tempered distributions

$$e^{i\langle k,x\rangle}, \qquad k \in \mathbb{S}^{n-1}_{\lambda_i}. \tag{5.69}$$

As a Hamel basis they generate a vector space the dimension of which is equal to the cardinality of $\mathbb{S}^{n-1}$. Of course, as a Schauder basis the functions with

$$k \in D \subset \mathbb{S}^{n-1}_{\lambda_i}, \tag{5.70}$$

where D is countable and dense, generate a dense subspace.

This example indicates that not all eigendistributions of H_1 might be physically relevant. Contrary to the cases of the black holes, where the selection of eigenvectors and eigendistributions was a natural process, only the temporal eigenvectors are naturally selected in the present situation and of course at least one matching spatial eigendistribution to obtain a solution of the wave equation. Hence, we could use H_0 to define the partition function. However, we believe this choice would be too restrictive, and we shall instead stipulate that we only pick at most

$$c|\lambda_i|^p \tag{5.71}$$

spatial eigendistributions in $\mathscr{E}_{\lambda_i}$, where c and p are arbitrary but fixed constants, i.e., we assume that

$$n(\lambda_i) \le c|\lambda_i|^p \qquad \forall\, i \in \mathbb{N}. \tag{5.72}$$

With this assumption it becomes evident that the results and conjectures of Theorem 5.4, Remark 5.5 and Theorem 5.6 are also valid in case of globally hyperbolic spacetimes with asymptotically Euclidean hypersurfaces.

6. The Friedmann universes with negative cosmological constants

In [5, Remark 6.11] we observed that, if the Cauchy hypersurface $\mathcal{S}_0$ is a space of constant curvature and if the wave equation (4.1) on page 25 is only considered for functions u which do not depend on x, then this equation is identical to the equation obtained by quantizing the Hamilton constraint in a Friedman universe without matter but including a cosmological constant. The equation is then the ODE

$$\frac{1}{32}\frac{n^2}{n-1}\ddot{u} - \frac{n}{2}Rt^{2-\frac{4}{n}}u + nt^2\Lambda u = 0, \qquad 0 < t < \infty, \tag{6.1}$$

where R is the scalar curvature of $\mathcal{S}_0$. We cannot apply our previous arguments to the solutions of this ODE. However, if we consider instead the more general equation (4.1), where u is also allowed to depend on x, which certainly is more general and accurate, then the previous arguments can be applied if the curvature $\tilde{\kappa}$ of $\mathcal{S}_0$ vanishes

$$\tilde{\kappa} = 0. \tag{6.2}$$

The scalar curvature, which is equal to

$$R = n(n-1)\tilde{\kappa}, \tag{6.3}$$

then vanishes too and

$$\mathcal{S}_0 = \mathbb{R}^n. \tag{6.4}$$

We are now in the situation which we analyzed at the end of the previous section, where now the spatial Hamiltonian is

$$H_1 = -(n-1)\Delta \tag{6.5}$$

and some spatial eigendistributions are shown in (5.69) on page 43. However, since we consider the quantized version of a Friedmann universe we shall look for radially symmetric eigendistributions, i.e., we look for smooth functions $v = v(x)$ satisfying

$$v(x) = \varphi(r) \tag{6.6}$$

such that

$$\Delta v = \ddot{\varphi} + (n-1)r^{-1}\dot{\varphi} = -\mu^2\varphi \quad \text{in} \quad r > 0, \tag{6.7}$$

where $\mu > 0$. Obviously, it is sufficient to assume $\mu = 1$, because, if φ is an eigenfunction for $\mu = 1$, then

$$\tilde{\varphi}(r) = \varphi(\mu r) \tag{6.8}$$

is an eigenfunction for the eigenvalue μ^2. Therefore, let us choose $\mu = 1$.

We shall express the solution φ with the help of a Bessel function J_ν. Let ψ be a solution of the Bessel equation

$$\ddot{\psi} + r^{-1}\dot{\psi} + (1 - r^{-2}\nu^2)\psi = 0, \tag{6.9}$$

where

$$\nu = \frac{n-2}{2}, \tag{6.10}$$

then the function

$$\varphi(r) = r^{-\nu}\psi \tag{6.11}$$

satisfies

$$r\ddot{\varphi} + (2\nu + 1)\dot{\varphi} + r\varphi = 0, \tag{6.12}$$

which is equivalent to (6.7) with $\mu = 1$. The Bessel equation (6.9) has the two independent solutions J_ν and Y_ν, the Bessel functions of first kind resp. of second kind. It is well known that the functions

$$r^{-\nu} J_\nu \tag{6.13}$$

can be expressed as a power series in the variable r^2, cf. [2, equ. (21), p. 420], i.e., the function

$$v(x) = \varphi(r) = r^{-\nu} J_\nu \tag{6.14}$$

is smooth in $\mathbb{R}^n$, while the functions

$$r^{-\nu} Y_\nu \tag{6.15}$$

have a singularity in $r = 0$. Hence, there exists exactly one smooth radially symmetric solution v of the eigenvalue equation

$$-\Delta v = \lambda^2 v, \qquad \lambda > 0, \tag{6.16}$$

which is given by

$$v = (\lambda r)^{-\nu} J_\nu(\lambda r). \tag{6.17}$$

This solution also vanishes at infinity, hence it is uniformly bounded and a tempered distribution.

A solution of the wave equation (4.1) on page 25, in case of a quantized Friedmann universe, is therefore given by a sequence

$$u_i = w_i(t) v_i(x), \qquad i \in \mathbb{N}, \tag{6.18}$$

where w_i is a temporal eigenfunction and v_i a spatial eigenfunction. The u_i are also eigenfunctions for the temporal Hamiltonian as well as for the spatial Hamiltonian. Each eigenvalue has multiplicity one. We have therefore proved:

6.1. **Theorem.** *The results in Theorem 5.4, Remark 5.5 and Theorem 5.4 are also valid, if the quantized spacetime $N = N^{n+1}$, $n \geq 3$, is a Friedmann universe without matter but with a negative cosmological constant Λ and with vanishing spatial curvature. The eigenvalues of the spatial Hamiltonian H_1 all have multiplicity one.*

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