SMOOTHNESS OF LIPSCHITZ SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

CLAUS GERHARDT

Contents

1. Lipschitz solutions are smooth

We consider the Dirichlet problem

$$\begin{aligned} Au + H(x,u,Du) &= 0, \\ u_{|_{|\partial\Omega}} &= \varphi, \end{aligned} \tag{1.1}$$

where

$$Au = -D_i(a^i(x,u,Du)) \tag{1.2}$$

is an elliptic quasilinear operator. The vector field a^i and H should satisfy

$$a^i \in C^{m-1,\alpha}(\bar\Omega \times \mathbb{R} \times \mathbb{R}^n), \tag{1.3}$$

and

$$H \in C^m(\bar\Omega \times \mathbb{R} \times \mathbb{R}^n), \tag{1.4}$$

such that $2 \le m \in \mathbb{N}$, $0 < \alpha < 1$.

Then we can prove:

1.1. **Theorem.** *Let $\Omega \Subset \mathbb{R}^n$ be open, connected with $\partial\Omega \in C^{m,\alpha}$, $2 \le m \in \mathbb{N}$, $0 < \alpha < 1$, and let $\varphi \in C^{m,\alpha}(\bar\Omega)$. Then any weak solution $u \in C^{0,1}(\bar\Omega)$ of (1.1) is of class $C^{m,\alpha}(\bar\Omega)$.*

Proof. (i) Let $u_0 \in C^{0,1}(\bar\Omega)$ be a weak solution such that

$$1 + |u_0| + |Du_0| \le M. \tag{1.5}$$

Date: June 27, 2011.

From [2, Section 1] and [1, Appendix II] we conclude that there exists a uniformly elliptic vector field $\tilde{a}^i \in C^{m-1,\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and a function $\tilde{H} \in C^m(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ such that

$$|\tilde{a}^{ij}| + \left|\frac{\partial \tilde{a}^i}{\partial x^j}\right| + \left|\frac{\partial \tilde{a}^i}{\partial t}\right| \le c, \tag{1.6}$$

$$|\tilde{a}^i| \le c(1 + |\xi|), \tag{1.7}$$

and

$$|\tilde{a}^{ij}|_{m-2,\alpha} \le c. \tag{1.8}$$

Furthermore, we have

$$|\tilde{H}|_{m,\alpha} \le c, \tag{1.9}$$

and

$$Au_0 + H(x, u_0, Du_0) = \tilde{A}u_0 + \tilde{H}(x, u_0, Du_0), \tag{1.10}$$

Hence, there exists a constant $\gamma > 0$ such that the operator

$$u \to \tilde{A}u + H(x, u, Du) + \gamma u \tag{1.11}$$

is coercive, i.e., there exists a constant $m > 0$ such that for any functions $u_i \in H^{1,2}(\Omega)$, $i = 1, 2$, with equal boundary values there holds

$$\begin{aligned}\langle \tilde{A}u_2 + \tilde{H}(x, u_2, Du_2) + \gamma u_2 - \tilde{A}u_1 - \tilde{H}(x, u_1, Du_1) - \gamma u_1, u_2 - u_1\rangle \\ \ge m\|u_2 - u_1\|^2_{1,2},\end{aligned} \tag{1.12}$$

and we conclude that the Dirichlet problem

$$\begin{aligned}\tilde{A}u + \tilde{H}(x, u, Du) + \gamma(u - u_0) &= 0, \\ u_{|_{|\partial\Omega}} &= \varphi,\end{aligned} \tag{1.13}$$

has the unique solution $u = u_0$.

(ii) Since $\partial\Omega \in C^{m,\alpha}$, we may extend u_0 in a neighbourhood of $\bar{\Omega}$ such that the extension is as smooth as u_0 is up to the maximal regularity $C^{m,\alpha}$. This will be important when we improve the regularity of u_0 successively. The improved regularity will then be reflected in the boundedness of the corresponding norms of a mollification of u_0, cf. the implication (1.17).

Let v_ϵ be a mollification of u_0, then we shall show in Lemma 1.2 below that the Dirichlet problem

$$\begin{aligned}\tilde{A}u_\epsilon + \tilde{H}(x, u_\epsilon, Du_\epsilon) + \gamma(u_\epsilon - v_\epsilon) &= 0, \\ u_{\epsilon|_{|\partial\Omega}} &= \varphi,\end{aligned} \tag{1.14}$$

has a unique solution $u_\epsilon \in C^{m,\alpha}(\bar{\Omega})$ such that

$$|u_\epsilon|_{m,\alpha,\bar{\Omega}} \le \text{const}, \tag{1.15}$$

where the constant depends on

$$|v_\epsilon|_{m-2,\alpha,\bar{\Omega}} + |v_\epsilon|_{1,\Omega} \tag{1.16}$$

but not explicitly on ϵ.

Thus, we conclude

$$u_0 \in C^{m-2,\alpha}(\bar{\Omega}) \cap C^{0,1}(\bar{\Omega}) \quad \Longrightarrow \quad |u_\epsilon|_{m,\alpha.\bar{\Omega}} \le \text{const} \tag{1.17}$$

uniformly in ϵ, and the u_ϵ converge in $C^2(\bar{\Omega})$ to a solution $u \in C^{m,\alpha}(\bar{\Omega})$ of (1.13), and we deduce further that

$$u = u_0, \tag{1.18}$$

i.e.,

$$u_0 \in C^{m-2,\alpha}(\bar{\Omega}) \cap C^{0,1}(\bar{\Omega}) \quad \Longrightarrow \quad u_0 \in C^{m,\alpha}(\bar{\Omega}). \tag{1.19}$$

By assumption u_0 is Lipschitz and therefore $u_0 \in C^{2,\alpha}(\bar{\Omega})$. A simple induction argument then yields the final result. $\square$

1.2. **Lemma.** *Under the assumptions of the preceding theorem the Dirichlet problem* (1.14) *has a solution* $u_\epsilon \in C^{m,\alpha}(\bar{\Omega})$ *satisfying the estimate* (1.15).

Proof. For simplicity we drop the subscript ϵ in u_ϵ resp. v_ϵ. Linearizing the operator in (1.14), the equation can be written in the form

$$-\tilde{a}^{ij} u_{ij} + a(x, u, Du) + \gamma(u - v) = 0, \tag{1.20}$$

where

$$\tilde{a}^{ij} = \tilde{a}^{ij}(x, u, Du) \tag{1.21}$$

is uniformly elliptic, $a \in C^{m-2,\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$

$$|a(x, t, \xi)| \le c(1 + |\xi|), \tag{1.22}$$

and

$$\frac{\partial a}{\partial t} + \gamma > 0, \tag{1.23}$$

i.e., the maximum principle can be applied to yield uniqueness of the linearized equation.

(i) Now, let $u \in C^{m,\alpha}(\bar{\Omega})$ be a solution (1.20) with boundary values φ, then we first deduce a priori estimates in the C^0-norm because of the maximum principle, and then a priori estimates in the C^1-norm,

$$|Du| \le c(|v|_1), \tag{1.24}$$

cf. [4, Theorem 15.2].

(ii) Since u is also a solution of (1.14) we obtain L^2-estimates

$$\|u\|_{2,2} \le c(|v|_1), \tag{1.25}$$

cf. PDE II, Kap. 1.6.

(iii) The De Giorgi-Nash estimates then yield a priori estimates in $C^{1,\lambda}(\bar{\Omega})$ for some $0 < \lambda < 1$. For simplicity we shall write α for λ having in mind that this α might not be the original α. However, once we have proved uniform C^2-estimates it will be obvious that we can switch back to the original meaning of α.

(iv) Applying then the Schauder estimates to the solution of (1.20), we infer

$$|u|_{2,\beta,\bar{\Omega}} \le c(|u|_{1,\alpha}, |v|_{0,\beta}, |v|_1) \tag{1.26}$$

for some

$$0 < \beta \le \alpha. \tag{1.27}$$

Applying the Schauder estimates again, but this time using the fact that $u \in C^2(\bar{\Omega})$, we deduce

$$|u|_{2,\alpha,\bar{\Omega}} \le c(|u|_{1,\alpha}, |v|_{0,\alpha}, |v|_1). \tag{1.28}$$

Using induction we then conclude

$$|u|_{m,\alpha,\bar{\Omega}} \le c(|u|_{1,\alpha}, |v_{m-2,\alpha}|, |v|_1). \tag{1.29}$$

(v) The preceding a priori estimates are also valid, if, instead of the Dirichlet problem (1.14), we consider solutions of the problems

$$\begin{aligned} \tilde{A}u + \sigma\tilde{H}(x,u,Du) + (1-\sigma)\frac{\partial\tilde{a}^i(x,u,Du)}{\partial x^i} + \gamma(u-\sigma v) &= 0, \\ u_{|_{|\partial\Omega}} &= \sigma\varphi, \end{aligned} \tag{1.30}$$

for fixed but arbitrary $0 < \sigma < 1$.

Choosing $\gamma > 0$ large enough, the new operator is also coercive independently of σ, because of (1.6), and the preceding a priori estimates are also independent of σ.

Then we conclude from [3, Theorem 3.1.2] that (1.14) has a solution $u_\epsilon \in C^{2,\alpha}(\bar{\Omega})$, and we deduce further that u_ϵ is also of class $C^{m,\alpha}$, because of the Schauder theory and the uniqueness, satisfying the estimate (1.29). □

Using the L^p-estimates instead of the Schauder estimates we can prove:

1.3. **Theorem.** *Let $\Omega \Subset \mathbb{R}^n$ be open, connected with $\partial\Omega \in C^2$, and let $\varphi \in H^{2,p}(\Omega)$, $n < p < \infty$. Then any weak solution $u \in C^{0,1}(\bar{\Omega})$ of (1.1) is of class $H^{2,p}(\Omega)$.*

Proof. Exercise; note that [3, Theorem 3.1.2] can also be applied in the present situation. □

References

[1] Claus Gerhardt, *Hypersurfaces of prescribed mean curvature over obstacles*, Math. Z. **133** (1973), 169–185, pdf file.

[2] ———, *Global $C^{1,1}$-regularity for solutions to quasi-linear variational inequalities*, Arch. Rat. Mech. Analysis **89** (1985), 83–92, pdf file.

[3] ———, *Existence theorems for quasilinear operators*, 2008, pdf file, Lecture notes.

[4] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.

Ruprecht-Karls-Universität, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany

E-mail address: gerhardt@math.uni-heidelberg.de

URL: http://www.math.uni-heidelberg.de/studinfo/gerhardt/