

QUANTUM COSMOLOGICAL FRIEDMAN MODELS WITH A YANG-MILLS FIELD AND POSITIVE ENERGY LEVELS

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ABSTRACT. We prove the existence of a spectral resolution of the Wheeler-DeWitt equation when the matter field is provided by a Yang-Mills field, with or without mass term, if the spatial geometry of the underlying spacetime is homothetic to \mathbb{R}^3 . The energy levels of the resulting quantum model, i.e., the eigenvalues of the corresponding self-adjoint Hamiltonian with a pure point spectrum, are strictly positive.

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1. INTRODUCTION

In a recent paper [2] we proved the existence of a spectral resolution of the Wheeler-DeWitt equation when the matter field is provided by a massive Yang-Mills field. The underlying spacetimes could be either spatially *closed*, i.e., spatially homothetic to S^3 , or *unbounded*, i.e., spatially homothetic to \mathbb{R}^3 .

However, the resulting quantum models had energy levels ranging from $-\infty$ to ∞ , due to the employed techniques.

In the present paper we prove, in case that the underlying spacetime is spatially homothetic to \mathbb{R}^3 , a different spectral resolution the energy levels of which are strictly positive.

As we have explained in [2, Introduction] solving the Wheeler-DeWitt equation comprises three steps: First, the Hamilton operators corresponding to the gravitational field and the matter field, respectively, have to be

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separated; second, for one of the operators a complete set of eigenfunctions has to be found, i.e., a *free* spectral resolution has to be proved without any constraints; third, for the remaining Hamilton operator then a *constrained* spectral resolution has to be found by looking at the Wheeler-DeWitt equation as an *implicit* eigenvalue problem.

In our previous paper we treated the open and closed spatial geometries simultaneously and, therefore, had to use the Hamilton operator corresponding to the gravitational field to solve the free eigenvalue problem and the Hamilton operator corresponding to the Yang-Mills field to solve the implicit eigenvalue problem. For this reason we also had to assume a massive Yang-Mills field, since the scalar factor representing the mass played the role of the implicit eigenvalue.

However, when assuming flat spatial sections a different approach is possible with positive energy levels. The technical difference is that now the gravitational Hamiltonian H_1 can be used to solve the implicit eigenvalue problem instead of the Hamiltonian of the Yang-Mills field.

The Wheeler-DeWitt equation had the form, cf. [2, Theorem 3.2],

$$(1.1) \quad H_2\psi - H_1\psi = 0,$$

where the wave function $\psi = \psi(r, y)$ belongs to a suitable subspace of $L^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C})$ and where

$$(1.2) \quad H_1\psi = -\ddot{\psi} - \bar{A}r^4\psi + 4\tilde{\kappa}r^2\psi,$$

$$(1.3) \quad H_2\psi = -c_1\psi'' + V\psi - \bar{\mu}y^2\psi.$$

Here, the variables r resp. y represent the scale factor resp. the Yang-Mills field, and a dot indicates differentiation with respect to r and a prime with respect to y . \bar{A} is a positive multiple of the cosmological constant Λ , $\tilde{\kappa}$ the spatial curvature, i.e., $\tilde{\kappa} \in \{0, 1\}$, c_1 a positive constant, $\bar{\mu}$ a positive multiple of the mass of the Yang-Mills field and V the potential

$$(1.4) \quad V = 2\alpha_M(\tilde{\kappa}y + y^2)^2,$$

where α_M is a positive coupling constant for the matter Lagrangian, cf. [2, equ. (1.13)].

H_1 is the Hamiltonian of the gravitational field and H_2 the Hamiltonian of the (massive) Yang-Mills field.

Contrary to the situation in [2] $\bar{\mu}$ will now be fixed, only subject to the requirement

$$(1.5) \quad \bar{\mu} < \bar{\mu}_0,$$

where $0 < \bar{\mu}_0$ is an extremal value such that the free eigenvalue problem

$$(1.6) \quad H_2\eta = \mu\eta$$

will have a *smallest* eigenvalue $\mu = \mu_0 = 0$ when $\bar{\mu} = \bar{\mu}_0$ and $\tilde{\kappa} = 0$.

In case $\bar{\mu} < \bar{\mu}_0$ and $\tilde{\kappa} = 0$ the smallest eigenvalue μ_0 will always be positive. We emphasize that especially the value $\bar{\mu} = 0$ is allowed which would remove the mass term in the Lagrangian.

Choosing $\tilde{\kappa} = 0$ the Hamilton operator H_1 in (1.2) has the form

$$(1.7) \quad H_1 u = -\ddot{u} - \bar{\Lambda} r^4 u$$

and for this operator we can solve an implicit eigenvalue problem by using a rescaling trick as in [3, Theorem 1.7].

We shall prove:

1.1. Theorem. *Assuming $\tilde{\kappa} = 0$ and $\bar{\mu}$ satisfying (1.5), there exists a self-adjoint operator H in the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C})$,*

$$(1.8) \quad H = H_2^{-1} \tilde{H}_1 = \tilde{H}_1 H_2^{-1},$$

where

$$(1.9) \quad \tilde{H}_1 \psi = -\ddot{\psi} + r^4 \psi,$$

with a pure point spectrum consisting of countably many eigenvalues λ_{ij} ,

$$(1.10) \quad \lambda_{ij} = \tilde{\lambda}_i \mu_j^{-1},$$

$\tilde{\lambda}_i$ resp. μ_j are the eigenvalues of the operators \tilde{H}_1 resp. H_2 , such that the properly rescaled eigenfunctions

$$(1.11) \quad \psi_{ij}(r, y) = \tilde{\psi}_{ij}(\lambda_{ij}^{-\frac{1}{2}} r, y)$$

are solutions of the Wheeler-DeWitt equation

$$(1.12) \quad H_2 \psi_{ij} - H_1 \psi_{ij} = 0,$$

where

$$(1.13) \quad H_1 \psi_{ij} = -\ddot{\psi}_{ij} - \bar{\Lambda}_{ij} r^4 \psi_{ij}$$

and

$$(1.14) \quad \bar{\Lambda}_{ij} = -\lambda_{ij}^{-3}.$$

The eigenvalues λ_{ij} are strictly monotone increasing in i and strictly monotone decreasing in j and they range from 0 to ∞

$$(1.15) \quad \lim_i \lambda_{ij} = \infty \quad \wedge \quad \lim_j \lambda_{ij} = 0.$$

The solutions of the corresponding Schrödinger equation, with initial values $\tilde{\psi}_0$ belonging to the span of the eigenfunctions, provide a dynamical development of the quantum model.

The theorem will be proved in the following two sections.

2. THE EIGENVALUE PROBLEMS

The Hamiltonian in the Wheeler-DeWitt equation (1.1) on page 2 is already separated, hence, a separation of variables is possible

$$(2.1) \quad \psi(r, y) = u(r)\eta(y), \quad (r, y) \in \mathbb{R}_+ \times \mathbb{R}.$$

We first solve the free eigenvalue problem for H_2

$$(2.2) \quad H_2\eta = \mu\eta,$$

where, for simplicity, we assume, without loss of generality, η to be real valued.

A complete set of eigenfunctions can be found with the help of a well-known variational principle, see e.g., [1] for details. Let \mathcal{H}_2 be the Hilbert space obtained by the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm

$$(2.3) \quad \|\eta\|_2^2 = \int_{\mathbb{R}} (|\eta'|^2 + y^4|\eta|^2).$$

Then the quadratic form

$$(2.4) \quad K(\eta) = \int_{\mathbb{R}} |\eta|^2$$

is compact in \mathcal{H}_2 , cf. [3, Lemma 6.8] for a proof in a similar situation, and the quadratic form

$$(2.5) \quad \langle H_2\eta, \eta \rangle + cK(\eta) = \int_{\mathbb{R}} (c_1|\eta'|^2 + V|\eta|^2 - \bar{\mu}y^2|\eta|^2) + cK(\eta)$$

is uniformly positive definite if the positive constant c is large enough.

Thus, we conclude:

2.1. Theorem. *There exist countably many eigenfunctions η_i with eigenvalues μ_i such that*

$$(2.6) \quad \mu_i < \mu_{i+1} \quad \forall i \in \mathbb{N}$$

and

$$(2.7) \quad \lim_i \mu_i = \infty.$$

The eigenfunctions (u_i) are dense in \mathcal{H}_2 as well as in $L^2(\mathbb{R})$ and the eigenvalues have multiplicities 1.

The theorem is valid for arbitrary values of $\bar{\mu}$ in (1.3) on page 2. Moreover, we can prove a very precise *mass gap* for that particular Yang-Mills Hamiltonian:

2.2. Theorem. *There exists exactly one $\bar{\mu}_0 > 0$ such that, when choosing $\bar{\mu} = \bar{\mu}_0$ in (1.3) on page 2, the corresponding smallest eigenvalue μ_0 satisfies*

$$(2.8) \quad \mu_0 = 0.$$

Choosing $\bar{\mu} < \bar{\mu}_0$ the corresponding smallest eigenvalue μ_0 is strictly positive

$$(2.9) \quad \mu_0 = \mu(\bar{\mu}) > 0.$$

Proof. (i) For $\bar{\mu} \in \mathbb{R}$ and $\eta \in \mathcal{H}_2$ consider the functional

$$(2.10) \quad J_{\bar{\mu}}(\eta) = \int_{\mathbb{R}} c_1 |\eta'|^2 + V|\eta|^2 - \bar{\mu} \int_{\mathbb{R}} y^2 |\eta|^2.$$

Define

$$(2.11) \quad \mu(\bar{\mu}) = \inf \{ J_{\bar{\mu}}(\eta) : \int_{\mathbb{R}} |\eta|^2 = 1, \eta \in \mathcal{H}_2 \}$$

and set

$$(2.12) \quad E = \{ \bar{\mu} : \mu(\bar{\mu}) \leq 0 \}.$$

We immediately deduce

$$(2.13) \quad E \neq \emptyset$$

and

$$(2.14) \quad \bar{\mu} \in E \implies \bar{\mu} > 0.$$

We also note that $\mu(\bar{\mu})$ is exactly the smallest eigenvalue μ_0 of the corresponding eigenvalue problem (2.2)

$$(2.15) \quad \mu_0 = \mu(\bar{\mu}).$$

Let

$$(2.16) \quad \bar{\mu}_0 = \inf E,$$

then $\bar{\mu}_0 \in E$ because of the compactness of the form (2.4) and hence

$$(2.17) \quad \bar{\mu}_0 > 0,$$

in view of (2.14).

(ii) Next, we claim that

$$(2.18) \quad \mu_0 = \mu(\bar{\mu}_0) = 0.$$

We argue by contradiction. Assume

$$(2.19) \quad \mu_0 < 0$$

and let η be a corresponding eigenfunction with unit L^2 -norm such that

$$(2.20) \quad J_{\bar{\mu}_0}(\eta) = \mu_0 < 0,$$

then we infer

$$(2.21) \quad J_{\bar{\mu}_0 - \delta}(\eta) \leq \frac{\mu_0}{2} < 0,$$

if $\delta > 0$ is small enough contradicting the definition of $\bar{\mu}_0$.

(iii) Let $\bar{\mu} < \bar{\mu}_0$, then for any $0 \neq \eta \in \mathcal{H}_2$

$$(2.22) \quad 0 \leq J_{\bar{\mu}_0}(\eta) < J_{\bar{\mu}}(\eta),$$

hence

$$(2.23) \quad \mu(\bar{\mu}) > 0.$$

(iv) To prove the uniqueness of $\bar{\mu}_0$ let $\bar{\mu}_1 \neq \bar{\mu}_0$ be another value such that

$$(2.24) \quad \mu(\bar{\mu}_1) = 0.$$

In view of (iii) there holds

$$(2.25) \quad \bar{\mu}_0 < \bar{\mu}_1,$$

hence

$$(2.26) \quad 0 \leq J_{\bar{\mu}_1}(\eta) < J_{\bar{\mu}_0}(\eta) \quad \forall 0 \neq \eta \in \mathcal{H}_2;$$

a contradiction. \square

Next, we consider the constrained eigenvalue problem for H_1 . Let $\mu > 0$ be one of the eigenvalues of H_2 , then we look at the implicit eigenvalue problem

$$(2.27) \quad H_1 u = -\ddot{u} - \bar{A} r^4 u = \mu u,$$

where \bar{A} or $-\bar{A}$ should play the role of an eigenvalue, i.e., it is more precisely an implicit eigenvalue problem for the operator

$$(2.28) \quad u \rightarrow -\ddot{u} - \mu u.$$

However, the quadratic form

$$(2.29) \quad K(u) = \int_{\mathbb{R}_+} r^4 |u|^2$$

is not compact relative to any reasonable energy form.

To solve (2.27) we have to use a rescaling trick as in [3, Theorem 1.7].

Let us first consider the Hamiltonian

$$(2.30) \quad \tilde{H}_1 u = -\ddot{u} + r^4 u$$

with corresponding energy form

$$(2.31) \quad \langle \tilde{H}_1 u, u \rangle = \int_{\mathbb{R}_+} (|\dot{u}|^2 + r^4 |u|^2) \equiv \|u\|_1^2$$

and define the real Hilbert space \mathcal{H}_1 as the completion of $C_c^\infty(\mathbb{R}_+)$ with respect to the norm $\|\cdot\|_1$.

The eigenvalue problem

$$(2.32) \quad \tilde{H}_1 \tilde{u} = \tilde{\lambda} \tilde{u}$$

is then solvable and we obtain an analogue of Theorem 2.1, namely:

2.3. Theorem. *There exist countably many eigenfunctions \tilde{u}_i with eigenvalues $\tilde{\lambda}_i$ such that*

$$(2.33) \quad \tilde{\lambda}_i < \tilde{\lambda}_{i+1} \quad \forall i \in \mathbb{N},$$

$$(2.34) \quad \tilde{\lambda}_0 > 0,$$

and

$$(2.35) \quad \lim_i \tilde{\lambda}_i = \infty.$$

The eigenfunctions (\tilde{u}_i) are dense in \mathcal{H}_1 as well as in $L^2(\mathbb{R}_+)$ and the eigenvalues have multiplicities 1.

2.4. Theorem. Let $\mu > 0$, then the pairs (\tilde{u}_i, λ_i) represent a complete set of eigenfunctions with eigenvalues

$$(2.36) \quad \lambda_i = \tilde{\lambda}_i \mu^{-1}$$

for the eigenvalue problem

$$(2.37) \quad \tilde{H}_1 u = \lambda \mu u.$$

The rescaled functions

$$(2.38) \quad u_i(r) = \tilde{u}_i(\lambda_i^{-\frac{1}{2}} r)$$

then satisfy

$$(2.39) \quad -\ddot{u} + \lambda_i^{-3} r^4 u_i = \mu u_i,$$

or, if we set

$$(2.40) \quad \bar{\Lambda}_i = -\lambda_i^{-3},$$

$$(2.41) \quad -\ddot{u} - \bar{\Lambda}_i r^4 u_i = \mu u_i.$$

3. THE SPECTRAL RESOLUTION

Let (μ, η) resp. (λ, \tilde{u}) satisfy

$$(3.1) \quad H_2 \eta = \mu \eta$$

resp.

$$(3.2) \quad \tilde{H}_1 \tilde{u} = \lambda \mu \tilde{u},$$

then

$$(3.3) \quad \tilde{\psi} = \tilde{u} \eta$$

solves

$$(3.4) \quad \tilde{H}_1 \tilde{\psi} = \lambda H_2 \tilde{\psi},$$

or equivalently, in view of Theorem 2.4 on page 7,

$$(3.5) \quad H_1 \psi - H_2 \psi = 0,$$

where

$$(3.6) \quad \psi = u \eta,$$

$$(3.7) \quad u(r) = \tilde{u}(\lambda^{-\frac{1}{2}} r),$$

$$(3.8) \quad H_1 \psi = -\ddot{\psi} - \bar{\Lambda} r^4 \psi,$$

and

$$(3.9) \quad \bar{\Lambda} = -\lambda^{-3},$$

i.e., ψ is a solution of the Wheeler-DeWitt equation.

Moreover,

$$(3.10) \quad \dot{\psi} = i\eta \quad \wedge \quad \psi' = u\eta',$$

hence,

$$(3.11) \quad \int_{\mathbb{R}_+ \times \mathbb{R}} |D\psi|^2 = \int_{\mathbb{R}_+} |\dot{u}|^2 \int_{\mathbb{R}} |\eta|^2 + \int_{\mathbb{R}_+} |u|^2 \int_{\mathbb{R}} |\eta'|^2,$$

and similarly,

$$(3.12) \quad \int_{\mathbb{R}_+ \times \mathbb{R}} |\psi|^2 y^p = \int_{\mathbb{R}_+} |u|^2 \int_{\mathbb{R}} |\eta|^2 y^p,$$

for $p = 2, 4$, as well as

$$(3.13) \quad \int_{\mathbb{R}_+ \times \mathbb{R}} |\psi|^2 r^4 = \int_{\mathbb{R}_+} |u|^2 r^4 \int_{\mathbb{R}} |\eta|^2.$$

Thus, ψ has bounded norm

$$(3.14) \quad \|\psi\|^2 = \int_{\mathbb{R}_+ \times \mathbb{R}} |D\psi|^2 + \int_{\mathbb{R}_+ \times \mathbb{R}} |\psi|^2 (r^4 + y^4).$$

Let \mathcal{H} be the completion of $C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ with respect to this norm, then \mathcal{H} can be viewed as a dense subspace of

$$(3.15) \quad \mathcal{H}_0 = L^2(\mathbb{R}_+ \times \mathbb{R})$$

and the eigenfunctions of (3.4) are complete in \mathcal{H} as well as \mathcal{H}_0 , where we note that the eigenfunctions $\tilde{\psi}_{ij}$ are products

$$(3.16) \quad \tilde{\psi}_{ij} = \tilde{u}_i \eta_j$$

with eigenvalues

$$(3.17) \quad \lambda_{ij} = \tilde{\lambda}_i \mu_j^{-1},$$

where we recall that $\tilde{\lambda}_i$ are the eigenvalues of the Hamiltonian \tilde{H}_1 , cf. Theorem 2.4 on page 7. Thus, the eigenvalues λ_{ij} are strictly monotone increasing in i and strictly monotone decreasing in j and they range from 0 to ∞

$$(3.18) \quad \lim_i \lambda_{ij} = \infty \quad \wedge \quad \lim_j \lambda_{ij} = 0.$$

The claim that the eigenfunctions are complete needs some verification.

3.1. Lemma. *The eigenfunctions $\tilde{\psi}_{ij}$ are complete in \mathcal{H} as well as in \mathcal{H}_0 .*

Proof. It suffices to prove the density in \mathcal{H} . The eigenfunctions are certainly complete in the closure of $C_c^\infty(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$ in \mathcal{H} , in view of (3.10) and (3.11), but $C_c^\infty(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$ is dense in \mathcal{H} as can be easily proved with the help of the Weierstraß approximation theorem. \square

From now on we shall assume that the functions are complex valued. Denote by A the operator

$$(3.19) \quad A = H_2^{-1} \tilde{H}_1$$

with domain $D(A) \subset \mathcal{H}_0$ equal to the subspace generated by its eigenfunctions $\tilde{\psi}_{ij}$.

We observe that A is well defined and that

$$(3.20) \quad H_2^{-1} \tilde{H}_1 = \tilde{H}_1 H_2^{-1}.$$

Moreover, one easily checks that H_2^{-1} and hence A are symmetric.

3.2. Lemma. *A is essentially self-adjoint in \mathcal{H}_0 .*

Proof. It suffices to prove that $R(A \pm i)$ is dense, which is evidently the case, since the eigenfunctions belong to $R(A \pm i)$. \square

Let H be the closure of A , then H is self-adjoint and the spectral resolution for the Wheeler-DeWitt equation accomplished, since there holds:

3.3. Lemma. *Let $(\psi, \lambda) \in \mathcal{H} \times \mathbb{R}_+$ be a solution of the Wheeler-DeWitt equation*

$$(3.21) \quad -\ddot{\psi} + \lambda^{-3} r^4 \psi - H_2 \psi = 0,$$

then there exists $(ij) \in \mathbb{N} \times \mathbb{N}$ such that

$$(3.22) \quad \lambda = \lambda_{ij}$$

and

$$(3.23) \quad \psi = \psi_{ij},$$

where

$$(3.24) \quad \psi_{ij}(r, y) = \tilde{\psi}_{ij}(\lambda_{ij}^{-\frac{1}{2}} r, y),$$

and $\tilde{\psi}_{ij}$ is an eigenfunction of H with eigenvalue λ_{ij} .

Proof. Define

$$(3.25) \quad \tilde{\psi}(r, y) = \psi(\lambda^{\frac{1}{2}} r, y),$$

then $\tilde{\psi}$ is a solution of

$$(3.26) \quad H\tilde{\psi} = \lambda\tilde{\psi},$$

hence the result.

Note that the eigenspaces of H are not necessarily one-dimensional. \square

The Schrödinger equation for H offers a dynamical development of the system provided the initial value is a finite superposition of eigenfunctions, since then the time dependent solutions are also solutions of the Wheeler-DeWitt equation, cf. the remarks at the end of [3, Section 8].

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