THE QUANTIZATION OF A KERR-ADS BLACK HOLE

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ABSTRACT. We apply our model of quantum gravity to a Kerr-AdS spacetime of dimension $2m+1, \geq 2$, where all rotational parameters are equal, resulting in a wave equation in a quantum spacetime which has a sequence of solutions that can be expressed as a product of stationary and temporal eigenfunctions. The stationary eigenfunctions can be interpreted as radiation and the temporal as gravitational waves. The event horizon corresponds in the quantum model to a Cauchy hypersurface that can be crossed by causal curves in both directions such that the information paradox does not occur. We also prove that the Kerr-AdS spacetime can be maximally extended by replacing in a generalized Boyer-Lindquist coordinate system the r variable by $\rho = r^2$ such that the extended spacetime has a timelike curvature singularity in $\rho = -a^2$.

CONTENTS

1. INTRODUCTION

In general relativity the Cauchy development of a Cauchy hypersurface $\mathcal{S}_0$ is governed by the Einstein equations, where of course the second fundamental form of $\mathcal{S}_0$ has also to be specified.

In the model of quantum gravity we developed in a series of papers [5, 6, 3, 7, 8, 9, 11] we pick a Cauchy hypersurface, which is then only considered to be a complete Riemannian manifold $(\mathcal{S}_0, g_{ij})$ of dimension $n \geq 3$, and define

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its quantum development to be special solutions of the wave equation

$$(1.1) \qquad \frac{1}{32}\frac{n^2}{n-1}\ddot{u} - (n-1)t^{2-\frac{4}{n}}\Delta u - \frac{n}{2}t^{2-\frac{4}{n}}Ru + nt^2\Lambda u = 0$$

defined in the spacetime

$$(1.2) \qquad\qquad\qquad Q = \mathcal{S}_0 \times (0,\infty).$$

The Laplacian is the Laplacian with respect to g_{ij}, R is the scalar curvature of the metric, $0 < t$ is the time coordinate defined by the derivation process of the equation and $\Lambda < 0$ a cosmological constant. If other physical fields had been present in the Einstein equations then the equation would contain further lower order terms, cf. [8], but a negative cosmological constant would always have to be present even if the Einstein equations would only describe the vacuum.

Using separation of variables we proved that there is a complete sequence of eigenfunctions v_j (or a complete set of eigendistributions) of a stationary eigenvalue problem and a complete sequence of eigenfunctions w_i of a temporal implicit eigenvalue problem, where Λ plays the role of an eigenvalue, such that the functions

$$(1.3) \qquad\qquad\qquad u(t,x) = w_i(t)v_j(x)$$

are solutions of the wave equation, cf. [7, Section 6] and [8, 9].

In a recent paper [10] we applied this model to quantize a Schwarzschild-AdS spacetime and we shall prove in the present paper that similar arguments can also be used to quantize a Kerr-AdS spacetime with a rotating black hole.

We consider an odd-dimensional Kerr-AdS spacetime N, $\dim N = 2m+1$, $m \geq 2$, where all rotational parameters are equal

$$(1.4) \qquad\qquad\qquad a_i = a \qquad \forall\, 1 \leq i \leq m,$$

and where we also set

$$(1.5) \qquad\qquad\qquad n = 2m.$$

Replacing the r coordinate in a generalized Boyer-Lindquist coordinate system by

$$(1.6) \qquad\qquad\qquad \rho = r^2$$

we shall prove that in the new coordinate system the metric is smooth in the interval

$$(1.7) \qquad\qquad\qquad -a^2 < \rho < \infty$$

and that the extended spacetime N has a timelike curvature singularity in $\rho = -a^2$, cf. Lemma 3.5 on page 17.

For the quantization we first assume that there is a non-empty interior black hole region B which is bounded by two horizons

$$(1.8) \qquad\qquad\qquad B = \{r_1 < r < r_2\}$$

where the outer horizon is the event horizon. Picking a Cauchy hypersurface in B of the form

$$(1.9) \qquad \{r = \text{const}\},$$

we shall prove that the induced metric of the Cauchy hypersurface can be expressed in the form

$$(1.10) \qquad ds^2 = d\tau^2 + \sigma_{ij} dx^i dx^j,$$

where

$$(1.11) \qquad -\infty < \tau < \infty,$$

$r = \text{const}$ and σ_{ij} is a smooth Riemannian metric on $\mathbb{S}^{2m-1}$ depending on r, a and the cosmological constant $\Lambda < 0$. The metric in (1.10) is free of any coordinate singularity, hence we can let r tend to r_2 such that the Cauchy hypersurfaces converge to a Riemannian manifold $\mathcal{S}_0$ which represents the event horizon at least topologically. Furthermore, the Laplacian of the metric in (1.10) comprises a harmonic oscillator with respect to τ which enables us to write the stationary eigenfunctions v_j in the form

$$(1.12) \qquad v_j(\tau, x) = \zeta(\tau)\varphi_j(x),$$

where φ_j is an eigenfunction of the elliptic operator

$$(1.13) \qquad -(n-1)\tilde{\Delta} - \frac{n}{2}R,$$

where

$$(1.14) \qquad \tilde{\Delta} = \Delta_M,$$

and ζ an eigenfunction of the harmonic oscillator the frequency of which are still to be determined.

Due to the presence of the harmonic oscillator we can now consider an *explicit* temporal eigenvalue problem, i.e., we consider the eigenvalue problem

$$(1.15) \qquad -\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w$$

with a fixed $\Lambda < 0$, where we choose Λ to be the cosmological constant of the Kerr-AdS spacetime.

The eigenvalue problem (1.15) has a complete sequence (w_i, λ_i) of eigenfunctions with finite energies λ_i such that

$$(1.16) \qquad 0 < \lambda_0 < \lambda_1 < \cdots$$

and by choosing the frequencies of ζ appropriately we can arrange that the stationary eigenvalues μ_j of v_j agree with the temporal eigenvalues λ_i. If this is the case then the eigenfunctions

$$(1.17) \qquad u = w_i v_j$$

will be a solution of the wave equation. More precisely we proved:

1.1. **Theorem.** *Let $(\varphi_j, \tilde{\mu}_j)$ resp. (w_i, λ_i) be eigenfunctions of the elliptic operator in* (1.13) *resp. the temporal eigenfunctions and, for a given index j, let λ_{i_0} be the smallest eigenvalue of the (λ_i) with the property*

$$\lambda_{i_0} \geq \tilde{\mu}_j, \tag{1.18}$$

then, for any $i \geq i_0$, there exists

$$\omega = \omega_{ij} \geq 0 \tag{1.19}$$

and corresponding ζ_{ij} satisfying

$$-\ddot{\zeta}_{ij} = \omega_{ij}^2 \tag{1.20}$$

such that

$$\lambda_i = \mu_{ij} = \omega_{ij}^2 + \tilde{\mu}_j \quad \forall\, i \geq i_0. \tag{1.21}$$

The functions

$$u_{ij} = w_i \zeta_{ij} \varphi_j \tag{1.22}$$

are then solutions of the wave equation with bounded energies satisfying

$$\lim_{t \to 0} u_{ij}(t) = \lim_{t \to \infty} u_{ij}(t) = 0 \tag{1.23}$$

and

$$u_{ij} \in C^\infty(\mathbb{R}_+^* \times \mathcal{S}_0) \cap C^{2,\alpha}(\bar{\mathbb{R}}_+^* \times \mathcal{S}_0) \tag{1.24}$$

for some

$$\frac{2}{3} \leq \alpha < 1. \tag{1.25}$$

Moreover, we have

$$\omega_{ij} > 0 \qquad \forall\, i > i_0. \tag{1.26}$$

If

$$\lambda_{i_0} = \tilde{\mu}_j, \tag{1.27}$$

then we define

$$\zeta_{i_0 j} \equiv 1. \tag{1.28}$$

1.2. **Remark.** (i) The event horizon corresponds to the Cauchy hypersurface $\{t = 1\}$ in Q and the open set

$$\{-a^2 < \rho < \rho_2\} \tag{1.29}$$

in N, where

$$\rho_2 = r_2^2, \tag{1.30}$$

to the region

$$\mathcal{S}_0 \times (0, 1), \tag{1.31}$$

while the part

$$\{\rho_2 < \rho < \infty\} \tag{1.32}$$

is represented by

$$(1.33) \qquad \mathcal{S}_0 \times (1, \infty).$$

The *timelike* black hole singularity corresponds to $\{t = 0\}$ which is a *spacelike* curvature singularity in the quantum spacetime provided we equip Q with a metric such that the hyperbolic operator is normally hyperbolic, cf. [8, Lemma 6.2]. Moreover, in the quantum spacetime the Cauchy hypersurface $\mathcal{S}_0$ can be crossed by causal curves in both directions, i.e., the information paradox does not occur.

(ii) The stationary eigenfunctions can be looked at as being radiation because they comprise the harmonic oscillator, while we consider the temporal eigenfunctions to be gravitational waves.

As it is well-known the Schwarzschild black hole or more specifically the extended Schwarzschild space has already been analyzed by Hawking [14] and Hartle and Hawking [13], see also the book by Wald [19], using quantum field theory, but not quantum gravity, to prove that the black hole emits radiation.

The metric describing a rotating black hole in a four-dimensional vacuum spacetime was first discovered by Kerr [17]. Carter [1] generalized the Kerr solution by describing a rotating black hole in a four-dimensional de Sitter or anti-de Sitter background. Higher dimensional solutions for a rotating black hole were given by Myers and Perry [18] in even-dimensional Ricci flat spacetimes and by Hawking, Hunter and Taylor [15] in five-dimensional spacetimes satisfying the Einstein equations with cosmological constant.

A general solution in all dimension was given in [12] by Gibbons et al. and we shall use their metric in odd dimensions, with all rotational parameters supposed to be equal, to define our spacetime N, though we shall maximally extend it.

1.3. Notations. We apply the summation convention and label coordinates with contravariant indices, e.g., μ^i. However, for better readability we shall usually write

$$(1.34) \qquad \mu_i^2$$

instead of

$$(1.35) \qquad (\mu^i)^2.$$

2. Preparations

We consider odd-dimensional Kerr-AdS spacetimes N, $\dim N = 2m + 1$, $m \geq 2$, assuming that all rotational parameters are equal

$$(2.1) \qquad a_i = a \neq 0, \qquad \forall\, 1 \leq i \leq m.$$

The Kerr-Schild form of the metric can then be expressed as

$$d\bar{s}^2 = -\frac{1 + l^2 r^2}{1 - a^2 l^2} dt^2 + \frac{r^2 dr^2}{(1 + l^2 r^2)(r^2 + a^2)}$$

(2.2)
$$+ \frac{r^2 + a^2}{1 - a^2 l^2} \sum_{i=1}^{m} (d\mu_i^2 + \mu_i^2 d\varphi_i^2)$$

$$+ \frac{2m_0}{U} \Big(\frac{1}{1 - a^2 l^2} (dt - a \sum_{i=1}^{m} \mu_i^2 d\varphi^i) + \frac{r^2 dr}{(1 + l^2 r^2)(r^2 + a^2)} \Big)^2,$$

where

$$(2.3) \qquad l^2 = -\frac{1}{m(2m-1)} \Lambda$$

and $\Lambda < 0$ is the cosmological constant such that the Einstein equations

$$(2.4) \qquad G_{\alpha\beta} + \Lambda \bar{g}_{\alpha\beta} = 0$$

are satisfied in N, m_0 is the mass of the black hole,

$$(2.5) \qquad U = (r^2 + a^2)^{m-1},$$

$$(2.6) \qquad \sum_{i=1}^{m} (d\mu_i^2 + \mu_i^2 d\varphi_i^2)$$

is the standard metric of $\mathbb{S}^{2m-1}$, where the φ^i are the azimuthal coordinates, the values of which have to be identified modulo 2π, and the μ^i are the latitudinal coordinates subject to the side-condition

$$(2.7) \qquad \sum_{i=1}^{m} \mu_i^2 = 1.$$

The μ^i also satisfy

$$(2.8) \qquad 0 \le \mu^i \le 1 \qquad \forall\, 1 \le i \le m.$$

The coordinates (t, r) are defined in

$$(2.9) \qquad -\infty < t < \infty$$

and

$$(2.10) \qquad 0 < r < \infty$$

respectively, cf. [12, Section 2 and Appendix B].

The horizons are hypersurfaces $\{r = \text{const}\}$, where $\rho = r^2$ satisfies the equation

$$(2.11) \qquad (1 + l^2 \rho)(\rho + a^2)^m - 2m_0 \rho = 0.$$

Let

$$(2.12) \qquad \Phi = \Phi(\rho)$$

be the polynomial on the left-hand side of (2.11), then Φ is strictly convex in $\mathbb{R}_+$ and we have

$$\tag{2.13} \Phi(0) > 0$$

and

$$\tag{2.14} \lim_{\rho \to \infty} \Phi(\rho) = \infty,$$

from which we deduce that the equation (2.11) is satisfied if and only if

$$\tag{2.15} \inf_{\mathbb{R}_+} \Phi \leq 0,$$

and in case

$$\tag{2.16} \inf_{\mathbb{R}_+} \Phi < 0$$

we have exactly two solutions otherwise only one. If there are two solutions r_i, $i = 1, 2$, such that

$$\tag{2.17} 0 < r_1 < r_2,$$

then the outer horizon is called *event horizon* and the black hole has an interior region

$$\tag{2.18} B = \{r_1 < r < r_2\}$$

in which the variable r is a time coordinate. If there is only one solution r_0, then B is empty and the black hole is called *extremal*.

We shall first quantize a black hole with $B \neq \emptyset$; the quantization of an extremal black hole is then achieved by approximation.

Thus, let us consider a non-extremal black hole and let $S \subset B$ be a space-like coordinate slice

$$\tag{2.19} S = S(r) = \{r = \text{const}\},$$

where r also denotes the constant value.

In view of (2.2), the induced metric can be expressed as

$$\tag{2.20}
\begin{aligned}
ds_S^2 &= \Big(\frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2} - \frac{1+l^2r^2}{1-a^2l^2}\Big)dt^2 - \frac{2m_0}{U}\frac{2a}{(1-a^2l^2)^2}\mu_i^2 dt d\varphi^i \\
&\quad + \Big(\frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}\Big)d\varphi^i d\varphi^j \\
&\quad + \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2,
\end{aligned}$$

from which we deduce

$$\tag{2.21} g_{tt} = \frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2} - \frac{1+l^2r^2}{1-a^2l^2},$$

$$\tag{2.22} g_{t\varphi^i} = g_{\varphi^i t} = -\frac{2m_0}{U}\frac{a}{(1-a^2l^2)^2}\mu_i^2,$$

and

$$(2.23) \qquad g_{\varphi^i\varphi^j} = \frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}.$$

The other components of the metric are either 0 or are represented by the line element

$$(2.24) \qquad \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^{m} d\mu_i^2,$$

note the constraint (2.7).

To eliminate the $g_{t\varphi^i}$ we shall introduce new coordinates. First, let us make the simple change by defining t' through

$$(2.25) \qquad ct' = t,$$

where $c \neq 0$ is a constant which will be specified later, and dropping the prime in the sequel, resulting in a replacement of the components in (2.21) and (2.22) by

$$(2.26) \qquad g_{tt} = c^2\Big(\frac{2m_0}{U}\frac{1}{(1-a^2l^2)^2} - \frac{1+l^2r^2}{1-a^2l^2}\Big)$$

respectively,

$$(2.27) \qquad g_{t\varphi^i} = g_{\varphi^i t} = -c\frac{2m_0}{U}\frac{a}{(1-a^2l^2)^2}\mu_i^2.$$

Next, we define new coordinates $(\tilde{t}, \tilde{\varphi}^i)$ by

$$(2.28) \qquad \alpha\tilde{t} = t$$

and

$$(2.29) \qquad \tilde{\varphi}^i = \varphi^i - a\gamma t,$$

where α, γ are non-vanishing constants to specified later, such that

$$(2.30) \qquad \varphi^i = \tilde{\varphi}^i + a\alpha\gamma\tilde{t}.$$

In the new coordinates the only interesting new components are

$$(2.31) \qquad \begin{aligned} g_{\tilde{t}\tilde{t}} &= g_{tt}\frac{\partial t}{\partial \tilde{t}}\frac{\partial t}{\partial \tilde{t}} + 2g_{t\varphi^i}\frac{\partial t}{\partial \tilde{t}}\frac{\partial \varphi^i}{\partial \tilde{t}} + g_{\varphi^i\varphi^j}\frac{\partial \varphi^i}{\partial \tilde{t}}\frac{\partial \varphi^j}{\partial \tilde{t}} \\ &= \alpha^2\big(g_{tt} + 2a\gamma\sum_i g_{t\varphi^i} + a^2\gamma^2\sum_{i,j}g_{\varphi^i\varphi^j}\big) \end{aligned}$$

and

$$(2.32) \qquad \begin{aligned} g_{\tilde{t}\tilde{\varphi}^i} &= g_{t\varphi^j}\frac{\partial t}{\partial \tilde{t}}\frac{\partial \varphi^j}{\partial \tilde{\varphi}^i} + g_{\varphi^k\varphi^l}\frac{\partial \varphi^k}{\partial \tilde{t}}\frac{\partial \varphi^l}{\partial \tilde{\varphi}^i} \\ &= \alpha\big(g_{t\varphi^i} + a\gamma\sum_k g_{\varphi^k\varphi^i}\big). \end{aligned}$$

We therefore deduce, in view of (2.23), (2.26) and (2.27),

$$(2.33) \quad g_{\tilde{t}\tilde{t}} = \alpha^2 \Big(\frac{2m_0}{U} \frac{1}{(1-a^2l^2)^2} (c - a^2\gamma)^2 + \frac{r^2 + a^2}{1 - a^2l^2} a^2\gamma^2 - \frac{1 + l^2r^2}{1 - a^2l^2} c^2 \Big)$$

and

$$(2.34) \quad g_{\tilde{t}\tilde{\varphi}^i} = \alpha \Big(\frac{2m_0}{U} \frac{a}{(1-a^2l^2)^2} (a^2\gamma - c) + \frac{r^2 + a^2}{1 - a^2l^2} a\gamma \Big) \mu_i^2.$$

Choosing now

$$(2.35) \qquad c = \Big(a^2 + \frac{U}{2m_0} (r^2 + a^2)(1 - a^2l^2) \Big) \gamma$$

we conclude

$$(2.36) \qquad g_{\tilde{t}\tilde{\varphi}^i} = 0.$$

Combining then (2.35) and (2.33) by setting $\gamma = 1$ we obtain

$$(2.37) \qquad \begin{aligned} g_{\tilde{t}\tilde{t}} = \alpha^2 \Big(&\frac{U}{2m_0} (r^2 + a^2)^2 + \frac{r^2 + a^2}{1 - a^2l^2} a^2 \\ &- \frac{1 + l^2r^2}{1 - a^2l^2} \Big(a^2 + \frac{U}{2m_0} (r^2 + a^2)(1 - a^2l^2) \Big)^2 \Big). \end{aligned}$$

Define

$$(2.38) \qquad \beta = \frac{U}{2m_0} (r^2 + a^2) - \frac{r^2}{1 + l^2r^2},$$

then

$$(2.39) \qquad \beta < 0 \qquad \text{in } B,$$

since the function Φ in (2.12) is negative in B. Writing

$$(2.40) \qquad \begin{aligned} a^2 + \frac{U}{2m_0} (r^2 + a^2)(1 - a^2l^2) &= a^2 + \frac{r^2}{1 + l^2r^2} (1 - a^2l^2) \\ &\quad + \beta(1 - a^2l^2) \\ &= \frac{r^2 + a^2}{1 + l^2r^2} + \beta(1 - a^2l^2), \end{aligned}$$

we infer

$$(2.41) \qquad g_{\tilde{t}\tilde{t}} = \alpha^2 (-\beta(r^2 + a^2) - \beta^2(1 + l^2r^2)(1 - a^2l^2)).$$

The term in the brackets vanishes on the event horizon and is strictly positive in B, in view of (2.39) and the identity

$$(2.42) \qquad \begin{aligned} &(r^2 + a^2) + \beta(1 + l^2r^2)(1 - a^2l^2) \\ &= (r^2 + a^2) + \frac{(r^2 + a^2)^m}{2m_0} (1 + l^2r^2)(1 - a^2l^2) - r^2(1 - a^2l^2) \\ &= a^2(1 + l^2) + \frac{(r^2 + a^2)^m}{2m_0} (1 + l^2r^2)(1 - a^2l^2) > 0. \end{aligned}$$

Hence, for any r satisfying

(2.43)
$$r_1 < r < r_2$$

we can choose $\alpha > 0$ such that

(2.44)
$$g_{\tilde{t}\tilde{t}} = 1.$$

Writing (τ, φ^i) instead of $(\tilde{t}, \tilde{\varphi}^i)$ we can then state

2.1. **Lemma.** *For any hypersurface*

(2.45)
$$S = S(r) \subset B$$

the induced metric can be expressed in the form

(2.46)
$$\begin{aligned}
ds_S^2 = {}& d\tau^2 + \Big(\frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}\Big)d\varphi^i d\varphi^j \\
& + \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2 \\
\equiv {}& d\tau^2 + \sigma_{ij}dx^i dx^j,
\end{aligned}$$

where

(2.47)
$$\sigma_{ij} = \sigma_{ij}(r, a, l)$$

is a smooth Riemannian metric on $\mathbb{S}^{2m-1}$ and τ ranges in $\mathbb{R}$, while in case

(2.48)
$$S = S(r) \subset N\backslash\bar{B},$$

the induced metric is Lorentzian of the form

(2.49)
$$\begin{aligned}
ds_S^2 = {}& -d\tau^2 + \Big(\frac{2m_0}{U}\frac{a^2}{(1-a^2l^2)^2}\mu_i^2\mu_j^2 + \frac{r^2+a^2}{1-a^2l^2}\mu_i^2\delta_{ij}\Big)d\varphi^i d\varphi^j \\
& + \frac{r^2+a^2}{1-a^2l^2}\sum_{i=1}^m d\mu_i^2 \\
\equiv {}& -d\tau^2 + \sigma_{ij}dx^i dx^j.
\end{aligned}$$

If $r < r_2$ tends to r_2, then the hypersurfaces $S(r)$ converge topologically to the event horizon and the induced metrics to the Riemannian metric

(2.50)
$$\begin{aligned}
ds_S^2 = {}& d\tau^2 + \frac{r^2+a^2}{1-a^2l^2}\big(\delta_{ij}d\mu^i d\mu^j + \mu_i^2\delta_{ij}d\varphi^i d\varphi^j\big) \\
& + a^2\frac{(1+l^2r^2)(r^2+a^2)}{r^2(1-a^2l^2)^2}\mu_i^2\mu_j^2 d\varphi^i d\varphi^j.
\end{aligned}$$

Proof. We only have to prove the case (2.48). However, the proof of this case is identical to the proof when (2.45) is valid by observing that then the term β in (2.41) is strictly positive. $\square$

3. The quantization

We are now in a position to argue very similar as in our former paper [10, Section 2]. For the convenience of the reader we shall repeat some of the arguments so that the results can be understood directly without having to look up the details in the reference.

The interior of the black hole is a globally hyperbolic spacetime and the slices $S(r)$ with

$$(3.1) \qquad r_1 < r < r_2$$

are Cauchy hypersurfaces. Let r tend to r_2 and let $\mathcal{S}_0$ be the resulting limit Riemannian manifold, i.e., topologically it is the event horizon but equipped with the metric in (2.50) which we shall write in the form

$$(3.2) \qquad ds^2 = d\tau^2 + \sigma_{ij} dx^i dx^j$$

as in (2.46) on page 10. By a slight abuse of language we shall also call $\mathcal{S}_0$ to be a Cauchy hypersurface though it is only the geometric limit of Cauchy hypersurfaces. However, $\mathcal{S}_0$ is a genuine Cauchy hypersurface in the quantum model which is defined by the equation (1.1) on page 2.

Let us now look at the stationary eigenvalue equation, where we recall that $n = 2m$,

$$(3.3) \qquad -(n-1)\Delta v - \frac{n}{2} R v = \mu v$$

in $\mathcal{S}_0$, where

$$(3.4) \qquad -(n-1)\Delta v = -(n-1)\ddot{v} - (n-1)\tilde{\Delta} v$$

and $\tilde{\Delta}$ is the Laplacian in the Riemannian manifold

$$(3.5) \qquad M = (\mathbb{S}^{n-1}, \sigma_{ij});$$

moreover the scalar curvature R is also the scalar curvature with respect to σ_{ij} in view of (3.2). Using separation of variables let us write

$$(3.6) \qquad v(\tau, x) = \zeta(\tau)\varphi(x)$$

to conclude that the left-hand side of (3.3) can be expressed in the form

$$(3.7) \qquad -(n-1)\ddot{\zeta}\varphi + \zeta\{-(n-1)\tilde{\Delta}\varphi - \frac{n}{2}R\varphi\}.$$

Hence, the eigenvalue problem (3.3) can be solved by setting

$$(3.8) \qquad v = \zeta\varphi_j,$$

where φ_j, $j \in \mathbb{N}$, is an eigenfunction of the elliptic operator

$$(3.9) \qquad -(n-1)\tilde{\Delta} - \frac{n}{2}R$$

such that

$$(3.10) \qquad -(n-1)\tilde{\Delta}\varphi_j - \frac{n}{2}R\varphi_j = \tilde{\mu}_j\varphi_j,$$

$$(3.11) \qquad \tilde{\mu}_0 < \tilde{\mu}_1 \le \tilde{\mu}_2 \le \cdots$$

and ζ is an eigenfunction of the harmonic oscillator. The eigenvalue of the harmonic oscillator can be arbitrarily positive or zero. We define it at the moment as

$$(3.12) \qquad\qquad\qquad \omega^2$$

where $\omega \geq 0$ will be determined later. For $\omega > 0$ we shall consider the real eigenfunction

$$(3.13) \qquad\qquad\qquad \zeta = \sin \omega \tau$$

which represents the ground state in the interval

$$(3.14) \qquad\qquad\qquad I_0 = (0, \frac{\pi}{\omega})$$

with vanishing boundary values. ζ is a solution of the variational problem

$$(3.15) \qquad\qquad \frac{\int_{I_0} |\dot{\vartheta}|^2}{\int_{I_0} |\vartheta|^2} \to \min \quad \forall\, 0 \neq \vartheta \in H_0^{1,2}(I_0)$$

in the Sobolev space $H_0^{1,2}(I_0)$.

Multiplying ζ by a constant we may assume

$$(3.16) \qquad\qquad\qquad \int_{I_0} |\zeta|^2 = 1.$$

Obviously,

$$(3.17) \qquad\qquad\qquad \mathcal{S}_0 = \mathbb{R} \times M$$

and though ζ is defined in $\mathbb{R}$ and is even an eigenfunction it has infinite norm in $L^2(\mathbb{R})$. However, when we consider a finite disjoint union of N open intervals I_j

$$(3.18) \qquad\qquad\qquad \Omega = \bigcup_{j=1}^{N} I_j,$$

where

$$(3.19) \qquad\qquad I_j = (k_j \frac{\pi}{\omega}, (k_j+1)\frac{\pi}{\omega}), \quad k_j \in \mathbb{Z},$$

then

$$(3.20) \qquad\qquad\qquad \zeta_N = N^{-\frac{1}{2}} \zeta$$

is a unit eigenfunction in Ω with vanishing boundary values having the same energy as ζ in I_0. Hence, it suffices to consider ζ only in I_0 since this configuration can immediately be generalized to arbitrary large bounded open intervals

$$(3.21) \qquad\qquad\qquad \Omega \subset \mathbb{R}.$$

We then can state:

3.1. **Lemma.** *There exists a complete sequence of unit eigenfunctions of the operator in* (3.9) *with eigenvalues $\tilde{\mu}_j$ such that the functions*

$$(3.22) \qquad\qquad v_j = \zeta\varphi_j,$$

where ζ is a constant multiple of the function in (3.13) *with unit $L^2(I_0)$ norm, are solutions of the eigenvalue problem* (3.3) *with eigenvalue*

$$(3.23) \qquad\qquad \mu_j = (n-1)\omega^2 + \tilde{\mu}_j.$$

The eigenfunctions v_j form an orthogonal basis for $L^2(I_0 \times M, \mathbb{C})$.

To solve the wave equation (1.1) on page 2 let us first consider the following eigenvalue problem

$$(3.24) \qquad\qquad -\frac{1}{32}\frac{n^2}{n-1}\ddot{w} + n|\Lambda|t^2 w = \lambda t^{2-\frac{4}{n}} w$$

in the Sobolev space

$$(3.25) \qquad\qquad H_0^{1,2}(\mathbb{R}_+^*).$$

Here,

$$(3.26) \qquad\qquad \Lambda < 0$$

can in principle be an arbitrary negative parameter but in the case of a Kerr-AdS black hole it seems reasonable to choose the cosmological constant of the Kerr-AdS spacetime.

The eigenvalue problem (3.24) can be solved by considering the generalized eigenvalue problem for the bilinear forms

$$(3.27) \qquad B(w,\tilde{w}) = \int_{\mathbb{R}_+^*} \{\frac{1}{32}\frac{n^2}{n-1}\bar{w}'\tilde{w}' + n|\Lambda|t^2\bar{w}\tilde{w}\}$$

and

$$(3.28) \qquad\qquad K(w,\tilde{w}) = \int_{\mathbb{R}_+^*} t^{2-\frac{4}{n}}\bar{w}\tilde{w}$$

in the Sobolev space $\mathcal{H}$ which is the completion of

$$(3.29) \qquad\qquad C_c^\infty(\mathbb{R}_+^*, \mathbb{C})$$

in the norm defined by the first bilinear form.

We then look at the generalized eigenvalue problem

$$(3.30) \qquad\qquad B(w,\varphi) = \lambda K(w,\varphi) \quad \forall\, \varphi \in \mathcal{H}$$

which is equivalent to (3.24).

3.2. **Theorem.** *The eigenvalue problem* (3.30) *has countably many solutions (w_i, λ_i) such that*

$$(3.31) \qquad\qquad 0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots,$$

$$(3.32) \qquad\qquad \lim \lambda_i = \infty,$$

and

(3.33) $$K(w_i, w_j) = \delta_{ij}.$$

The w_i are complete in $\mathcal{H}$ as well as in $L^2(\mathbb{R}_+^*)$.

Proof. The quadratic form K is compact with respect to the quadratic form B as one can easily prove, cf. [2, Lemma 6.8], and hence a proof of the result, except for the strict inequalities in (3.31), can be found in [4, Theorem 1.6.3, p. 37]. Each eigenvalue has multiplicity one since we have a linear ODE of order two and all solutions satisfy the boundary condition

(3.34) $$w_i(0) = 0.$$

The kernel is two-dimensional and the condition (3.34) defines a one-dimensional subspace. Note, that we considered only real valued solutions to apply this argument. $\qquad\square$

We are now ready to define the solutions of the wave equation (1.1).

3.3. **Theorem.** *Let $(\varphi_j, \tilde{\mu}_j)$ resp. (w_i, λ_i) be eigenfunctions of the elliptic operator in (3.9) resp. the temporal eigenfunctions and let λ_{i_0} be the smallest eigenvalue of the (λ_i) with the property*

(3.35) $$\lambda_{i_0} \geq \tilde{\mu}_j,$$

then, for any $i \geq i_0$, there exists

(3.36) $$\omega = \omega_{ij} \geq 0$$

and corresponding ζ_{ij} satisfying

(3.37) $$-\ddot{\zeta}_{ij} = \omega_{ij}^2 \zeta_{ij}$$

such that

(3.38) $$\lambda_i = \mu_{ij} = (n-1)\omega_{ij}^2 + \tilde{\mu}_j \quad \forall\, i \geq i_0.$$

The functions

(3.39) $$u_{ij} = w_i \zeta_{ij} \varphi_j$$

are then solutions of the wave equation with bounded energies satisfying

(3.40) $$\lim_{t \to 0} w_{ij}(t) = \lim_{t \to \infty} w_{ij}(t) = 0$$

and

(3.41) $$w_{ij} \in C^\infty(\mathbb{R}_+^* \times \mathcal{S}_0) \cap C^{2,\alpha}(\bar{\mathbb{R}}_+^* \times \mathcal{S}_0)$$

for some

(3.42) $$\frac{2}{3} \leq \alpha < 1.$$

Moreover, we have

(3.43) $$\omega_{ij} > 0 \qquad \forall\, i > i_0.$$

If

$$(3.44) \qquad \lambda_{i_0} = \tilde{\mu}_j,$$

then we define

$$(3.45) \qquad \zeta_{i_0 j} \equiv 1.$$

Proof. The proof is obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.4. **Remark.** (i) By construction the temporal and spatial energies of the solutions of the wave equation have to be equal.

(ii) The stationary solutions comprising a harmonic oscillator can be looked at a being radiation while we consider the temporal solutions to be gravitational waves.

(iii) If one wants to replace the bounded Interval I_0 by $\mathbb{R}$ then the eigenfunctions ζ_{ij} have to be replaced by eigendistributions. An appropriate choice would be

$$(3.46) \qquad \zeta_{ij} = e^{i\omega_{ij}\tau};$$

also see [9] for a more general setting.

The hyperbolic operator defined by the wave equation (1.1) on page 2 can be defined in the spacetime

$$(3.47) \qquad Q = \mathcal{S}_0 \times \mathbb{R}_+^*$$

which can be equipped with the Lorentzian metrics

$$(3.48) \qquad d\bar{s}^2 = -\frac{32(n-1)}{n^2}dt^2 + g_{ij}dx^i dx^j$$

as well as with the metric

$$(3.49) \qquad d\tilde{s}^2 = -\frac{32(n-1)}{n^2}dt^2 + \frac{1}{n-1}t^{\frac{4}{n}-2}g_{ij}dx^i dx^j,$$

where g_{ij} is the metric defined on $\mathcal{S}_0$ and the indices now have the range $1 \leq i,j \leq n$. In both metrics Q is globally hyperbolic provided $\mathcal{S}_0$ is complete, which is the case for the metric defined in (3.2). The hyperbolic operator is symmetric in the first metric but not normally hyperbolic while it is normally hyperbolic but not symmetric in the second metric. Normally hyperbolic means that the main part of the operator is identical to the Laplacian of the spacetime metric.

Hence, if we want to describe quantum gravity not only by an equation but also by the metric of a spacetime then the metric in (3.49) has to be chosen. In this metric Q has a curvature singularity in $t = 0$, cf. [8, Remark 6.3]. The Cauchy hypersurface $\mathcal{S}_0$ then corresponds to the hypersurface

$$(3.50) \qquad \{t = 1\}$$

which also follows from the derivation of the quantum model where we consider a fiber bundle E with base space $\mathcal{S}_0$ and the elements of the fibers were Riemann metrics of the form

$$(3.51) \qquad g_{ij}(t,x) = t^{\frac{4}{n}} \sigma_{ij}(x)$$

where σ_{ij} were metrics defined in $\mathcal{S}_0$ and t is the time coordinate that we use in Q, i.e.,

$$(3.52) \qquad g_{ij}(1,x) = \sigma_{ij}(x).$$

In the present situation we used the symbol g_{ij} to denote the metric on $\mathcal{S}_0$ since σ_{ij} is supposed to be a metric on $\mathbb{S}^{2m-1}$.

Thus the event horizon is characterized by the Cauchy hypersurface

$$(3.53) \qquad \{t = 1\}.$$

If $a = 0$, i.e., in case we consider a Schwarzschild-AdS black hole as in [10], then we shall obviously assume that the black hole singularity

$$(3.54) \qquad \{r = 0\}$$

corresponds to the curvature singularity

$$(3.55) \qquad \{t = 0\}$$

of Q, i.e., the open black hole region is described in the quantum model by

$$(3.56) \qquad \mathcal{S}_0 \times (0,1)$$

and the open exterior region by

$$(3.57) \qquad \mathcal{S}_0 \times (1,\infty).$$

If $a \neq 0$, then there is no curvature singularity in $r = 0$, only a coordinate singularity in our present coordinate system. Indeed, if we choose generalized Boyer-Lindquist coordinates, cf. [12, equation (3.1)], the metric has the form

$$
\begin{aligned}
(3.58) \quad d\bar{s}^2 = &-\frac{1 + l^2 r^2}{1 - a^2 l^2} d\tau^2 + \frac{U r^2 dr^2}{(1 + l^2 r^2)(r^2 + a^2)U - 2m_0 r^2} \\
&+ \frac{r^2 + a^2}{1 - a^2 l^2} \sum_{i=1}^{m} (d\mu_i^2 + \mu_i^2 (d\varphi_i + l^2 d\tau)^2) \\
&+ \frac{2m_0}{U} \Big(d\tau - \frac{a}{1 - a^2 l^2} \sum_{i=1}^{m} \mu_i^2 d\varphi^i\Big)^2.
\end{aligned}
$$

Then, defining

$$(3.59) \qquad \rho = r^2,$$

such that

$$(3.60) \qquad d\rho = 2r dr$$

we obtain new coordinates in which the metric is smooth up to $\rho = 0$, indeed the metric is even smooth in the interval

$$(3.61) \qquad -a^2 < \rho < \infty.$$

In $\rho = -a^2$ there is curvature singularity:

3.5. **Lemma.** *The extended spacetime N has a timelike curvature singularity in $\rho = -a^2$.*

Proof. The fact that the curvature singularity is timelike follows immediately from (2.49) on page 10, where we proved that outside the black hole region the hypersurfaces

$$(3.62) \qquad \{\rho = \text{const}\}$$

are timelike.

To prove the existence of a curvature singularity we first consider the case $m \geq 3$. Looking at the metric in (3.58) we observe that the components with respect to the coordinates μ_i form a diagonal matrix without any cross terms with the other coordinates, namely,

$$(3.63) \qquad \frac{\rho + a^2}{1 - a^2 l^2} \sum_{i=1}^{m} d\mu_i^2,$$

where the μ_i are subject to the side-condition

$$(3.64) \qquad \sum_{i=1}^{m} \mu_i^2 = 1,$$

i.e., (3.63) represents the metric of a sphere of radius

$$(3.65) \qquad \sqrt{\frac{\rho + a^2}{1 - a^2 l^2}}$$

embedded in $\mathbb{R}^m$ and the corresponding sectional curvatures in N are defined independently of the other components of the metric in N and they obviously become unbounded when ρ tends to $-a^2$, since the sectional curvature σ_p in a point $p \in N$ of a plane spanned by two linearly independent vectors in

$$(3.66) \qquad T_p(\mathbb{S}^{m-1}) \hookrightarrow T_p(N)$$

is equal to

$$(3.67) \qquad \frac{1 - a^2 l^2}{\rho + a^2}.$$

Secondly, in case $m = 2$, we used the package GREAT [16] in Mathematica to compute the squared Riemannian curvature tensor in dimension 5 and obtained

$$(3.68) \qquad \bar{R}_{\alpha\beta\gamma\delta}\bar{R}^{\alpha\beta\gamma\delta} = \frac{96 m_0^2 \left(3a^2 - \rho\right)\left(a^2 - 3\rho\right)}{\left(\rho + a^2\right)^6} + 40 l^4$$

completing the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since the curvature singularity is *timelike* and not *spacelike* as the singularity of a Schwarzschild-AdS spacetime or the singularity in our quantum spacetime, equipped with the metric in (3.49), it is easily avoidable. Despite this difference, we stipulate that the region in (3.57) corresponds to

$$\{\rho_2 < \rho < \infty\}, \tag{3.69}$$

where

$$r_2^2 = \rho_2, \tag{3.70}$$

and the region in (3.56) to

$$\{-a^2 < \rho < \rho_2\}. \tag{3.71}$$

3.6. **Remark.** The time coordinate τ in a generalized Boyer-Lindquist coordinate system is a time function in

$$N \backslash \bar{B}, \tag{3.72}$$

where N is the extended Kerr-AdS spacetime. We proved it directly with the help of Mathematica, if $\dim N = 5$, by proving

$$\bar{g}^{\alpha\beta}\tau_\alpha\tau_\beta = \bar{g}^{00} < 0. \tag{3.73}$$

For a proof in any odd dimension it will be sufficient to prove that the slices

$$\{\tau = \text{const}\} \tag{3.74}$$

are spacelike in the region specified in (3.72). Looking at the metric (3.58) we immediately see, by setting $d\tau = 0$, that the induced metric is Riemannian.

3.7. **Remark.** When we have an extremal black hole with mass m_0' and corresponding radius r_0 for the event horizon, then the function $\Phi = \Phi(\rho)$ in (2.12), where

$$\rho = r^2, \tag{3.75}$$

satisfies

$$0 = \Phi(\rho_0) = \inf \Phi, \tag{3.76}$$

hence

$$\Phi'(\rho_0) = 0. \tag{3.77}$$

From the definition of Φ we then conclude that any black hole with mass

$$m_0 > m_0', \tag{3.78}$$

while the other parameters remain equal, will have an interior region. Hence, our previous arguments could then be applied to yield a quantum model depending on the Riemannian metric in (2.50) on page 10. Letting m_0 tend to m_0' the corresponding radii of the event horizons will then converge to r_0 leading to a quantum model for an extremal black hole.

3.8. **Remark.** In the quantum model of the black hole the event horizon is a regular Cauchy hypersurface and can be crossed in both directions by causal curves hence no information paradox can occur.

REFERENCES

[1] Brandon Carter, *Hamilton-Jacobi and Schrödinger separable solutions of Einstein's equations*, Comm. Math. Phys. **10** (1968), no. 4, 280–310, http://projecteuclid.org/euclid.cmp/1103841118.

[2] Claus Gerhardt, *Quantum cosmological Friedman models with an initial singularity*, Class. Quantum Grav. **26** (2009), no. 1, 015001, arXiv:0806.1769, doi:10.1088/0264-9381/26/1/015001.

[3] ———, *A unified quantum theory II: gravity interacting with Yang-Mills and spinor fields*, 2013, arXiv:1301.6101.

[4] ———, *Partial differential equations II*, Lecture Notes, University of Heidelberg, 2013, pdf file.

[5] ———, *The quantization of gravity in globally hyperbolic spacetimes*, Adv. Theor. Math. Phys. **17** (2013), no. 6, 1357–1391, arXiv:1205.1427, doi:10.4310/ATMP.2013.v17.n6.a5.

[6] ———, *A unified quantum theory I: gravity interacting with a Yang-Mills field*, Adv. Theor. Math. Phys. **18** (2014), no. 5, 1043–1062, arXiv:1207.0491, doi:10.4310/ATMP.2014.v18.n5.a2.

[7] ———, *A unified field theory I: The quantization of gravity*, (2015), arXiv:1501.01205.

[8] ———, *A unified field theory II: Gravity interacting with a Yang-Mills and Higgs field*, (2016), arXiv:1602.07191.

[9] ———, *Deriving a complete set of eigendistributions for a gravitational wave equation describing the quantized interaction of gravity with a Yang-Mills field in case the Cauchy hypersurface is non-compact*, (2016), arXiv:1605.03519.

[10] ———, *The quantization of a black hole*, (2016), arXiv:1608.08209.

[11] ———, *The quantum development of an asymptotically Euclidean Cauchy hypersurface*, (2016), arXiv:1612.03469.

[12] G. W. Gibbons, H. Lu, Don N. Page, and C. N. Pope, *The General Kerr-de Sitter metrics in all dimensions*, J. Geom. Phys. **53** (2005), 49–73, arXiv:hep-th/0404008, doi:10.1016/j.geomphys.2004.05.001.

[13] J. B. Hartle and S. W. Hawking, *Path-integral derivation of black-hole radiance*, Phys. Rev. D **13** (1976), no. 8, 2188–2203, 10.1103/PhysRevD.13.2188.

[14] S. W. Hawking, *Particle creation by black holes*, Communications in Mathematical Physics **43** (1975), no. 3, 199–220, doi:10.1007/BF02345020.

[15] S. W. Hawking, C. J. Hunter, and Marika Taylor, *Rotation and the AdS / CFT correspondence*, Phys. Rev. **D59** (1999), 064005, arXiv:hep-th/9811056, doi:10.1103/PhysRevD.59.064005.

[16] Tristan Hübsch, *General Relativity, Einstein and All That (GREAT)*, 2003, http://library.wolfram.com/infocenter/MathSource/4781/.

[17] Roy P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Phys. Rev. Lett. **11** (1963), 237–238, doi:10.1103/PhysRevLett.11.237.

[18] Robert C. Myers and M. J. Perry, *Black Holes in Higher Dimensional Space-Times*, Annals Phys. **172** (1986), 304, doi:10.1016/0003-4916(86)90186-7.

[19] Robert M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago Lectures in Physics, University of Chicago Press, Chicago, IL, 1994.

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