AN ENERGY GAP FOR YANG-MILLS CONNECTIONS

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Abstract. Consider a Yang-Mills connection over a Riemann manifold $M = M^n$, $n \geq 3$, where M may be compact or complete. Then its energy must be bounded from below by some positive constant, if M satisfies certain conditions, unless the connection is flat.

Contents

1. Introduction

We consider the problem: When is a Yang-Mills connection non-flat? Of course, the trivial answer $F_{\mu\lambda} \not\equiv 0$ is unsatisfactory. Bourguignon and Lawson proved in [3, Theorem C], among other results, that any Yang-Mills connection over S^n, $n \geq 3$, the field strength of which satisfies the pointwise estimate

$$F^2 = -\operatorname{tr}(F_{\mu\lambda}F^{\mu\lambda}) < \binom{n}{2} \tag{1.1}$$

is flat.

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We want to prove that under certain assumptions on the base space M, which is supposed to be a Riemannian manifold of dimension $n \ge 3$, the energy of a Yang-Mills connection has to satisfy

$$\Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}} \ge \kappa_0 > 0, \tag{1.2}$$

where κ_0 depends only on the Sobolev constants of M, n and the dimension of the Lie group $\mathcal{G}$, unless the connection is flat.

Here,

$$|F| = \sqrt{F^2}, \tag{1.3}$$

and we also call the left-hand side of (1.2) energy though this label is only correct when $n = 4$. However, this norm is also the crucial norm, which has to be (locally) small, used to prove regularity of a connection, cf. [4, Theorem 1.3].

The exponent $\frac{n}{2}$ naturally pops up when Sobolev inequalities are applied to solutions of differential equations satisfied by the field strength or the energy density of a connection in the adjoint bundle.

We distinguish two cases: M compact and M complete and non-compact. When M is compact, we require

$$\bar R_{\alpha\beta}\varLambda^\alpha_\lambda\varLambda^{\beta\lambda} - \tfrac12 \bar R_{\alpha\beta\mu\lambda}\varLambda^{\alpha\beta}\varLambda^{\mu\lambda} \ge c_0 \varLambda_{\alpha\beta}\varLambda^{\alpha\beta} \tag{1.4}$$

for all skew-symmetric $\varLambda_{\alpha\beta} \in T^{0,2}(M)$, where $0 < c_0$, while for non-compact M the weaker assumption

$$\bar R_{\alpha\beta}\varLambda^\alpha_\lambda\varLambda^{\beta\lambda} - \tfrac12 \bar R_{\alpha\beta\mu\lambda}\varLambda^{\alpha\beta}\varLambda^{\mu\lambda} \ge 0 \tag{1.5}$$

and in addition

$$\Big(\int_M u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le c_1 \int_M |Du|^2 \qquad \forall\, u \in H^{1,2}(M) \tag{1.6}$$

should be satisfied.

1.1. **Remark.** (i) If M is a space of constant curvature

$$\bar R_{\alpha\beta\mu\lambda} = K_M(\bar g_{\alpha\mu}\bar g_{\beta\lambda} - \bar g_{\alpha\lambda}\bar g_{\beta\mu}), \tag{1.7}$$

then

$$\bar R_{\alpha\beta}\varLambda^\alpha_\lambda\varLambda^{\beta\lambda} - \tfrac12 \bar R_{\alpha\beta\mu\lambda}\varLambda^{\alpha\beta}\varLambda^{\mu\lambda} = (n-2)K_M \varLambda_{\alpha\beta}\varLambda^{\alpha\beta}. \tag{1.8}$$

In case $n = 2$ the curvature term therefore vanishes, and this result is also valid for an arbitrary two-dimensional Riemannian manifold, since the curvature tensor then has the same structure as in (1.7) though K_M is not necessarily constant.

(ii) If $M = \mathbb{R}^n$, $n \geq 3$, the conditions (1.5) and (1.6) are always valid.

1.2. **Theorem.** *Let $M = M^n$, $n \geq 3$, be a compact Riemannian manifold for which the condition* (1.4) *with $c_0 > 0$ holds. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or satisfies* (1.2) *for some constant $\kappa_0 > 0$ depending on the Sobolev constants of M, n, c_0, and the dimension of the Lie group.*

1.3. **Theorem.** *Let $M = M^n$, $n \geq 3$, be complete, non-compact and assume that the conditions* (1.5) *and*(1.6) *hold. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or the estimate* (1.2) *is valid. The constant $\kappa_0 > 0$ in* (1.2) *depends on the constant c_1 in* (1.6)*, n, and the dimension of the Lie group.*

2. The compact case

Let $(P, M, \mathcal{G}, \mathcal{G})$ be a principal fiber bundle where $M = M^n$, $n \geq 3$ is a compact Riemannian manifold with metric $\bar{g}_{\alpha\beta}$ and $\mathcal{G}$ a compact, semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $f_c = (f^a_{cb})$ be a basis of $\operatorname{ad}\mathfrak{g}$ and

$$A_\mu = f_c A^c_\mu \tag{2.1}$$

a Yang-Mills connection in the adjoint bundle $(E, M, \mathfrak{g}, \operatorname{Ad}(\mathcal{G}))$.

The curvature tensor of the connection is given by

$$R^a{}_{b\mu\lambda} = f^a_{cb} F^c_{\mu\lambda}, \tag{2.2}$$

where

$$F_{\mu\lambda} = f_c F^c_{\mu\lambda} \tag{2.3}$$

is the field strength of the connection, and

$$F^2 \equiv \gamma_{ab} F^a_{\mu\lambda} F^{b\mu\lambda} = R_{ab\mu\lambda} R^{ab\mu\lambda} \tag{2.4}$$

the energy density of the connection—at least up to a factor $\frac{1}{4}$.

Here, γ_{ab} is the Cartan-Killing metric acting on elements of the fiber $\mathfrak{g}$, and Latin indices are raised or lowered with respect to the inverse γ^{ab} or γ_{ab}, and Greek indices with respect to the metric of M.

2.1. **Definition.** The adjoint bundle E is vector bundle; let E^* be the dual bundle, then we denote by

$$T^{r,s}(E)=\Gamma(\underbrace{E\otimes\cdots\otimes E}_{r}\otimes\underbrace{E^*\otimes\cdots\otimes E^*}_{s}) \tag{2.5}$$

the sections of the corresponding tensor bundle.

Thus, we have

$$F^a_{\mu\lambda}\in T^{1,0}(E)\otimes T^{0,2}(M). \tag{2.6}$$

Since A_μ is a Yang-Mills connection it solves the Yang-Mills equation

$$F^{a\alpha}{}_{\lambda;\alpha}=0, \tag{2.7}$$

where we use Einstein's summation convention, a semi-colon indicates covariant differentiation, and where we stipulate that a covariant derivative is always a *full* tensor, i.e.,

$$F^a_{\mu\lambda;\alpha}=F^a_{\mu\lambda,\alpha}+f^a_{bc}A^b_\alpha F^c_{\mu\lambda}-\bar\Gamma^\gamma_{\alpha\mu}F^a_{\gamma\lambda}-\bar\Gamma^\gamma_{\alpha\lambda}F^a_{\mu\gamma}, \tag{2.8}$$

where $\bar\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the Riemannian connection; a comma indicates partial differentiation.

Before we formulate the crucial lemma let us note that $\bar R_{\alpha\beta\gamma\delta}$ resp. $\bar R_{\alpha\beta}$ symbolize the Riemann curvature tensor resp. the Ricci tensor of $\bar g_{\alpha\beta}$.

2.2. **Lemma.** *Let A_μ be a Yang-Mills connection, then its energy density F^2 solves the equation*

$$\begin{aligned}&-\tfrac14\Delta F^2+\tfrac12F_{a\mu\lambda;\alpha}F^{a\mu\lambda}{}_;{}^\alpha+\bar R_{\beta\mu}F^{a\beta}{}_\lambda F_a{}^{\mu\lambda}-\tfrac12\bar R_{\alpha\beta\mu\lambda}F_a{}^{\alpha\beta}F^{a\mu\lambda}\\&=-f^a_{cb}F^c_{\alpha\mu}F^{b\alpha}{}_\lambda F_a{}^{\mu\lambda}.\end{aligned} \tag{2.9}$$

Proof. Differentiating (2.7) covariantly with respect to x^μ and using the Ricci identities we obtain

$$\begin{aligned}0&=-F^{a\alpha}{}_{\lambda;\alpha\mu}\\&=-F^{a\alpha}{}_{\lambda;\mu\alpha}+R^a{}_{b\alpha\mu}F^{b\alpha}{}_\lambda+\bar R^\alpha{}_{\beta\alpha\mu}F^{a\beta}{}_\lambda+\bar R^\beta{}_{\lambda\mu\alpha}F^{a\alpha}{}_\beta.\end{aligned} \tag{2.10}$$

On the other hand, differentiating the second Bianchi identities

$$0=F^a_{\alpha\lambda;\mu}+F^a_{\mu\alpha;\lambda}+F^a_{\lambda\mu;\alpha} \tag{2.11}$$

we infer

$$0=F^{a\alpha}{}_{\lambda;\mu\alpha}+F^a{}_\mu{}^\alpha{}_{;\lambda\alpha}+\Delta F^a_{\lambda\mu}, \tag{2.12}$$

and we deduce further

$$-\Delta F^a_{\mu\lambda} F_a{}^{\mu\lambda} = -2F^{a\alpha}{}_{\lambda;\mu\alpha} F_a{}^{\mu\lambda}. \tag{2.13}$$

In view of (2.10) we then conclude

$$\begin{aligned} 0 = -\tfrac{1}{2}\Delta F^a_{\mu\lambda} F_a{}^{\mu\lambda} + R^a{}_{b\alpha\mu} F^{b\alpha}{}_{\lambda} F_a{}^{\mu\lambda} + \bar R_{\beta\mu} F^{a\beta}{}_{\lambda} F_a{}^{\mu\lambda} \\ + \bar R^{\beta}{}_{\lambda\mu\alpha} F^{a\alpha}{}_{\beta} F_a{}^{\mu\lambda}, \end{aligned} \tag{2.14}$$

which is equivalent to

$$\begin{aligned} 0 = -\tfrac{1}{2}\Delta F^a_{\mu\lambda} F_a{}^{\mu\lambda} + f^a_{cb} F^c_{\alpha\mu} F^{b\alpha}{}_{\lambda} F_a{}^{\mu\lambda} + \bar R_{\beta\mu} F^{a\beta}{}_{\lambda} F_a{}^{\mu\lambda} \\ - \bar R_{\alpha\mu\beta\lambda} F^{a\alpha\beta} F_a{}^{\mu\lambda}, \end{aligned} \tag{2.15}$$

in view of (2.2).

Finally, using the first Bianchi identities,

$$\bar R_{\alpha\beta\mu\lambda} + \bar R_{\alpha\mu\lambda\beta} + \bar R_{\alpha\lambda\beta\mu} = 0, \tag{2.16}$$

we deduce

$$\bar R_{\alpha\beta\mu\lambda} F^{a\alpha\beta} F_a{}^{\mu\lambda} + \bar R_{\alpha\mu\lambda\beta} F^{a\alpha\beta} F_a{}^{\mu\lambda} + \bar R_{\alpha\lambda\beta\mu} F^{a\alpha\beta} F_a{}^{\mu\lambda} = 0, \tag{2.17}$$

and hence

$$\bar R_{\alpha\beta\mu\lambda} F^{a\alpha\beta} F_a{}^{\mu\lambda} = 2\bar R_{\alpha\mu\beta\lambda} F^{a\alpha\beta} F_a{}^{\mu\lambda}, \tag{2.18}$$

from which the equation (2.9) immediately follows. □

Proof of Theorem 1.2 on page 3. Define

$$u = F^2, \tag{2.19}$$

then

$$\bar R_{\beta\mu} F^{a\beta}{}_{\lambda} F_a{}^{\mu\lambda} - \tfrac{1}{2}\bar R_{\alpha\beta\mu\lambda} F_a{}^{\alpha\beta} F^{a\mu\lambda} \ge c_0 u, \tag{2.20}$$

where $c_0 > 0$, in view of the assumption (1.4) on page 2.

Multiplying (2.9) with u and integrating by parts we obtain

$$\tfrac{3}{8}\int_M |Du|^2 + c_0 \int_M u^2 \le c \int_M \sqrt{u}\, u^2, \tag{2.21}$$

where we used the simple estimate

$$|Du|^2 \le 4F_{a\mu\lambda;\alpha} F^{a\mu\lambda}{}_{;}{}^{\alpha} u \tag{2.22}$$

and where c depends on n and the dimension of $\mathfrak{g}$; note that

$$f_c \in \mathrm{SO}(\mathfrak{g}, \gamma_{ab}). \tag{2.23}$$

The integral on the right-hand side of (2.21) is estimated by

$$\int_M \sqrt{u}u^2 \le \Big(\int_M u^{\frac{n}{4}}\Big)^{\frac{2}{n}} \Big(\int_M u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}, \tag{2.24}$$

where

$$\Big(\int_M u^{\frac{n}{4}}\Big)^{\frac{2}{n}} = \Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}}. \tag{2.25}$$

Applying then the Sobolev inequality

$$\Big(\int_M u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le c_1 \int_M |Du|^2 + c_2 \int_M u^2, \tag{2.26}$$

cf. [1], we obtain

$$\Big(\int_M u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le c_3 \Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_M u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}, \tag{2.27}$$

where c_3 depends on c_1, c_2, c_0 and c. Hence, we deduce $u \equiv 0$ or

$$c_3^{-1} \le \Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}}. \tag{2.28}$$

Setting

$$\kappa_0 = c_3^{-1} \tag{2.29}$$

finishes the proof. □

3. The non-compact case

We now suppose that $M = M^n$ is a complete, non-compact Riemannian manifold. Then there holds

$$H^{1,2}(M) = H_0^{1,2}(M), \tag{3.1}$$

i.e., the test functions $C_c^\infty(M)$ are dense in the Sobolev space $H^{1,2}(M)$, see [1, Lemme 4] or [2, Theorem 2.6].

Since we do not a priori know

$$F^2 \in H^{1,2}(M), \tag{3.2}$$

but only

$$F^2 \in H^{1,2}_{\text{loc}}(M), \tag{3.3}$$

the preceding proof has to be modified.

Let $\eta = \eta(t)$ be defined through

$$\eta(t) = \begin{cases} 1, & t \le 1, \\ (2-t)^q, & 1 \le t \le 2, \\ 0, & t \ge 2, \end{cases} \tag{3.4}$$

where

$$q = \max(1, \tfrac{8}{n}). \tag{3.5}$$

Fix a point $x_0 \in M$ and let r be the Riemannian distance function with center in x_0

$$r(x) = d(x_0, x). \tag{3.6}$$

Then r is Lipschitz such that

$$|Dr| = 1 \tag{3.7}$$

almost everywhere.

For $k \ge 1$ define

$$\eta_k(x) = \eta(k^{-1} r). \tag{3.8}$$

The functions

$$u^{p-1}\eta_k^p, \tag{3.9}$$

where

$$p = \tfrac{n}{4}, \tag{3.10}$$

then have compact support, and multiplying (2.9) on page 4 with $u^{p-1}\eta_k^p$ yields

$$\begin{aligned} (\tfrac{p}{4} + \tfrac{1}{8} - \epsilon) \int_M |Du|^2 u^{p-2} \eta_k^p \le c\Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}} \Big(\int_M (u\eta_k)^{\frac{n}{n-2}p}\Big)^{\frac{n-2}{n}} \\ + c_\epsilon \int_M |D\eta_k|^2 \eta_k^{p-2} u^p, \end{aligned} \tag{3.11}$$

where $0 < \epsilon$ is supposed to be small.

Furthermore, there holds

$$\begin{aligned} \int_M |D(u\eta_k)^{\frac{p}{2}}|^2 &= \frac{p^2}{4} \int_M |Du\eta_k + uD\eta_k|^2 (u\eta_k)^{p-2} \\ &\le (1+\epsilon)\frac{p^2}{4} \int_M |Du|^2 u^{p-2} \eta_k^p + c_\epsilon \frac{p^2}{4} \int_M |D\eta_k|^2 \eta_k^{p-2} u^p. \end{aligned} \tag{3.12}$$

Now, choosing ϵ so small such that

$$(1+\epsilon)\frac{p^2}{4} \le p(\tfrac{p}{4}+\tfrac{1}{8}-\epsilon) \tag{3.13}$$

and setting

$$\varphi = (u\eta_k)^{\frac{p}{2}} \tag{3.14}$$

we obtain

$$\int_M |D\varphi|^2 \le pc\Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int_M \varphi^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} + c_\epsilon \int_M |D\eta_k|^2\eta_k^{p-2}u^p, \tag{3.15}$$

where c_ϵ is a new constant.

We furthermore observe that

$$|D\eta_k|^2\eta_k^{p-2} \le q^2k^{-2}(2-k^{-1}r)^{qp-2}, \tag{3.16}$$

subject to

$$1 \le k^{-1}r \le 2. \tag{3.17}$$

In view of (3.5) and (3.10)

$$qp-2 \ge 0 \tag{3.18}$$

and hence

$$|D\eta_k|^2\eta_k^{p-2} \le q^2k^{-2}. \tag{3.19}$$

Applying now the Sobolev inequality (1.6) on page 2 to φ and choosing

$$\kappa_0 = (c_1cp)^{-1} \tag{3.20}$$

we conclude $|F| \equiv 0$, if

$$\Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < \kappa_0. \tag{3.21}$$

Indeed, if the preceding inequality is valid, then we deduce from (3.15)

$$\Big(1-\kappa_0^{-1}\Big(\int_M |F|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big)\Big(\int_M |\varphi|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le c_\epsilon q^2k^{-2}\int_M |F|^{\frac{n}{2}}. \tag{3.22}$$

In the limit $k\to\infty$ we obtain

$$\Big(\int_M |u|^{\frac{pn}{n-2}}\Big)^{\frac{n-2}{n}} \le 0. \tag{3.23}$$

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