

PINCHING ESTIMATES FOR DUAL FLOWS IN HYPERBOLIC AND DE SITTER SPACE

CLAUS GERHARDT

ABSTRACT. We prove pinching estimates for dual flows provided the curvature function used in the inverse flow in de Sitter space is convex.

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1. INTRODUCTION

In a recent paper [2] we considered dual flows in \mathbb{S}^{n+1} , where one flow is a direct flow

$$(1.1) \quad \dot{x} = -F\nu$$

while the other is an inverse flow

$$(1.2) \quad \dot{x} = \tilde{F}^{-1}\nu,$$

where F is a smooth curvature function defined in Γ_+ and \tilde{F} its inverse

$$(1.3) \quad \tilde{F}(\kappa_i) = \frac{1}{F(\kappa_i^{-1})}.$$

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We assumed that F is symmetric, monotone, positive, homogeneous of degree 1, and that both F and \tilde{F} are concave; F was also supposed to be strictly concave, cf. [2, Definition 3.1] for a precise definition. Using the Gauß map in \mathbb{S}^{n+1} which maps closed, strictly convex hypersurfaces M to their polar sets \tilde{M} , we proved that the flow hypersurfaces of the above flows are polar sets of each other, if the initial hypersurfaces have this property, hence the name dual flows. Thus, if one problem is solvable for arbitrarily closed, strictly convex initial hypersurfaces the dual problem is also solvable. We were therefore able to prove that the direct flows contract to a round point and the inverse flows to an equator such that, after an appropriate rescaling, both flows converge to a geodesic sphere exponentially fast.

Let us emphasize that the result for the inverse flow could only be achieved because of the duality. We could tackle the inverse problem directly. However, Makowski and Scheuer [3] independently proved that the inverse flow converges to an equator in $C^{1,\alpha}$.

There also exist Gauß maps between hyperbolic space \mathbb{H}^{n+1} and de Sitter space N and vice versa mapping closed, strictly convex hypersurfaces of one space to closed, strictly convex hypersurfaces of the other space, cf. Section 2 or [1, Chapter 10.4]. Moreover, the duality between direct flows in one space and inverse flows in the other is also valid, cf. [4]. When a direct flow in \mathbb{H}^{n+1} with closed, strictly convex initial hypersurface M contracts to a point x_0 the dual flow in N will expand to a totally geodesic hypersurface which is isometric to \mathbb{S}^n . After applying an isometry such that the point x_0 is the Beltrami point the hypersurfaces of the dual flow are all contained in N_- and their limit hypersurface will be the slice

$$(1.4) \quad \{\tau = 0\},$$

compare Section 2.

As in the case of the dual flows in \mathbb{S}^{n+1} , the direct flow is easier to solve. A major ingredient in the proof is a pinching estimate for the principle curvatures. In [4] pinching estimates could be derived provided the initial hypersurface in \mathbb{H}^{n+1} is horoconvex, i.e., the principal curvatures satisfy the estimate

$$(1.5) \quad \kappa_i \geq 1.$$

We shall prove in this paper that pinching estimates can be derived provided the curvature function used in the inverse flow in N is convex, monotone and positive without requiring that the principal curvatures of the initial

hypersurface in N satisfies the estimate

$$(1.6) \quad \kappa_i \leq 1$$

which would be equivalent to the relation (1.5) in \mathbb{H}^{n+1} . Hence, pinching estimates can be proved for merely strictly convex hypersurfaces.

1.1. Theorem. *Let $F \in C^\infty(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ be monotone, positive, convex, symmetric and homogeneous of degree 1, then the solution hypersurfaces of the flow*

$$(1.7) \quad \dot{x} = -F^{-1}\nu$$

in N with strictly convex, spacelike, closed initial hypersurface M_0 satisfy the pinching estimate

$$(1.8) \quad \frac{\kappa_n}{\kappa_1} \leq c_0,$$

where the κ_i are labelled such that

$$(1.9) \quad \kappa_1 \leq \dots \leq \kappa_n$$

and where c_0 only depends on M_0 and $F(1, \dots, 1)$.

In [4] Yu has shown that the converge results described above are also valid for this configuration.

2. ISOMETRIES IN HYPERBOLIC AND DE SITTER SPACE AND THE GAUSS MAPS

Hyperbolic and de Sitter space are „spheres“ in the Minkowski space $\mathbb{R}^{n+1,1}$ defined by

$$(2.1) \quad \mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1,1} : \langle x, x \rangle = -1 \quad \wedge \quad x^0 > 0\}$$

and

$$(2.2) \quad N = \{x \in \mathbb{R}^{n+1,1} : \langle x, x \rangle = 1\}.$$

A point $x \in \mathbb{R}^{n+1,1}$ has coordinates $x = (x^a)$, $0 \leq a \leq n+1$, where x^0 is the time coordinate.

The isometries are the elements of the linear Lorentz group $\mathcal{O}(n+1, 1)$ in $\mathbb{R}^{n+1,1}$. For completeness we shall repeat the simple proof that $\mathcal{O}(n+1, 1)$ acts transitively in these spaces.

2.1. Lemma. *$\mathcal{O}(n+1, 1)$ acts transitively in \mathbb{H}^{n+1} and N .*

Proof. Let $x = (x^a)$ be an arbitrary point in \mathbb{H}^{n+1} . Using a rotation around the x^0 -axis it can be mapped to the point $x_0 = (x^0, x^1, 0, \dots, 0)$. The map acting like

$$(2.3) \quad \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}, \quad \theta \in \mathbb{R},$$

in the coordinates (x^0, x^1) and like the identity for the rest, is an isometry. Since

$$(2.4) \quad x^0 = \sqrt{1 + |x^1|^2}$$

we can choose θ such that

$$(2.5) \quad (x^0, x^1, 0, \dots, 0) \rightarrow p_0 = (1, 0, \dots, 0)$$

which is known as the *Beltrami point*.

By a similar proof, any point $x \in N$ can be mapped to $(0, 1, 0, \dots, 0)$ by an isometry respecting the time orientation. \square

The Gauß maps

Let $M \subset \mathbb{H}^{n+1}$ be a closed, strictly convex hypersurface, then M can be considered to be a submanifold in $\mathbb{R}^{n+1,1}$ of codimension 2, i.e.,

$$(2.6) \quad x : M_0 \rightarrow \mathbb{R}^{n+1,1}.$$

The Gaussian formula for M then looks like

$$(2.7) \quad x_{ij} = g_{ij}x - h_{ij}\tilde{x},$$

where g_{ij} is the induced metric, h_{ij} the second fundamental form of M considered as a hypersurface in \mathbb{H}^{n+1} , and \tilde{x} is the representation of the (exterior¹) normal vector $\nu = (\nu^\alpha)$ of M in $T(\mathbb{H}^{n+1})$ as a vector in $T(\mathbb{R}^{n+1,1})$.

The mapping

$$(2.8) \quad \tilde{x} : M_0 \rightarrow N$$

is then the embedding of a strictly convex, closed, spacelike hypersurface \tilde{M} . We call this mapping the *Gauß map*. In [1, Theorem 10.4.4] we have proved:

¹Notice that M is orientable, since it is strictly convex, and that for any closed, connected, orientable, immersed hypersurface in \mathbb{H}^{n+1} an exterior normal vector can be unambiguously defined.

2.2. Theorem. *Let $x : M_0 \rightarrow M \subset \mathbb{H}^{n+1}$ be a closed, connected, strictly convex hypersurface of class C^m , $m \geq 3$, then the Gauß map \tilde{x} in (2.8) is the embedding of a closed, spacelike, achronal, strictly convex hypersurface $\tilde{M} \subset N$ of class C^{m-1} .*

Viewing \tilde{M} as a codimension 2 submanifold in $\mathbb{R}^{n+1,1}$, its Gaussian formula is

$$(2.9) \quad \tilde{x}_{ij} = -\tilde{g}_{ij}\tilde{x} + \tilde{h}_{ij}x,$$

where \tilde{g}_{ij} , \tilde{h}_{ij} are the metric and second fundamental form of the hypersurface $\tilde{M} \subset N$, and $x = x(\xi)$ is the embedding of M which also represents the future directed normal vector of \tilde{M} . The second fundamental form \tilde{h}_{ij} is defined with respect to the future directed normal vector, where the time orientation of N is inherited from $\mathbb{R}^{n+1,1}$.

The second fundamental forms of M , \tilde{M} and the corresponding principal curvatures κ_i , $\tilde{\kappa}_i$ satisfy

$$(2.10) \quad h_{ij} = \tilde{h}_{ij} = \langle \tilde{x}_i, x_j \rangle$$

and

$$(2.11) \quad \tilde{\kappa}_i = \kappa_i^{-1}.$$

There also exists an inverse Gauß map mapping strictly convex hypersurfaces in N to strictly convex hypersurfaces in \mathbb{H}^{n+1} :

2.3. Theorem. *Let $\tilde{M} \subset N$ be a closed, connected, spacelike, strictly convex, embedded hypersurface of class C^m , $m \geq 3$, such that, when viewed as a codimension 2 submanifold in $\mathbb{R}^{n+1,1}$, its Gaussian formula is*

$$(2.12) \quad \tilde{x}_{ij} = -\tilde{g}_{ij}\tilde{x} + \tilde{h}_{ij}x,$$

where $\tilde{x} = \tilde{x}(\xi)$ is the embedding, x the future directed² normal vector, and \tilde{g}_{ij} , \tilde{h}_{ij} the induced metric and the second fundamental form of the hypersurface in N . Then we define the Gauß map as $x = x(\xi)$

$$(2.13) \quad x : \tilde{M} \rightarrow \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1,1}.$$

The Gauß map is the embedding of a closed, connected, strictly convex hypersurface M in \mathbb{H}^{n+1} .

²The time orientation is inherited from the ambient space $\mathbb{R}^{n+1,1}$.

Let g_{ij} , h_{ij} be the induced metric and second fundamental form of M , then, when viewed as a codimension 2 submanifold, M satisfies the relations

$$(2.14) \quad x_{ij} = g_{ij}x - h_{ij}\tilde{x},$$

$$(2.15) \quad h_{ij} = \tilde{h}_{ij} = \langle x_i, \tilde{x}_j \rangle,$$

and

$$(2.16) \quad \kappa_i = \tilde{\kappa}_i^{-1},$$

where κ_i , $\tilde{\kappa}_i$ are the corresponding principal curvatures.

For a proof see [1, Theorem 10.4.5].

When the Beltrami point is an interior point of the convex body defined by $M \subset \mathbb{H}^{n+1}$, then the image of the Gauß map $\tilde{x}(M)$ is contained in

$$(2.17) \quad N_+ = \{x \in N : \tau > 0\},$$

where τ is the time coordinate in N which is naturally defined by the embedding

$$(2.18) \quad N \hookrightarrow \mathbb{R}^{n+1,1}.$$

More precisely, we have proved in [1, Theorem 10.4.9]

2.4. Theorem. *The Gauß maps provide a bijective relation between the connected, closed, strictly convex hypersurfaces $M \subset \mathbb{H}^{n+1}$ having the Beltrami point in the interior of their convex bodies and the spacelike, closed, connected, strictly convex hypersurfaces $\tilde{M} \subset N_+$.*

The geodesic spheres with center in the Beltrami point are mapped onto the coordinate slices $\{\tau = \text{const}\}$.

2.5. Remark. Our default definition for the second fundamental of a spacelike hypersurface in Lorentzian manifolds relies on the past directed normal vector and not on the future directed. Hence, we switch the light cone in N such that the images of the Gauß map are strictly convex with respect to the past directed normal vector. But then, the coordinate used to prove the previous theorem is no longer future directed. In order to get a future directed coordinate system we have to replace the time function τ by $-\tau$. Then the image $\tilde{x}(M)$ is contained in N_- instead of N_+ ; for details see the remarks at the end of [1, page 307].

3. CONVEX CURVATURE FUNCTIONS

Let $F \in C^\infty(\Gamma_+)$ be monotone, convex, homogeneous of degree 1 and normalized such that

$$(3.1) \quad F(1, \dots, 1) = 1.$$

Then the following estimates are valid

$$(3.2) \quad \frac{1}{n}H \leq F \leq \kappa_n$$

and

$$(3.3) \quad \sum_i F_i \leq 1,$$

where

$$(3.4) \quad F_i = \frac{\partial F}{\partial \kappa^i}$$

and where the κ_i are labelled such that

$$(3.5) \quad \kappa_1 \leq \dots \leq \kappa_n.$$

The F_i satisfy the same ordering

$$(3.6) \quad F_1 \leq \dots \leq F_n.$$

Let \tilde{F} be the inverse of F

$$(3.7) \quad \tilde{F}(\kappa_i) = F^{-1}(\kappa_i^{-1}),$$

the \tilde{F} is of class (K) and hence strictly concave, cf. [1, Lemma 2.2.12] and [2, Lemma 3.6].

From (3.6) we infer

$$(3.8) \quad F_i \kappa_i \leq F_j \kappa_j, \quad \text{if } \kappa_i \leq \kappa_j,$$

deducing further

$$(3.9) \quad \frac{1}{n}\kappa_n \leq F = \sum_i F_i \kappa_i \leq nF_n \kappa_n,$$

and therefore,

$$(3.10) \quad \frac{1}{n^2} \leq F_n \leq 1.$$

Define the relation

$$(3.11) \quad F \approx \kappa_i$$

if there are uniform constants c_1, c_2 such that

$$(3.12) \quad 0 < c_1 \leq \frac{F}{\kappa_i} \leq c_2,$$

then

$$(3.13) \quad F \approx \kappa_n \quad \wedge \quad \tilde{F} \approx \kappa_1.$$

Finally, we deduce from (3.9) and the homogeneity of F that

$$(3.14) \quad 1 \leq \sum_i \tilde{F}_i = F^{-2} \sum_i F_i \kappa_i^2 \leq F^{-1} \kappa_n \leq n$$

where the arguments of \tilde{F} are κ_i^{-1} .

Similarly we have

$$(3.15) \quad 1 \geq \sum_i F_i = \tilde{F}^{-2} \tilde{F}_i \kappa_i^2 \geq \tilde{F}^{-1} \kappa_1 = F \kappa_n^{-1} \geq \frac{1}{n}$$

4. PINCHING ESTIMATES

Let $N = N^{n+1}$ be the de Sitter space with metric given by

$$(4.1) \quad d\bar{s}^2 = -d\tau^2 + \cosh^2 \tau \sigma_{ij} dx^i dx^j,$$

where $\tau \in \mathbb{R}$ and σ_{ij} is the standard metric of \mathbb{S}^n . We also assume that τ is a future directed time function; occasionally we also write x^0 instead of τ . N is space of constant curvature $K_N = 1$.

In the following we shall formulate equations by using the general symbol K_N for the curvature because of transparency having in mind that $K_N = 1$.

We consider the inverse curvature flow

$$(4.2) \quad \dot{x} = -F^{-1}\nu \equiv \Phi\nu,$$

where ν is the past directed normal and F is a curvature function defined in Γ_+ . $\Phi = \Phi(r)$ is the real function

$$(4.3) \quad \Phi = -r^{-1}, \quad r > 0.$$

We shall consider smooth curvature functions F which are monotone, convex, and homogeneous of degree 1. The initial hypersurface M_0 of the flow should be closed, spacelike and strictly convex. As we pointed out in Section 2 we may assume, by applying an isometry,

$$(4.4) \quad M_0 \subset \{\tau < 0\}.$$

Spacelike hypersurfaces can be written as graphs over \mathbb{S}^n , hence the leaves $M(t)$ of the flow can be expressed as

$$(4.5) \quad M(t) = \text{graph } u(t, \cdot)|_{\mathbb{S}^n}.$$

Let

$$(4.6) \quad v^2 = 1 - \cosh^{-2} u \sigma^{ij} u_j u_i,$$

the u satisfies the scalar curvature equation

$$(4.7) \quad \dot{u} = \frac{\partial u}{\partial t} = -v\Phi > 0,$$

hence we conclude

4.1. Lemma. *During the evolution the leaves $M(t)$ stay in compact set, or more precisely,*

$$(4.8) \quad \inf_{M_0} u \leq u < \sinh \pi.$$

Proof. We shall prove that

$$(4.9) \quad \inf_{M(t)} u < 0 \quad \forall t \in [0, T^*),$$

from which the result would follow, since

$$(4.10) \quad |Du|^2 = \sigma^{ij} u_i u_j < \cosh^2 u$$

and hence

$$(4.11) \quad \text{osc } u(t, \cdot) < \sinh \pi.$$

Indeed, assume

$$(4.12) \quad \inf_{M(t)} u \geq 0$$

for some t , then there would exist $\xi \in M(t)$ such that the second fundamental form in that point would satisfy

$$(4.13) \quad 0 < h_{ij} \leq \bar{h}_{ij} \leq 0,$$

where \bar{h}_{ij} is the second form of the coordinate slice

$$(4.14) \quad \{\tau = \inf_{M(t)} u\},$$

a contradiction. □

4.2. **Remark.** In [4] it is proved that the flow will converge to an umbilical maximal hypersurface M . Since any closed umbilical maximal hypersurface in N is isometric to

$$(4.15) \quad \{\tau = 0\},$$

an isometric image of the flow will converge to the maximal coordinate slice, and, in view of the scalar curvature equation (4.7), we can then deduce that the whole transformed flow is contained in N_- .

The following evolution equation is well known

$$(4.16) \quad \Phi' - \dot{\Phi} F^{ij} \Phi_{;ij} = -\dot{\Phi} F^{ij} h_{ik} h_j^k \Phi + K_N \dot{\Phi} F^{ij} g_{ij} \Phi,$$

where

$$(4.17) \quad \Phi' = \frac{d}{dt} \Phi$$

and

$$(4.18) \quad \dot{\Phi} = \frac{d}{dr} \Phi(r),$$

see e.g. [1, Lemma 2.3.4].

From (4.16) we infer

$$(4.19) \quad \dot{F} - \dot{\Phi} F^{ij} F_{;ij} = \dot{\Phi} F^{ij} h_{ik} h_j^k F - 2F^{-1} \dot{\Phi} F^{ij} F_i F_j - K_N \dot{\Phi} F^{ij} g_{ij} F.$$

Let $(\tilde{h}^{ij}) = (h_{ij})^{-1}$, then the mixed tensor \tilde{h}_j^i satisfies the evolution equation

$$(4.20) \quad \begin{aligned} \dot{\tilde{h}}_j^i - \dot{\Phi} F^{kl} \tilde{h}_{j;kl}^i &= \dot{\Phi} F^{kl} h_{kr} h_l^r \tilde{h}_j^i - \{\dot{\Phi} F - \Phi\} \delta_j^i \\ &- K_N \{\dot{\Phi} F + \Phi\} \tilde{h}_{kj} \tilde{h}^{ki} + K_N \dot{\Phi} F^{kl} g_{kl} \tilde{h}_j^i \\ &- \{\dot{\Phi} F^{pq,kl} h_{pq;r} h_{kl;s} + 2\dot{\Phi} F^{kl} \tilde{h}^{pq} h_{pk;r} h_{ql;s} + \ddot{\Phi} F_r F_s\} \tilde{h}^{is} \tilde{h}_j^r, \end{aligned}$$

cf. [1, Lemma 2.4.3].

Finally, let $\eta = \eta(\tau)$ be a solution of the ordinary differential equation

$$(4.21) \quad \dot{\eta} = -\frac{\bar{H}}{n} \eta,$$

where \bar{H} is the mean curvature of the slices

$$(4.22) \quad \{x^0 = \tau\},$$

then the function

$$(4.23) \quad \chi = \tilde{v} \eta(u),$$

where $\tilde{v} - v^{-1}$, satisfies

$$(4.24) \quad \dot{\chi} - \dot{\Phi}F^{ij}\chi_{ij} = -\dot{\Phi}F^{ij}h_{ik}h_j^k\chi + \{\dot{\Phi}F + \Phi\}\frac{\bar{H}}{n}\chi,$$

cf. [1, Lemma 10.5.5]. In case of the de Sitter spacetime we can choose

$$(4.25) \quad \eta = \cosh \tau,$$

then

$$(4.26) \quad \dot{\eta} = -\left(-\frac{\sinh \tau}{\cosh \tau}\right)\eta.$$

Applying the maximum principle we deduce from (4.8) and (4.24)

4.3. Lemma. *The quantities χ and \tilde{v} are uniformly bounded during the evolution.*

From the equations (4.19) and (4.24) we infer

4.4. Lemma. *F is uniformly bounded during the evolution.*

Proof. Let $0 < T < T^*$ be arbitrary and apply the maximum principle to the function

$$(4.27) \quad w = \log F + \log \chi$$

in

$$(4.28) \quad Q_T = [0, T] \times \mathbb{S}^n$$

assuming that there exist (t_0, ξ_0) , $0 < t_0 \leq T$ such that

$$(4.29) \quad w(t_0, \xi_0) = \sup_{Q_T} w,$$

then we have in (t_0, ξ_0)

$$(4.30) \quad 0 \leq -K_N \dot{\Phi}F^{ij}g_{ij} < 0,$$

a contradiction; hence

$$(4.31) \quad w \leq \sup_{M_0} w \quad \text{in} \quad Q_T$$

from which the result immediately follows, since χ is uniformly positive

$$(4.32) \quad 0 < c_1 \leq \chi \leq c_2 < \infty.$$

□

We are now ready to prove the pinching estimates. First, let us observe that it suffices to prove

$$(4.33) \quad \frac{F}{\kappa_1} \leq \text{const},$$

where the principal curvatures are labelled

$$(4.34) \quad \kappa_1 \leq \cdots \leq \kappa_n,$$

in view of (3.13) on page 8.

4.5. Theorem. *Let F be monotone, convex, and homogeneous of degree 1, then*

$$(4.35) \quad \kappa_1^{-1} F \leq \text{const}$$

during the evolution.

Proof. We shall show that the function

$$(4.36) \quad w = \log F + \log \tilde{H} - 2nt,$$

where

$$(4.37) \quad \tilde{H} = \tilde{h}^{ij} g_{ij},$$

is uniformly bounded from above during the evolution by applying the maximum principle and using the fact that $T^* < \infty$; the latter can easily be checked by comparing with the evolution of a coordinate slice, in view of (4.9).

Let $0 < T < T^*$ be arbitrary and suppose there exists (t_0, ξ_0) , $0 < t_0 \leq T$, such that

$$(4.38) \quad w(t_0, \xi_0) = \sup_{QT} w.$$

Applying the maximum principle we obtain

$$(4.39) \quad \begin{aligned} 0 &\leq 2F^{-2} F^{ij} h_{ik} h_j^k - 2nF^{-1} - 2n \\ &< 2n - 2n = 0, \end{aligned}$$

in view of (3.14) on page 8, a contradiction, proving the theorem. Here, we also used that the last term on the right-hand side of equation (4.20) is non-positive, cf. [1, equ. (2.2.43) and Lemma 2.2.14]. \square

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RUPRECHT-KARLS-UNIVERSITÄT, INSTITUT FÜR ANGEWANDTE MATHEMATIK, IM NEUENHEIMER FELD 294, 69120 HEIDELBERG, GERMANY

E-mail address: gerhardt@math.uni-heidelberg.de

URL: <http://www.math.uni-heidelberg.de/studinfo/gerhardt/>