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Evolutionary Surfaces of Prescribed Mean Curvature*

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0. INTRODUCTION

Surfaces of prescribed mean curvature have been studied intensively in the past from various view points: parametric and non-parametric surfaces, smooth and generalized surfaces. But only time-independent surfaces have been considered since no physical problem has been known involving time-dependent surfaces of prescribed mean curvature except the still unsolved and even non-attacked corresponding hyperbolic problem.

The *parabolic* equation

$$\dot{u} + Au + H(x, u) = 0 \qquad \text{in} \quad \Omega \times (0, T) \tag{0.1}$$

with the minimal surface operator A as differential operator, where

$$Au = -D_i(a^i(Du)), \qquad a^i(p) = p^i \cdot (1 + |p|^2)^{-1/2} \tag{0.2}$$

is much easier to handle: the corresponding Dirichlet problem with boundary data φ has a classical smooth solution if the boundary $\partial\Omega$ is of class C^2 and satisfies Serrin's condition (cf. [13])

$$|H(x, \varphi(x))| \leq (n-1) \cdot H_{n-1}(x) \qquad \forall x \in \partial\Omega, \tag{0.3}$$

where H_{n-1} is the mean curvature of $\partial\Omega$ with respect to the inward normal vector. The proof is almost identical to that in the stationary case, since the maximum principle is still valid. Difficulties first arise, if one considers generalized solutions or tries to prove interior gradient estimates.

As far as we know Lichnewsky and Temam have first considered generalized evolutionary minimal surfaces: the solution satisfies the equation in the weak

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sense as $H^{1,2}_{loc}(\Omega)$ functions and a Dirichlet boundary condition in a generalized sense, cf. [9, 10].

Brakke [1] has looked at the problem from the view point of geometric measure theory.

The reason for this interest in evolutionary surfaces of prescribed mean curvature is, that they model the physical system of the motion of grain boundaries in annealing pure metal (cf. [1] for further details).

We consider solutions of (0.1) satisfying a rather general boundary condition, namely,

$$-a^i \cdot \nu_i \in \beta(x, u - \varphi) \qquad \text{on} \quad \partial\Omega \times (0, T), \tag{0.4}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit normal vector to $\partial\Omega$ and $\beta(x, \cdot)$ a maximal monotone graph with

$$|\beta| \leqq 1. \tag{0.5}$$

φ is a given function on $\partial\Omega$. The corresponding stationary problem has been studied in [4], where we had to assume that the mean curvature function $H = H(x, \tau)$ satisfies

$$\frac{\partial H}{\partial \tau} \geqq \kappa > 0. \tag{0.6}$$

In the evolutionary case we are allowed to assume the less rigorous condition

$$\frac{\partial H}{\partial \tau} \geqq 0 \tag{0.7}$$

in view of the presence of the term $\dot{u}$.

We prove that the solutions of the boundary value problem (0.1), (0.4) are uniformly Lipschitz continuous if $|\partial\beta/\partial x|$ is bounded and $|\beta| \leqq \lambda < 1$.

The main part of the paper is organized in five sections. In Section 1 we prove useful a priori estimates for $|u|$ and $|\dot{u}|$. In Section 2 we consider a weak formulation of the problem and prove the existence of generalized solutions (in the $BV(\Omega)$ sense); we also prove in this section that solutions of (0.1) are uniformly bounded, provided Ω satisfies an internal sphere condition, without assuming any boundary condition. Gradient estimates are established in the Sections 3 (local estimates) and 4 (boundary estimates). In Section 5, finally, we consider the attainability of stationary solutions.

We are very grateful to Roger Temam for having acquainted us with this problem and for his interest in our results.

1. Notation and Preliminary Results

Ω is a bounded open set of $\mathbb{R}^n$, $n \geqq 2$, with Lipschitz-boundary $\partial\Omega$; sometimes we shall assume that $\partial\Omega$ or parts of it are smoother. The cylinder Q_T is defined by $Q_T = \Omega \times (0, T)$, $0 < T < \infty$. Functions u defined in Q_T are always real-valued and are denoted by u, $u(x, t)$, or $u(t)$, where $u(t) = u(\cdot, t)$. The Sobolev spaces or L^p-spaces are denoted by $H^{m,p}(\Omega)$ or $L^p(\Omega)$, as usually, with norms $\|\cdot\|_{m,p}$ or $\|\cdot\|_p \equiv \|\cdot\|_{0,p}$, respectively. If V is a Banach space, then $L^p(0, T; V)$ has a natural meaning.

The function class $BV(\Omega)$ (functions of bounded variation) consists of those functions $v \in L^1(\Omega)$ the distributional derivatives of which are measures. For $v \in BV(\Omega)$ we define

$$\begin{aligned}\int_\Omega (1 + |Dv|^2)^{1/2}\,dx \\ = \sup\left\{\int_\Omega (g^0 \cdot v + D_i g^i v)\,dx\colon g^0,\dots, g^n \in C_c^\infty(\Omega), |(g^0,\dots, g^n)| \leqq 1\right\}.\end{aligned} \tag{1.1}$$

Here and in the following we use the summation convention to sum over repeated indices from 1 to n, unless otherwise stated.

Functions $v \in BV(\Omega)$ have a trace $t(v)$ on $\partial\Omega$, such that $t(v) \in L^1(\partial\Omega)$ (cf. [12]). We shall always write v instead of $t(v)$. We note that the symbol $\mathscr{H}_k$, $k \in \mathbb{R}_+$, is used for the k-dimensional Hausdorff measure, and the functions $H = H(x, \tau)$ and $\beta = \beta(x, \tau)$ are Lipschitz continuous and monotone in τ, i.e.

$$\frac{\partial H}{\partial \tau}, \frac{\partial \beta}{\partial \tau} \equiv \beta' \geqq 0. \tag{1.2}$$

Sometimes β is only assumed to be Lipschitz continuous in x, and to be a maximal monotone graph in τ.

The boundary values φ are mostly supposed to satisfy

$$\varphi, \dot{\varphi} \in L^\infty(\partial\Omega \times (0, T)). \tag{1.3}$$

In contrast to the convention to denote the minimal surface operator by A, which we generally obey, we let $A = -D_i(a^i)$, $a^i = a^i(p)$, be an arbitrary elliptic quasilinear differential operator in this section.

For smooth solutions u of the evolution equation

$$\begin{aligned} \dot{u} + Au + H(x, u) &= 0 \qquad \text{in } Q_T, \\ u_0 &= u(0), \\ -a^i \cdot \nu_i &= \beta(x, u - \varphi) \qquad \text{in } \partial\Omega \times (0, T), \end{aligned} \tag{1.4}$$

we shall prove some a priori estimates, which will also hold if we replace the boundary condition by a Dirichlet condition $u|_{\partial\Omega} = \varphi$, as we shall state without proof. The estimates will not be sharp in the sense that we assume the least strong assumptions.

1.1 LEMMA. *Let u be a solution of* (1.4). *Then*

$$\sup_{Q_T} |\dot{u}| \leqslant \max(\sup_{\Omega} |\dot{u}(0)|, \sup_{\partial\Omega\times(0,T)} |\dot{\varphi}|). \tag{1.5}$$

Proof. Differentiate the equation (1.4) with respect to t to get

$$\ddot{u} - D_i(a^{ij}D_j\dot{u}) + \frac{\partial H}{\partial \tau} \cdot \dot{u} = 0 \tag{1.6}$$

and the boundary condition

$$-a^{ij}\nu_i D_j \dot{u} = \beta' \cdot (\dot{u} - \dot{\varphi}), \qquad \text{where} \quad a^{ij} = \frac{\partial a^i}{\partial p^j}. \tag{1.7}$$

Let k be the constant on the right hand-side of (1.5), and let $\eta = \operatorname{sgn} \dot{u} \cdot \max(|\dot{u}| - k, 0)$. Then multiplying (1.6) with η and integrating over Q_t, $0 < t \leqq T$, we obtain

$$1/2 \int_\Omega |\eta|^2 \, dx + \int_0^t \int_\Omega a^{ij}DD \, dx \, d\tau \leqslant 0 \text{ in view of (1.3), (1.7),} \tag{1.8}$$

and the definition of k, hence the result. The condition

$$\sup_\Omega |\dot{u}(0)| < \infty \tag{1.9}$$

is satisfied provided $u_0 \in C^2(\bar{\Omega})$.

1.2. *Remark.* The boundedness of $|\dot{u}|$ immediately implies an a priori bound for $|u|$, though a more natural bound can be obtained by using (1.4) directly with a test-function $\eta = \operatorname{sgn} u \cdot \max(|u| - k, 0)$, $k \geqq k_0$, where k_0 is sufficiently large. In that case one has to make some assumptions concerning the structure of the a^i: uniformly elliptic would be sufficient, but also the a^i's of the minimal surface operator are allowed.

1.3. LEMMA. *Let u be a solution of* (1.4), *and assume that the a^i's are the components of the gradient of a differentiable function f. Then*

$$\begin{aligned} &\int_0^t \int_\Omega |\dot{u}|^2 \, dx d\tau + \int_\Omega f(Du(t)) \, dx + \int_\Omega \int_0^{u(t)} H(x, \tau) \, d\tau \, dx \\ &\quad + \int_{\partial\Omega} j(x, u(t) - \varphi(t)) \, d\mathscr{H}_{n-1} = \int_\Omega f(Du_0) \, dx + \int_\Omega \int_0^{u_0} H(x, \tau) \, d\tau \, dx \\ &\quad + \int_{\partial\Omega} j(x, u_0 - \varphi(0)) \, d\mathscr{H}_{n-1} + \int_0^t \int_\Omega a^i \cdot D_i\dot{\varphi} \, dx \, d\tau \\ &\quad + \int_0^t \int_\Omega H(x, u) \cdot \dot{\varphi} \, dx \, d\tau, \end{aligned} \tag{1.10}$$

where

$$j(x, t) = \int_0^t \beta(x, \tau)\, d\tau. \tag{1.11}$$

Proof. Multiply (1.4) with $\dot{u} - \dot{\varphi}$ and integrate over Q_t.

1.4. LEMMA. *Let u_i, $i = 1, 2$, be a solution of the equation*

$$\begin{aligned} \dot{u}_i + Au_i + H(x, u_i) &= 0 \\ u_i(0) &= u_{0i}. \end{aligned} \tag{1.12}$$

Then

$$| u_1 - u_2 | \leqslant \max(\sup_{\Omega} | u_{01} - u_{02} |, \sup_{\partial\Omega\times(0,T)} | u_1 - u_2 |). \tag{1.13}$$

Proof. The estimate is known as the weak maximum principle for parabolic equations. To indicate the proof, denote the right-hand side of (1.13) by k_0 and multiply the difference of the two equations resulting from (1.12) with $\eta = \operatorname{sgn}(u_1 - u_2) \cdot \max(| u_1 - u_2 | - k_0, o)$. Then using the monotonicity of the operator

$$A + H(x, \cdot) \tag{1.14}$$

one immediately gets

$$\eta(t) \equiv 0 \qquad \forall t, \tag{1.15}$$

hence the result.

1.5. LEMMA. *Let u_i, $i = 1, 2$, be a solution of* (1.4) *with data u_{0i} and φ_i. Then*

$$| u_1 - u_2 | \leqslant \max(\sup_{\Omega} | u_{01} - u_{02} |, \sup_{\partial\Omega\times(0,T)} | \varphi_1 - \varphi_2 |). \tag{1.16}$$

The proof is same as above using the monotonicity of β.

1.6. *Remark.* The lemma is also valid if β is a bounded monotone graph, A the minimal surface operator, and $u_i \in L^1(0, T; H^{1,1}(\Omega))$, $i = 1, 2$, are weak solutions of (1.4). The proof does not change.

2. A WEAK FORMULATION OF THE PROBLEM AND THE EXISTENCE OF $BV(\Omega)$ SOLUTIONS

Let u be a solution of (1.4) assuming $a^i = D_i f$ and $\beta(x, \cdot) = (\partial/\partial\tau) j(x, \cdot)$, where j is a uniformly Lipschitz continuous convex function; precisely we mean:

β is the subdifferential of j and thus a maximal monotone graph, the boundary condition is now interpreted as

$$-a^i \cdot \nu_i \in \beta(x, u - \varphi). \tag{2.1}$$

If the a^i are uniformly elliptic, $a^i \in C^1$, $\partial\Omega \in C^2$, β, $\partial\beta/\partial x$, $\dot{\varphi}$, u_0 bounded, $u_0 \in H^{1,2}(\Omega)$, and $\varphi \in L^2(0, T; H^{2,2}(\Omega))$, then there exists a solution $u \in L^2(0, T; H^{2,2}(\Omega)) \cap L^\infty(Q_T)$ of the corresponding evolutionary problem (1.4) (see e.g. [2] for a proof). Moreover, consider the scalar product

$$\int_\Omega a^i(Du) \cdot D_i v \, dx + \int_\Omega H(x, u) \cdot v \, dx + \int_{\partial\Omega} \beta(x, u - \varphi) v \, d\mathscr{H}_{n-1} \qquad \text{for} \quad v \in H^{1,2}(\Omega). \tag{2.2}$$

This relation defines an element of the dual space of $H^{1,2}(\Omega)$ which belongs to the subdifferential (or is the subdifferential if one regards the whole set $\beta(x, u - \varphi)$) of the convex function

$$\Phi(v) + \int_{\partial\Omega} j(x, v - \varphi) \, d\mathscr{H}_{n-1} \equiv \int_\Omega f(Dv) \, dx + \int_\Omega \int_0^v H(x, \tau) \, d\tau \, dx + \int_{\partial\Omega} j(x, v - \varphi) \, d\mathscr{H}_{n-1} \tag{2.3}$$

at the point $v = u$.

Hence we obtain from (1.4) the equivalent formulation

$$\int_\Omega \dot{u}(v - u) \, dx + \Phi(v) + \int_{\partial\Omega} j(x, v - \varphi) \, d\mathscr{H}_{n-1} \geqq \Phi(u) + \int_{\partial\Omega} j(x, u - \varphi) \, d\mathscr{H}_{n-1} \qquad \forall v \in H^{1,2}(\Omega). \tag{2.4}$$

Integrating over $[0, T]$ this yields

$$\int_0^T \Phi(u) \, dt + \int_0^T \int_{\partial\Omega} j(x, u - \varphi) \, d\mathscr{H}_{n-1} \, dt \leqslant \int_0^T \Phi(v) \, dt + \int_0^T \int_{\partial\Omega} j(x, v - \varphi) \, d\mathscr{H}_{n-1} \, dt + \int_0^T \int_\Omega \dot{u}(v - u) \, dx \, dt \tag{2.5}$$

for any $v \in L^2(0, T; H^{1,2}(\Omega))$.

A solution $u \in L^2(0, T; H^{1,2}(\Omega))$ of (2.5) satisfying

$$\dot{u} \in L^2(Q_T), \tag{2.6}$$

and the initial condition

$$u(0) = u_0\,, \tag{2.7}$$

will be looked at as a weak solution of the evolutionary problem (1.4).

Using the simple relation

$$\begin{aligned}\int_0^T \int_\Omega (\dot v - \dot u)(v - u)\,dx\,dt &= \tfrac{1}{2}\int_\Omega |\,v(T) - u(T)|^2\,dx - \tfrac{1}{2}\int_\Omega |\,v(0) - u_0\,|^2\,dx \\ &\geqslant -\tfrac{1}{2}\int_\Omega |\,v(0) - u_0\,|^2\,dx \end{aligned}\tag{2.8}$$

we deduce from (2.5) the apparently weaker formulation

$$\begin{aligned}-\tfrac{1}{2}\int_\Omega |\,v(0) - u_0\,|^2\,dx + \int_0^T \Phi(u)\,dt + \int_0^T\int_{\partial\Omega} j(x, u - \varphi)\,d\mathscr{H}_{n-1}\,dt \\ \leqslant \int_0^T \Phi(v)\,dt + \int_0^T\int_{\partial\Omega} j(x, v - \varphi)\,d\mathscr{H}_{n-1}\,dt + \int_0^T\int_\Omega \dot v(v - u)\,dx\,dt\end{aligned}\tag{2.9}$$

for any $v \in L^2(0, T; H^{1,2}(\Omega))$ satisfying $\dot v \in L^2(Q_T)$.

The solution u is now only thought to be of class $L^2(0, T; H^{1,2}(\Omega))$. However, from (2.9) we immediately conclude

2.1. LEMMA. *Let u be a solution of* (2.9) *then u satisfies* (2.7) *and*

$$\int_0^T \int_\Omega |\,\dot u\,|^2\,dx\,dt \leqslant \Phi(u_0) + \sup_{[0,T]} \int_{\partial\Omega} j(x, u_0 - \varphi)\,d\mathscr{H}_{n-1}\,. \tag{2.10}$$

Moreover, u is a solution of (2.5).

Proof. We follow the proof of Lemma II.3 in [2].

Let u_ϵ, $\epsilon > 0$, be a solution of

$$\begin{aligned} u_\epsilon + \epsilon\cdot\dot u_\epsilon &= u \\ u_\epsilon(0) &= u_0\,, \end{aligned}\tag{2.11}$$

i.e. let

$$u_\epsilon = e^{-t/\epsilon}u_0 + 1/\epsilon\cdot\int_0^t e^{(\tau-t)/\epsilon}u(\tau)\,d\tau. \tag{2.12}$$

Then we may insert u_ϵ in (2.9). Writing u_ϵ as a convex combination

$$u_\epsilon = e^{-t/\epsilon}u_0 + (1 - e^{-t/\epsilon})\cdot\frac{1}{\epsilon(1 - e^{-t/\epsilon})}\cdot\int_0^t e^{(\tau-t)/\epsilon}u(\tau)\,d\tau, \tag{2.13}$$

we obtain from the convexity of Φ and Jensen's inequality

$$\Phi(u_\epsilon) \leqslant e^{-t/\epsilon}\Phi(u_0) + 1/\epsilon \cdot \int_0^t e^{(\tau-t)/\epsilon}\Phi(u(\tau))\, d\tau. \tag{2.14}$$

The same estimate also holds for the boundary integral. Thus we get from (2.9)

$$\begin{aligned}
&\epsilon \int_0^T \int_\Omega |\dot{u}_\epsilon|^2\, dx\, dt + \int_0^T \Phi(u)\, dt + \int_0^T \int_{\partial\Omega} j(x, u - \varphi)\, d\mathscr{H}_{n-1}\, dt \\
&\quad \leqslant \left\{\Phi(u_0) + \sup_{[0,T]} \int_{\partial\Omega} j(x, u_0 - \varphi)\, d\mathscr{H}_{n-1}\right\} \cdot \int_0^T e^{-t/\epsilon}\, dt \\
&\qquad + 1/\epsilon \cdot \int_0^T \int_0^t e^{(\tau-t)/\epsilon} \left\{\Phi(u(\tau)) + \int_{\partial\Omega} j(x, u(\tau) - \varphi(\tau))\, d\mathscr{H}_{n-1}\right\} d\tau\, dt \\
&\quad \leqslant \epsilon \cdot (1 - e^{-T/\epsilon}) \cdot \left\{\Phi(u_0) + \sup_{[0,T]} \int_{\partial\Omega} j(x, u_0 - \varphi)\, d\mathscr{H}_{n-1}\right\} \\
&\qquad + \int_0^T \Phi(u)\, dt + \int_0^T \int_{\partial\Omega} j(x, u - \varphi)\, d\mathscr{H}_{n-1}\, dt,
\end{aligned} \tag{2.15}$$

implying

$$\int_0^T \int_\Omega |\dot{u}_\epsilon|^2\, dx\, dt \leqslant \Phi(u_0) + \sup_{[0,T]} \int_{\partial\Omega} j(x, u_0 - \varphi)\, d\mathscr{H}_{n-1}$$

The first conclusions of the lemma are now evident in view of the definition of u_ϵ.

To proof the last claim, we insert

$$v_\tau = \tau u + (1 - \tau)\, v, \qquad 0 \leqq \tau \leqq 1,$$

in (2.9), where v is an arbitrary possible test-function satisfying

$$v(0) = u_0\,. \tag{2.17}$$

In view of the convexity of the integrands we conclude

$$\begin{aligned}
&\left\{\int_0^T \Phi(u)\, dt + \int_0^T \int_{\partial\Omega} j(x, u - \varphi)\, d\mathscr{H}_{n-1}\, dt\right\} \cdot (1 - \tau) \\
&\quad \leqslant (1 - \tau) \cdot \left\{\int_0^T \Phi(v)\, dt + \int_0^T \int_{\partial\Omega} j(x, v - \varphi)\, d\mathscr{H}_{n-1}\, dt\right\} \\
&\qquad + (1 - \tau) \cdot \int_0^T \int_\Omega \dot{v}_\tau \cdot (v - u)\, dx\, dt.
\end{aligned} \tag{2.18}$$

Dividing by $(1 - \tau)$ for $\tau \neq 1$, and letting $\tau \to 1$, we obtain the result provided $\dot{v} \in L^2(Q_T)$ and satisfies (2.17). If we choose v only subject to the conditions of (2.5), then the final conclusion of the lemma follows via approximation.

2.2. *Remark.* For later reference we note that the solution u of (2.5), (2.9) also solves (2.4).

Before applying these results to evolutionary surfaces of prescribed mean curvature, we prove the following theorem, which can be looked at as an analogue of Concus' and Finn's estimate of capillary surfaces (cf. [3]).

2.3. THEOREM. *Let B be a ball of radius R, $B = B_R(x_0)$, in $\mathbb{R}^n$, and let $u \in L^\infty(0, T; C^{0,1}(B))$ be a solution of*

$$\begin{aligned} \dot{u} + Au + H(x, u) &= 0 \\ u(0) &= u_0 \end{aligned} \tag{2.19}$$

satisfying $\dot{u} \in L^2(B \times (0, T))$, where A is the minimal surface operator, and where H is non-decreasing in the second variable. Then, the estimate

$$|u| \leqslant \sup_B |u_0| + R + T \cdot \left(\frac{n}{R} + \sup_B |H(\cdot, 0)|\right) \tag{2.20}$$

is valid in $B \times (0, T)$.

Proof. Choose $0 < R_0 < R_1 < R$, and assume the right-hand side of (2.20) to be finite. We are going to prove the estimate in $B_{R_0} = B_{R_0}(x_0)$ knowing that $u(t)$ is uniformly Lipschitz in B_{R_0} for a.e. t. Since the estimate will hold for all $R_0 < R$, it will also be valid for R. Moreover, we shall only prove the estimate for u, the estimate for $-u$ can be proved similarly.

For convenience assume $x_0 = 0$, and consider in B_{R_0} the upper hemisphere

$$\delta(x, t) = -(R_1^2 - |x|^2)^{1/2} + M_0 + t.M, \tag{2.21}$$

where $M_0 = \sup_B |u_0| + R_1$, and $M = n/R_1 + \sup_B |H(.,0)|$. Then we have

$$A\delta = -\frac{n}{R_1}, \tag{2.22}$$

$$\dot{\delta} = M, \tag{2.23}$$

and

$$a^i(D\delta) \cdot \nu_i = \frac{2 \cdot R_0}{(R_1^2 + 3 \cdot R_0^2)^{1/2}} \qquad \text{on } \partial B_{R_0}, \tag{2.24}$$

where ν is the outward unit normal. From (2.22), (2.23) we deduce

$$\dot{\delta} + A\delta + H(x, \delta) \geqq \dot{\delta} + A\delta + H(x, 0) \geqq 0. \tag{2.25}$$

On the other hand we know

$$\sup_{B_{R_0}\times(0,T)} |Du| \leqq L < \infty, \tag{2.26}$$

hence

$$|a_i(Du)\cdot \nu_i| \leqslant \frac{L}{(1+L^2)^{1/2}} < 1 \qquad \text{on } \partial B_{R_0}. \tag{2.27}$$

Choosing R_1 close enough to R_0 such that

$$a_i(D\delta)\cdot \nu_i \geqslant \frac{L}{(1+L^2)^{1/2}} \qquad \text{on } \partial B_{R_0}, \tag{2.28}$$

the weak maximum principle shows that

$$u \leqslant \delta \qquad \text{in } B_{R_0}\times(0,T), \tag{2.29}$$

hence the result.

We note that the weak maximum principle is applicable, since our assumptions imply $u \in L^2(0, T; H^{2,2}(B_{R_0}))$, so that the boundary condition (2.27) makes sense for a.e. t.

2.4. Definition. Let Ω be an open set with Lipschitz boundary. We say that Ω satisfies an internal sphere condition (ISC) of radius R, if any point $x \in \Omega$ is contained in a ball B of radius R such that $B \subset \Omega$.

Bounded open sets with boundary of class C^2 satisfy an ISC.

2.5. Theorem. *Let $\Omega \subset \mathbb{R}^n$, be a bounded open set with Lipschitz boundary satisfying an ISC of radius R. Let $H = H(x, \tau)$ satisfy the conditions stated in Section* 1, *and let $j = j(x, \tau)$ be measurable in $\partial\Omega \times \mathbb{R}$, convex and non-expansive in τ, i.e.*

$$|j(x,\tau_1) - j(x,\tau_2)| \leqq |\tau_1 - \tau_2|, \tag{2.30}$$

such that

$$j(x, 0) \in L^\infty(\partial\Omega). \tag{2.31}$$

Let $\varphi \in L^\infty(\partial\Omega \times (0, T))$ and $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ be given data. Let Φ be the convex function

$$\Phi(v) = \int_\Omega (1 + |Dv|^2)^{1/2}\, dx + \int_\Omega \int_0^v H(x, \tau)\, d\tau\, dx. \tag{2.32}$$

Then, there exists a unique function $u \in L^1(0, T; BV(\Omega))$ satisfying (2.20),

$$\dot u \in L^2(Q_T), \qquad u(0) = u_0, \tag{2.33}$$

and

$$\int_0^t \int_\Omega \dot{u}(v-u)\,dx\,d\tau + \int_0^t \Phi(v)\,d\tau + \int_0^t \int_{\partial\Omega} j(x, v-\varphi)\,d\mathscr{H}_{n-1}\,d\tau$$
$$\geqslant \int_0^t \Phi(u)\,d\tau + \int_0^t \int_{\partial\Omega} j(x, u-\varphi)\,d\mathscr{H}_{n-1}\,d\tau \tag{2.34}$$
$$\forall v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T)$$

and for any $t \in [0, T]$. Moreover, if $\dot{\varphi}$ exists in the sense that $\dot{\varphi} \in L^1(0, T; BV(\Omega))$, then we have

$$u \in L^\infty(0, T; BV(\Omega)). \tag{2.35}$$

On the other hand, if we assume $u_0 \in C^2(\bar{\Omega})$ and $\dot{\varphi} \in L^\infty(\partial\Omega \times (0, T))$, then

$$|\dot{u}| \leqslant \max(\sup_\Omega |Au_0 + H(x, u_0)|, \sup_{\partial\Omega\times(0,T)} |\dot{\varphi}|) \tag{2.36}$$

and

$$\int_\Omega \dot{u}(v-u)\,dx + \Phi(v) + \int_{\partial\Omega} j(x, v-\varphi)\,d\mathscr{H}_{n-1}$$
$$\geqslant \Phi(u) + \int_{\partial\Omega} j(x, u-\varphi)\,d\mathscr{H}_{n-1} \qquad \forall v \in BV(\Omega) \tag{2.37}$$

for a.e. $t \in (0, T)$.

Proof. First, we assume the data u_0 and φ to be smooth. For each $\epsilon > 0$ we consider the convex function

$$\Phi_\epsilon(v) = \Phi(v) + \epsilon \cdot 1/2 \int_\Omega |Dv|^2\,dx \qquad \text{in place of } \Phi. \tag{2.38}$$

Then, we know from the previous considerations that there exist solutions $u_\epsilon \in L^2(0, T; H^{1,2}(\Omega))$ of the corresponding problems (1.4), (2.4), (2.5), or (2.9), satisfying the estimates (1.5) and (2.10) uniformly in ϵ. Therefore, we can assume

$$\|\dot{u}_\epsilon\|_{L^2(Q_T)} + \|u_\epsilon\|_{L^\infty(Q_T)} \leqslant \text{const} \tag{2.39}$$

uniformly in ϵ; actually even stronger estimates are valid, but it will be convenient to draw the following conclusions only under these mild assumptions for later reference.

Thus, a subsequence of the $u_\epsilon s$ (not relabelled) converges weakly in $L^2(Q_T)$ to some function u, satisfying the estimate (2.39). Moreover, for the same subsequence we have

$$\dot{u}_\epsilon \rightharpoonup \dot{u} \qquad \text{in} \quad L^2(Q_T), \tag{2.40}$$

$$u_\epsilon(t) \rightharpoonup u(t) \qquad \text{in} \quad L^2(\Omega), \tag{2.41}$$

for any $t \in [0, T]$, and

$$u(0) = u_0 . \tag{2.42}$$

Indeed, from the relation

$$1/2 \cdot \int_\Omega |u_\epsilon(\tau) - u_0|^2\, dx = \int_0^\tau \int_\Omega \dot{u}_\epsilon(u_\epsilon - u_0)\, dx\, ds \leqslant c \cdot \tau^{1/2} \tag{2.43}$$

we deduce

$$\int_0^t \int_\Omega |u_\epsilon(\tau) - u_0|^2\, dx\, d\tau \leqslant 2 \cdot c \cdot \int_0^t \tau^{1/2}\, d\tau = \tfrac{4}{3} \cdot c \cdot t^{3/2} \tag{2.44}$$

and hence

$$\int_0^t \int_\Omega |u(\tau) - u_0|^2\, dx\, d\tau \leqslant \tfrac{4}{3} \cdot c \cdot t^{3/2}, \tag{2.45}$$

which gives (2.42). The relation (2.41) then follows from

$$\int_\Omega (u_\epsilon(t) - u_0) \cdot v\, dx = \int_0^t \int_\Omega \dot{u}_\epsilon \cdot v\, dx\, d\tau \tag{2.46}$$

valid for any $v \in C_c(\Omega)$, and from the fact that this identity also holds if u_ϵ is replaced by u in view of (2.42).

Furthermore, from (2.4) we conclude that for a.e. t the $BV(\Omega)$-norm of $u_\epsilon(t)$ is bounded; since the imbedding $BV(\Omega)$ in $L^p(\Omega)$, $1 \leqslant p < n/n - 1$, is compact, we derive in view of (2.41)

$$u_\epsilon(t) \to u(t) \qquad \text{in } L^p(\Omega), \quad 1 \leqslant p < n/n - 1 \qquad \text{for a.e. } t. \tag{2.47}$$

On the other hand, we know that the functional

$$\Phi + \int_{\partial\Omega} j(x, \cdot - \varphi)\, d\mathscr{H}_{n-1} \tag{2.48}$$

is lower semicontinuous in $BV(\Omega)$ with respect to convergence in $L^1(\Omega)$ (cf. [5; Theorem 2.1]), i.e.

$$\Phi(u) + \int_{\partial\Omega} j(x, u - \varphi)\, d\mathscr{H}_{n-1} \leqslant \underline{\lim} \left\{\Phi(u_\epsilon) + \int_{\partial\Omega} j(x, u_\epsilon - \varphi)\, d\mathscr{H}_{n-1}\right\} \tag{2.49}$$

for fixed t. Applying Fatou's lemma, which is possible since the terms on the right-hand side of (2.49) are either non-negative or bounded, we get

$$\begin{aligned} &\int_0^t \Phi(u)\, d\tau + \int_0^t \int_{\partial\Omega} j(x, u - \varphi)\, d\mathscr{H}_{n-1}\, d\tau \\ &\qquad \leqslant \underline{\lim} \left\{\int_0^t \Phi(u_\epsilon)\, d\tau + \int_0^t \int_{\partial\Omega} j(x, u_\epsilon - \varphi)\, d\mathscr{H}_{n-1}\, d\tau\right\} \end{aligned} \tag{2.50}$$

for any $t \in [0, T]$. Now, using the relation (2.4) with Φ_ϵ and u_ϵ, integrating it over $[0, t]$ and observing that

$$\int_0^t \int_\Omega \dot{u}_\epsilon(v - u_\epsilon)\, dx\, d\tau = \int_0^t \int_\Omega \dot{u}_\epsilon \cdot v\, dx\, d\tau - 1/2 \cdot \int_\Omega (| u_\epsilon(t)|^2 - | u_0 |^2)\, dx \tag{2.51}$$

we obtain from (2.40), (2.41), and (2.50)

$$\begin{aligned} &\int_0^t \int_\Omega \dot{u} \cdot v\, dx\, d\tau + \int_0^t \Phi(v)\, d\tau + \int_0^t \int_{\partial\Omega} j(x, v - \varphi)\, d\mathscr{H}_{n-1}\, d\tau \\ &\quad \geqslant \int_0^t \Phi(u)\, d\tau + \int_0^t \int_{\partial\Omega} j(x, u - \varphi)\, d\mathscr{H}_{n-1}\, d\tau + 1/2 \cdot \int_\Omega (| u(t)|^2 - | u_0 |^2)\, dx \end{aligned} \tag{2.52}$$

for any $v \in L^2(0, T; H^{1,2}(\Omega))$, which is equivalent to (2.34) if v is restricted to that function class. If v is chosen as in (2.34), then for a.e. t, $v(t) \in BV(\Omega)$ and can be extended as a $H^{1,1}$-function with compact support outside Ω, say in a ball B with $\Omega \Subset B$, such that, if $\tilde{v}$ is the extension,

$$\int_B | D\tilde{v} |\, dx + \int_B | \tilde{v} |\, dx \leqslant c \cdot \left\{ \int_\Omega | Dv |\, dx + \int_\Omega | v |\, dx \right\}. \tag{2.53}$$

Consider now a sequence of mollifications (in x) v_ϵ of $\tilde{v}$. We have

$$\int_\Omega (1 + | Dv_\epsilon |^2)^{1/2}\, dx \leqslant c \cdot \left\{ \int_\Omega | Dv |\, dx + \int_\Omega | v |\, dx \right\}, \tag{2.54}$$

$$| j(x, v_\epsilon - \varphi)| \leqslant | j(x, -\varphi)| + | v_\epsilon |, \tag{2.55}$$

and, as is proved in [6; Appendix I], the relations

$$\int_{\partial\Omega} | v_\epsilon - v |\, d\mathscr{H}_{n-1} \to 0 \tag{2.56}$$

and

$$\int_\Omega (1 + | Dv_\epsilon |^2)^{1/2}\, dx \to \int_\Omega (1 + | Dv |^2)^{1/2}\, dx \tag{2.57}$$

are valid. Since evidently

$$v_\epsilon \rightharpoonup v \qquad \text{in} \quad L^2(Q_T) \tag{2.58}$$

we can apply Lebesgue's dominated convergence theorem to conclude (2.34), provided we can find a dominator for the term

$$\int_\Omega \int_0^{v_\epsilon} H(x, \tau)\, d\tau\, dx. \tag{2.59}$$

This can most easily be seen by observing that on account of the boundedness of u, we are now free to assume that H is bounded too; hence a dominator of (2.59) is

$$c \cdot \int_\Omega |v| \, dx \tag{2.60}$$

with a suitable constant c.

Thus, (2.33), (2.34) are established provided the data u_0 and φ were smooth. To prove (2.20) with these assumptions, we refer to Section 3, where we shall prove that the solution u which we have obtained is of class $L^\infty(0, T; C^{0,1}(\Omega))$ and satisfies the parabolic equation

$$\begin{aligned} \dot{u} + Au + H(x, u) &= 0, \\ u(0) &= u_0, \end{aligned} \tag{2.61}$$

where A is the minimal surface operator. Since Ω satisfies an ISC of radius R, we can therefore apply Theorem 2.3 to get the estimate (2.20).

If φ and u_0 only satisfy the assumptions stated in the theorem, then we approximate them by smooth functions and apply the convergence process being described in detail previously. To prove (2.35) we note that the estimate (1.10) valid for the u_ϵ's will hold uniformly in ϵ, if $\dot{\varphi} \in L^1(0, T; BV(\Omega))$ and we consider a suitable approximation of φ.

Finally, to prove (2.36) and (2.37), we first observe that (2.36) follows from (1.5) evaluated for u_ϵ. Moreover (2.37) is definitely valid for the u_ϵ's with Φ_ϵ in place of Φ. Integrating then the corresponding inequality over $[t_1, t_2]$ we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_\Omega \dot{u}_\epsilon (v - u_\epsilon) \, dx \, d\tau + \int_{t_1}^{t_2} \Phi(v) \, d\tau + \int_{t_1}^{t_2} \int_{\partial\Omega} j(x, v - \varphi) \, d\mathscr{H}_{n-1} \, d\tau \\ \geqslant \int_{t_1}^{t_2} \Phi(u_\epsilon) \, d\tau + \int_{t_1}^{t_2} \int_{\partial\Omega} j(x, u_\epsilon - \varphi) \, d\mathscr{H}_{n-1} \, d\tau. \end{aligned} \tag{2.62}$$

Consider the subsequence (not relabelled) satisfying (2.47). Let $1 < p < n/n - 1$ be fixed. In view of Lebesgue's dominated convergence theorem we have

$$u_\epsilon \to u \qquad \text{in} \quad L^p(Q_T). \tag{2.63}$$

On the other hand, since $\dot{u}_\epsilon$ is uniformly bounded we know

$$\dot{u}_\epsilon \rightharpoonup \dot{u} \qquad \text{in} \quad L^{p'}(Q_T), \quad 1/p + 1/p' = 1. \tag{2.64}$$

Therefore, we get

$$\int_{t_1}^{t_2} \int_\Omega \dot{u}_\epsilon (v - u_\epsilon) \, dx \, d\tau \to \int_{t_1}^{t_2} \int_\Omega \dot{u} (v - u) \, dx \, d\tau, \tag{2.65}$$

and we deduce from (2.62) and the known lower semicontinuity of the right-hand side of that inequality

$$\int_{t_1}^{t_2}\int_\Omega \dot u(v-u)\,dx\,d\tau + \int_{t_1}^{t_2}\Phi(v)\,d\tau + \int_{t_1}^{t_2}\int_{\partial\Omega} j(x, v-\varphi)\,d\mathscr{H}_{n-1}\,d\tau$$
$$\geqq \int_{t_1}^{t_2}\Phi(u)\,d\tau + \int_{t_1}^{t_2}\int_{\partial\Omega} j(x, u-\varphi)\,d\mathscr{H}_{n-1}\,d\tau. \tag{2.66}$$

This is equivalent to (2.37) in view of the arbitrariness of t_1, t_2.

It remains to prove the uniqueness of u. Let u_1, u_2 be two solutions of (2.33), (2.34). Adding the corresponding inequalities with v replaced by u_2 resp. u_1, we obtain for any t

$$\int_0^t\int_\Omega \dot u_1(u_2-u_1)\,dx\,d\tau + \int_0^t\int_\Omega \dot u_2(u_1-u_2)\,dx\,d\tau \geqq 0, \tag{2.67}$$

or equivalently,

$$\int_\Omega |\, u_1(t)-u_2(t)|^2\,dx = 0 \qquad \forall t\in[0,T], \tag{2.68}$$

hence the result.

3. Interior Gradient Estimates

Let A be the minimal surface operator and let u_ϵ, for $\epsilon > 0$, be a smooth solution of the equation

$$\begin{aligned}\dot u_\epsilon + Au_\epsilon - \epsilon\,\Delta u_\epsilon + H(x, u_\epsilon) &= 0 \qquad \text{in } Q_T,\\ u_\epsilon(0) &= u_0,\end{aligned} \tag{3.1}$$

where u_0 is Lipschitz continuous, and where H satisfies the conditions of Section 1.

We shall prove local a priori estimates for $|\, Du_\epsilon|$, i.e. estimates of $|\, Du_\epsilon|$ in $Q_T' = \Omega' \times (0, T)$, $\Omega' \Subset \Omega$, depending only on Ω', T, $|\, Du_0|$, $\sup_{Q_T}|\, u_\epsilon|$, $\sup_{Q_T}|\, \dot u_\epsilon|$, and known quantities.

Differentiating (3.1) with respect to $W^{-1}D_k u_\epsilon \cdot D_k$, $W = (1 + |\, Du_\epsilon|^2)^{1/2}$, we obtain

$$\dot W - \epsilon\,\Delta W - D_i(a^{ij}D_jW) + a^{ij}a^{kl}D_kD_iu_\epsilon\cdot D_lD_ju_\epsilon$$
$$+\,\epsilon\cdot a^{kl}D_lD_iu_\epsilon\cdot D_kD_iu_\epsilon + \frac{\partial H}{\partial x^k}\cdot\frac{D_ku_\epsilon}{W} + \frac{\partial H}{\partial \tau}\cdot\frac{|\, Du_\epsilon|^2}{W} = 0. \tag{3.2}$$

Several terms in this equation are non-negative: the fourth, fifth, and seventh; only for the first non-negative term one needs some reflection to recognize it:

3.1. LEMMA. *Let $A = (a^{ij})$ and $B = (b_{ij})$ be symmetric matrices, where A is positive semi-definite. Then*

$$\operatorname{tr}(AB \cdot AB) = a^{ij}a^{kl}b_{ki}b_{lj} \geqq 0. \tag{3.3}$$

Proof. There exists an orthogonal matrix O such that

$$D = O^*AO \tag{3.4}$$

is diagonal with non-negative eigenvalues λ^i, $i = 1,..., n$. Let

$$C = O^*BO = (c_{ij}). \tag{3.5}$$

Then, we have

$$\begin{aligned} \operatorname{tr}(AB \cdot AB) &= \operatorname{tr}(O^*AB \cdot ABO) \\ &= \operatorname{tr}(O^*AOO^*BO \cdot O^*AOO^*BO) \\ &= \operatorname{tr}(DC \cdot DC) \\ &= \lambda^i c_{ik} \cdot \lambda^k c_{ki} \\ &= \lambda^i\lambda^k \cdot | c_{ik} |^2 \geqq 0, \end{aligned} \tag{3.6}$$

since B (and therefore C, too) is symmetric.

From (3.2) we can thus deduce the key inequality for the forth-coming considerations

$$\dot{W} - \epsilon \Delta W - D_i(a^{ij}D_jW) + \frac{\partial H}{\partial x^k} \cdot \frac{D_k u_\epsilon}{W} \leqq 0. \tag{3.7}$$

For the proof of the gradient estimates we need several definitions and lemmata.

We denote by $\mathscr{S}$ the graph of u_ϵ, and by $\delta = (\delta_1 ,..., \delta_{n+1})$ the usual differential operators on the surface, i.e. for $g \in C^1(\Omega \times \mathbb{R})$

$$\delta_i g = D_i g - \nu^i \cdot \sum_{k=1}^{n+1} \nu^k \cdot D_k g, \qquad i = 1,..., n + 1, \tag{3.8}$$

where for the moment we let $\nu = (\nu^1,..., \nu^{n+1})$ be the normal vector of $\mathscr{S}$

$$\nu = -\frac{1}{W}(D_1 u_\epsilon ,..., D_n u_\epsilon , -1). \tag{3.9}$$

We note the important relations

$$a^{ij}D_i g D_j g = W^{-1} \cdot | \delta g |^2 \tag{3.10}$$

and

$$a^{ij}D_i g D_j f \leqq W^{-1} \cdot | \delta g | \cdot | Df | \tag{3.11}$$

valid for any functions $g, f \in C^1(\Omega)$.

The volume element of the surface is

$$d\mathscr{H}_n = W\, dx. \tag{3.12}$$

We first prove

3.2. LEMMA. *Let u_ϵ be a solution of* (3.1). *Then, we have*

$$Du_\epsilon \in L^p(Q'_T) \qquad \forall 1 \leqslant p < \infty \tag{3.13}$$

for any $\Omega' \Subset \Omega$ with uniformly bounded L^p-norm depending on Ω', T, p, $\| u_\epsilon \|_{L^\infty(Q_T)}$, $\| u_0 \|_{L^p(\Omega)}$, and known quantities.

Proof. Let η, $0 \leqq \eta \leqq 1$, be a cut-off function with $\operatorname{supp} \eta \Subset \Omega$. Multiply equation (3.1) with $u_\epsilon \cdot \eta^2$ and integrate to get

$$\int_\Omega | u_\epsilon(t)|^2 \cdot \eta^2\, dx + \epsilon \cdot \int_0^t \int_\Omega | Du_\epsilon |^2 \cdot \eta^2\, dx\, d\tau + \int_0^t \int_\Omega \frac{| Du_\epsilon |^2}{W} \cdot \eta^2\, dx\, d\tau \leqq \text{const}, \tag{3.14}$$

the constant depending on η, $\| u_\epsilon \|_{L^\infty(Q_T)}$, $\| u_0 \|_{L^2(\Omega)}$, and known quantities.

Thus, we conclude

$$\int_0^t \int_\Omega W \cdot \eta^2\, dx\, d\tau + \epsilon \cdot \int_0^t \int_\Omega | Du_\epsilon |^2\, dx\, d\tau \leqq \text{const}. \tag{3.15}$$

Now, look at inequality (3.7). Multiplying it with $W^{p-1} \cdot \eta^2$, $1 < p < \infty$, we obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega W^p \cdot \eta^2\, dx + \epsilon \cdot \int_\Omega | DW |^2 \cdot | W |^{p-2} \cdot \eta^2\, dx \\ &\quad + \int_\Omega a^{ij} D_i W D_j W \cdot W^{p-2} \cdot \eta^2\, dx \leqslant c \cdot \int_\Omega W^{p-1} \cdot \eta^2\, dx + c \cdot \int_\Omega | \delta\eta |^2 \\ &\quad \cdot W^{p-1}\, dx + \epsilon \cdot c \cdot \int_\Omega | D\eta |^2 \cdot W^p\, dx, \end{aligned} \tag{3.16}$$

in view of (3.11), where c depends on p and known quantities.

Moreover, since

$$| D | W |^{p/2} |^2 \leqq p^2/4 \cdot | DW |^2 \cdot W^{p-2} \tag{3.17}$$

we conclude

$$\sup_{[0,T]} \int_\Omega | W^{p/2} \cdot \eta |^2 \, dx + \epsilon \cdot \int_0^T \int_\Omega | D(W^{p/2} \cdot \eta)|^2 \, dx \, dt$$
$$\leqslant c \cdot \int_0^T \int_\Omega W^{p-1} \cdot \eta^2 \, dx \, dt + c \cdot \int_\Omega | W_0 |^p \cdot \eta^2 \, dx$$
$$+ c \cdot \int_0^T \int_\Omega | \delta\eta |^2 \cdot W^{p-1} \, dx \, dt + c \cdot \epsilon \cdot \int_0^T \int_\Omega W^p \cdot | D\eta |^2 \, dx \, dt. \tag{3.18}$$

From the Sobolev imbedding theorem

$$\left(\int_\Omega | v |^q \, dx\right)^{1/q} \leqq c \cdot \left(\int_\Omega | Dv |^2 \, dx\right)^{1/2} \qquad \forall v \in H_0^{1,2}(\Omega) \tag{3.19}$$

where

$$q = \begin{cases} 2n/(n-2), & \text{if} \quad n \geqslant 3 \\ < \infty, & \text{if} \quad n = 2 \end{cases} \tag{3.20}$$

we deduce from (3.18)

$$\sup_{[0,T]} \int_\Omega W^p \cdot \eta^2 \, dx + \epsilon \cdot \int_0^T \left(\int_\Omega (W^p \cdot \eta^2)^q \, dx\right)^{1/q} dt$$
$$\leqq c \cdot \int_\Omega W_0{}^p \cdot \eta^2 \, dx + c \cdot \int_0^T \int_\Omega (| \eta |^2 + | \delta\eta |^2) \, W^{p-1} \, dx \, dt$$
$$+ \epsilon \cdot c \cdot \int_0^T \int_\Omega | D\eta |^2 \cdot W^p \, dx \, dt, \tag{3.21}$$

where q is some fixed real number greater than 1.

Using interpolation inequalities for L^p-spaces we derive for some fixed q_0, $1 < q_0 < q$,

$$\sup_{[0,T]} \int_\Omega W^p \cdot \eta^2 \, dx + \epsilon^{1/q_0} \left(\int_0^T \int_\Omega (W^p \cdot \eta^2)^{q_0} \, dx \, dt\right)^{1/q_0}$$
$$\leqq c \cdot \int_\Omega W_0{}^p \cdot \eta^2 \, dx + c \cdot \int_0^T \int_\Omega (| \eta |^2 + | \delta\eta |^2) \cdot W^{p-1} \, dx \, dt$$
$$+ \epsilon \cdot c \cdot \int_0^T \int_\Omega | D\eta |^2 \cdot W^p \, dx \, dt. \tag{3.22}$$

We can now draw the following conclusion: assuming

$$\epsilon \cdot \int_0^T \int_{\Omega'} W^p \, dx \, dt \leqq c(\Omega'), \tag{3.23}$$

then

$$\sup_{[0,T]} \int_{\Omega''} W^p \, dx + \epsilon \cdot \int_0^T \int_{\Omega''} W^{p \cdot q_0} \, dx \, dt \leqq c(\Omega'', \Omega') \tag{3.24}$$

for any $\Omega'' \Subset \Omega' \Subset \Omega$.

The lemma is therefore proved taking (3.15) into account.

3.3. LEMMA. *Let u_ϵ be a solution of* (3.1). *Then, we have for* $0 < \epsilon \leqq 1$

$$\sup_{Q'_T} | Du_\epsilon |^2 \leqq c/\epsilon, \tag{3.25}$$

where the constant depends on Q_T, $\| Du_0 \|_{L^\infty(\Omega)}$, $\| u \|_{L^\infty(Q_T)}$, and known quantities.

Proof. For $k \geqq k_0 \geqq \| Du_0 \|_{L^\infty(\Omega)}$ define $W_k = \max(W - k, 0)$, and for fixed $x_0 \in \Omega$

$$A(k, r) = \{(x, t) \in Q_T : | x - x_0 | < r, W_k > 0\}.$$

We denote by $| A(k, r)|$ the Lebesgue measure of $A(k, r)$. Let $0 < R < 1$ be such that $B_R(x_0) \Subset \Omega$, and for $0 < \rho < r < R$ let η, $0 \leqq \eta \leqq 1$, be a cut-off function satisfying

$$\eta(x) = \begin{cases} 0, & | x - x_0 | > r \\ 1, & | x - x_0 | \leqq \rho \end{cases}, \qquad | D\eta | \leqq \frac{2}{| r - \rho |^2}.$$

Then, we multiply (3.7) with $W_k \cdot \eta^2$ to get

$$\frac{d}{dt} \int_\Omega W_k^2 \cdot \eta^2 \, dx + \epsilon \cdot \int_\Omega | DW_k |^2 \cdot \eta^2 \, dx$$
$$\leqq c \cdot \int_{B_r} W_k \, dx + \frac{c}{| r - \rho |^2} \cdot \int_{B_r} W_k^2 \, dx. \tag{3.26}$$

By similar considerations which we used to deduce (3.21) from (3.18), we derive from (3.26)

$$\sup_{[0,T]} \int_{B_\rho} W_k^2 \, dx + \epsilon \cdot \int_0^T \left(\int_{B_\rho} W_k^{2 \cdot q} \, dx \right)^{1/q} dt$$
$$\leqq c \cdot \int_0^T \int_{B_r} W_k \, dx \, dt + \frac{c}{| r - \rho |^2} \cdot \int_0^T \int_{B_r} W_k^2 \, dx \, dt. \tag{3.27}$$

Apply now once again the interpolation inequalities for L^p-spaces

$$\left(\int_{B_\rho} W_k^{2 \cdot q_0} \, dx \right)^{1/q_0} \leqq \left(\int_{B_\rho} W_k^{2 \cdot q} \, dx \right)^{a/q} \cdot \left(\int_{B_\rho} W_k^2 \, dx \right)^{(1-a)} \tag{3.28}$$

with

$$1/q_0 = a/q + 1 - a.$$

Choosing $1 < q_0 < q$ such that $q_0 \cdot a = 1$, we deduce from (3.27), (3.28)

$$\epsilon^{1/q_0} \left(\int_0^T \int_{B_\rho} W_k^{2\cdot q_0} \, dx \, dt \right)^{1/q_0} \leq c \cdot \int_0^T \int_{B_r} W_k \, dx \, dt$$

$$+ \frac{c}{|r-\rho|^2} \cdot \int_0^T \int_{B_r} W_k^2 \, dx \, dt. \tag{3.29}$$

In view of Lemma 3.2 we finally obtain for $h > k \geqq k_0$, where k_0 is so large that $|A(k_0, R)| \leqq 1$,

$$|h-k|^2 \cdot |A(h,\rho)| \leqq \frac{c \cdot \epsilon^{-1/q_0}}{|r-\rho|^2} \cdot |A(k,r)|^{\gamma + (q_0-1)/q_0}, \tag{3.30}$$

for arbitrary γ, $0 < \gamma < 1$, where the constant c then depends on γ, too. Choosing γ so large that $\gamma + (q_0 - 1)/q_0$ is greater than 1 we deduce from a lemma due to Stampacchia (cf. [15; Lemma 5.1]) that

$$\sup_{B_{R/2}(x_0)\times(0,T)} |W| \leqq k_0 + \epsilon^{-1/2q_0} \cdot c/R \cdot |A(k_0, R)|^{\gamma - 1/q_0}, \tag{3.31}$$

hence the result.

3.4. Lemma. *Let $k \geqq 4$. Then, we have*

$$\log W - \log k \leqq \frac{W-k}{W^{1/2}} \qquad \forall W \geqq k. \tag{3.32}$$

Proof. Simple calculus.

We are now ready to prove the gradient estimates.

3.5. Theorem. *Let u_ϵ be a solution of (3.1). Then, for any $\Omega' \Subset \Omega$, the estimate*

$$\sup_{Q_T'} |Du_\epsilon| \leqq \text{const} \tag{3.33}$$

holds uniformly in ϵ, where the constant depends on the quantities mentioned at the beginning of this section.

Proof. Without loss of generality we may assume that u_0 is smooth, so that $u_\epsilon \in C^2(Q_T)$. Let η, $0 \leqq \eta \leqq 1$, be a cut-off function with compact support in Ω, and let $k \geqq k_0 \geqq W_0$.

Then, multiply (3.7) with $W_k \cdot \eta^2$ to get

$$\frac{d}{dt}\int_\Omega W_k^2 \cdot \eta^2\, dx + \int_\Omega a^{ij} D_i W_k \cdot D_j W_k \eta^2\, dx$$
$$\leq c \cdot \int_\Omega W_k \cdot \eta^2\, dx + c \cdot \int_\Omega |\delta_n|^2 \cdot W_k\, dx + c \cdot \int_\Omega |D\eta|^2 \cdot W_k^2\, dx. \tag{3.34}$$

Introducing now $w = \log W$ and $w_{\bar{k}} = \max(w - \bar{k}, 0)$, $\bar{k} = \log k$, we derive from (3.34)

$$\sup_{[0,T]}\int_\Omega W_k^2 \cdot \eta^2\, dx + \int_0^T\int_\Omega |\delta w_{\bar{k}}|^2 \cdot \eta^2 \cdot W\, dx\, dt$$
$$\leq c \cdot \int_0^T\int_\Omega W_k(\eta^2 + |\delta\eta|^2)\, dx\, dt + c \cdot \int_0^T\int_\Omega |D\eta|^2 \cdot W_k^2\, dx\, dt. \tag{3.35}$$

Regarding the functions as being defined on the surface $\mathscr{S} = \mathscr{S}(t)$ rather than on Ω, we conclude in view of Lemma 3.4

$$\sup_{[0,T]}\int_{\mathscr{S}} w_{\bar{k}}^2 \cdot \eta^2\, d\mathscr{H}_n + \int_0^T\int_{\mathscr{S}} |\delta(w_{\bar{k}} \cdot \eta)|^2\, d\mathscr{H}_n\, dt$$
$$\leq c \cdot \int_0^T\int_{\mathscr{S}} (|\eta|^2 + |\delta\eta|^2 + |D\eta|^2) \cdot W_k\, d\mathscr{H}_n\, dt. \tag{3.36}$$

Now let $x_0 \in \Omega$ be arbitrary but fixed, and let $0 < \rho < r < R < 1$ be such

$$A(\bar{k}, r) = \{(x, t) \in Q_T : |x - x_0| < r, w_{\bar{k}} > 0\}.$$

We define

$$|A(k, r)| = \int_0^T\int_{\mathscr{S}} \tfrac{1}{2}\chi A(\bar{k}, r)\, d\mathscr{H}_n\, dt \tag{3.37}$$

in contrast to the previous case.

We are going to prove that $|A(\bar{k}, R/2)| = 0$ if $\bar{k}$ is sufficiently large using the same conclusions as in the proof of Lemma 3.3. We only have to replace the ordinary Sobolev imbedding theorem by the corresponding imbedding theorem for functions defined on the graph of a surface of prescribed mean curvature, or of a perturbation of such a surface: in [16; Lemma 3.1] the following inequality is proved

3.6. Lemma. *Let $u_\epsilon \in C^2(\Omega)$ be a solution of*

$$Au_\epsilon - \epsilon\, \Delta u_\epsilon = f_\epsilon \qquad in \quad \Omega, \tag{3.38}$$

and assume that for $0 < \epsilon \leqq 1$, $|f_\epsilon|$ *and* $\epsilon . |Du_\epsilon|$ *are uniformly bounded in* $B_{2R}(x_0)$, $B_{2R}(x_0) \subset \Omega$. *Then,*

$$\left(\int_{\mathscr{S}} |v|^{p/p-1} \, d\mathscr{H}_n\right)^{(p-1)/p} \leqslant c \cdot \int_{\mathscr{S}} |\delta v| \, d\mathscr{H}_n \tag{3.39}$$

for any $v \in C_c^1(B_R(x_0))$, *where the constant* c *depends on* R *and on the bounds for the quantities mentioned above.*

We may apply this lemma in view of Lemma 3.3 and in view of our assumption that

$$\sup_{Q_T} |\dot{u}_\epsilon| \leqq \text{const.} \tag{3.40}$$

As usual we obtain from (3.39)

$$\left(\int_{\mathscr{S}} |v|^q \, d\mathscr{H}_n\right)^{1/q} \leqq c \cdot \left(\int_{\mathscr{S}} |\delta v|^2 \, d\mathscr{H}_n\right)^{1/2} \tag{3.41}$$

for those functions v, where

$$q = \begin{cases} 2n/n-2, & \text{if} \quad n \geqq 3 \\ <\infty, & \text{if} \quad n = 2. \end{cases} \tag{3.42}$$

Reasoning now as at the end of the proof of Lemma 3.3, we have only to replace W_k by $w_{\bar{k}}$, $|A(k, r)|$ by $|A(\bar{k}; r)|$, noting that $W_k = 0$ is equivalent to $w_{\bar{k}} = 0$, we obtain the desired conclusion.

3.6. *Remark.* We note that the assumption (3.40) can be verified according to Lemma 1.1.

4. Boundary Estimates

In the following we shall be interested in a priori estimates near the boundary for the gradient of smooth solutions of the evolutionary boundary value problem

$$\begin{aligned} \dot{u} + Au + H(x, u) &= 0 && \text{in} \quad Q_{T_0}, \\ -a^i \cdot \nu_i &= \beta(x, u - \varphi) && \text{on} \quad \partial\Omega \times (0, T_0), \\ u(0) &= u_0, \end{aligned} \tag{4.1}$$

where $\partial\Omega$ is of class C^3 (we write T_0 instead of T in this section to avoid ambiguity in the following). $H = H(x, \tau)$, $\beta = \beta(x, \tau)$, and $\varphi = \varphi(x, t)$ are given Lipschitz continuous functions satisfying the usual weak monotonicity conditions in the τ-variable.

Moreover, we assume

$$|\beta| \leqq 1 - a, \qquad a > 0. \tag{4.2}$$

We suppose that this last assumption can be weakened to

$$|\beta| \leqslant 1 \tag{4.3}$$

if $\beta = \beta(\tau)$, but actually we cannot prove this; it merely seems to be a technical question. From the forth-coming considerations it will be clear that the most general condition (4.3) will be sufficient if we assume H to be strictly monotone, i.e. $\partial H/\partial\tau \geqslant \kappa > 0$, but we shall neither prove nor state this in detail.

We are going to prove a priori estimates for the tangential derivatives of u near the boundary depending linearly on the square-root of the normal derivative. If (4.2) holds, this will give an estimate for $|Du|$ on $\partial\Omega \times (0, T_0)$. By the maximum principle or by applying the interior estimates up to the boundary (which is possible then, since W_k vanishes on $\partial\Omega \times (0, T_0)$ if k is large) we then obtain a global gradient bound.

The method of proof is similar if not almost identical to that given in [4], where we have treated the case of a stationary solution of (4.1). That proof in turn depends on the techniques developed in [14].

In contrast to the stationary case we have not to assume H to be strictly monotone, provided (4.2) is valid, for we have the very strong term "$\dot{u}$" taking the role of "$\partial H/\partial\tau \geqq \kappa > 0$".

For the convenience of the reader we repeat some necessary definitions and assumptions.

d will denote the distance function, $d(x) = \operatorname{dist}(x, \partial\Omega)$. We shall work in a neighbourhood $\Omega_\delta = B_\delta(x_0) \cap \Omega$ of a point $x_0 \in \partial\Omega$. δ will be assumed small enough to ensure that d is of class C^3 in Ω_δ. We define $Q_{\delta,T_0} = \Omega_\delta \times (0, T_0)$.

Let L, M, and N be constants such that

$$\sup_{Q_{\delta,T_0}} \left|\frac{\partial}{\partial x}\beta(x, u - \varphi)\right| \leqq L, \tag{4.4}$$

$$\sup_{Q_{\delta,T_0}} \left\{|u| + |\dot{u}| + |H(x, u)| + \left|\frac{\partial}{\partial x} H(x, u)\right|\right\} \leqslant M, \tag{4.5}$$

and

$$\sup_{\partial\Omega\cap\partial\Omega_\delta\times(0,T_0)} \left|\frac{\partial}{\partial x}\varphi\right| + \sup_{\Omega_\delta} |Du_0| \leqq N. \tag{4.6}$$

We define $(Du)_T(x)$ to be the tangential derivative of u relative to the hypersurface $\{\xi \in \Omega_\delta : d(x) = d(\xi)\}$, i.e.

$$(Du)_T(x) = Du(x) - [Du(x) \cdot Dd(x)] \cdot Dd(x)\cdot \tag{4.7}$$

v_T is defined on Ω_δ by

$$v_T = (1 + |(Du)_T|^2)^{1/2}. \tag{4.8}$$

δ is assumed small enough to ensure that we can introduce local coordinates $y = y(x)$ in Ω_δ which "flatten" $\partial\Omega$ near x_0. We may choose $y = (y^1, \ldots, y^n)$ to be a diffeomorphism from Ω_δ to $\mathbb{R}^n$ such that

$$\begin{aligned} y^i &\in C^3(\Omega_\delta), \qquad i = 1, \ldots, n-1, \\ y^n &= d \qquad \text{on } \Omega_\delta, \end{aligned} \tag{4.9}$$

and such that the transposed Jacobian matrix J (i.e. the matrix with ith row $\partial y/\partial x^i$) satisfies

$$J^*(x)\, J(x) = (e^{ij}(y)), \qquad x \in \Omega_\delta, \tag{4.10}$$

where

$$e^{in} = 0, \qquad i = 1, \ldots, n-1, \qquad e^{nn} = 1, \tag{4.11}$$

and

$$\lambda \cdot |\xi|^2 \leq e^{ij}(y)\, \xi_i \xi_j, \qquad \xi \in \mathbb{R}^n, \quad y \in G_\delta, \tag{4.12}$$

for some positive constant λ, where G_δ is the image of Ω_δ under the transformation $y = y(x)$.

Λ will denote a constant such that

$$\lambda^{-1/2} + |Dy| + |D^2 y| + |D^3 y| \leq \Lambda \text{ uniformly in } \Omega_\delta. \tag{4.13}$$

If f is a function defined on Ω_δ, then $\tilde{f}$ is defined on G_δ by $\tilde{f}(y) = f(x)$. For functions $f, g \in C^1(\Omega_\delta)$ we have

$$D_{x^i} f(x) \cdot D_{x^i} g(x) = e^{ij}(y) \cdot D_{y^i} \tilde{f}(y) \cdot D_{y^j} \tilde{g}(y) \tag{4.14}$$

in view of the definition of the e^{ij}'s. For brevity we shall only write $D_i f$ if it is clear which partial derivative is meant; this is e.g. always the case if the function f has a tilde or hat, then $D_i \tilde{f} = D_{y^i} f$.

μ will denote the Jacobian of the transformation $y \to x$, i.e.

$$\mu(y) = (\det(e^{ij}(y)))^{-1/2}, \qquad y \in G_\delta. \tag{4.15}$$

We also introduce the following functions on G_δ

$$\tilde{v}(y) = (1 + e^{ij} D_i \tilde{u} \cdot D_j \tilde{u})^{1/2} = v(x) \equiv (1 + |Du(x)|^2)^{1/2},$$

$$\tilde{v}_T(y) = \left(1 + \sum_{i,j=1}^{n-1} e^{ij} D_i \tilde{u} \cdot D_j \tilde{u}\right)^{1/2} = v_T(x),$$

$$
\begin{aligned}
\hat{v}_T(y) &= \left(1 + \sum_{i=1}^{n-1} |D_i\tilde{u}|^2\right)^{1/2} \\
\tilde{\nu}^i(y) &= e^{ij}(D_j\tilde{u}/\tilde{v}), \qquad i = 1,\dots, n, \\
\chi &= (\tilde{v}_T/\tilde{v})^2, \\
g^{ij} &= e^{ij} - \tilde{\nu}^i \cdot \tilde{\nu}^j, \qquad i, j = 1,\dots, n.
\end{aligned} \tag{4.16}
$$

We note that the symbol v now denotes the same function which we called in the previous sections W; the functions $\tilde{\nu}^i$ are *not* the transformed components of the exterior normal ν at $\partial\Omega$. We introduce these two symbols only to make a comparison of the proofs (in the stationary and instationary case) more easily.

Note the relation

$$\chi = g^{nn} = 1 - |\tilde{\nu}^n|^2. \tag{4.17}$$

In terms of the transformed coordinates (4.1) becomes

$$
\begin{aligned}
\mu \cdot \dot{\tilde{u}} - D_i(\mu \cdot \tilde{\nu}^i) + \mu \cdot \tilde{H}(y, \tilde{u}) = 0 &\qquad \text{in} \quad \tilde{Q}_{\delta,T_0} \\
-\tilde{\nu}^n = \tilde{\beta}(y, \tilde{u} - \tilde{\varphi}) &\qquad \text{on} \quad \Gamma \times (0, T_0),
\end{aligned} \tag{4.18}
$$

where $\tilde{Q}_{\delta,T_0} = G_\delta \times (0, T_0)$ and $\Gamma = G_\delta \cap \{y \in \mathbb{R}^n : y^n = 0\}$.

This can most easily be seen by writing (4.1) in integral form, namely,

$$
\int_{\Omega_\delta} \dot{u} \cdot \zeta\, dx + \int_{\Omega_\delta} a^i \cdot D_i\zeta\, dx + \int_{\Omega_\delta} H(x, u) \cdot \zeta\, dx + \int_{\delta\Omega} \beta(x, u - \varphi) \cdot \zeta\, d\mathscr{H}_{n-1} = 0 \tag{4.19}
$$

for all Lipschitz functions with compact support in $B_\delta(x_0)$.

Making the transformation $y = y(x)$ we obtain

$$
\int_{G_\delta} \dot{\tilde{u}} \cdot \tilde{\zeta} \cdot \mu\, dy + \int_{G_\delta} \tilde{v}^{-1} \cdot e^{ij} D_i\tilde{u} \cdot D_j\tilde{\zeta} \cdot \mu\, dy + \int_{G_\delta} \tilde{H}(y, \tilde{u}) \cdot \tilde{\zeta} \cdot \tilde{\mu}\, dy + \int_\Gamma \tilde{\beta}(y, \tilde{u} - \tilde{\varphi}) \cdot \tilde{\zeta} \cdot \mu\, d\mathscr{H}_{n-1} = 0 \tag{4.20}
$$

or equivalently

$$
\int_{G_\delta} \mu \cdot \dot{\tilde{u}} \cdot \tilde{\zeta}\, dy + \int_{G_\delta} \mu \cdot \tilde{\nu}^i \cdot D_i\tilde{\zeta}\, dy + \int_{G_\delta} \mu \cdot \tilde{H}(y, \tilde{u}) \cdot \tilde{\zeta}\, dy + \int_\Gamma \mu \cdot \tilde{\beta}(y, \tilde{u} - \tilde{\varphi}) \cdot \tilde{\zeta}\, d\mathscr{H}_{n-1} = 0 \tag{4.21}
$$

for all Lipschitz continuous functions $\tilde{\zeta}$ with support in U_δ, $U_\delta = \{y(x): x \in B_\delta(x_0)\}$ The relation (4.18) then follows immediately.

We now present some inequalities which will be useful in the following

$$D_l\tilde{\nu}^i = D_l\gamma^{mi}\gamma^{mj}(D_j\tilde{u}/\tilde{v}) + \tilde{v}^{-1}g^{ij}D_lD_j\tilde{u} + \gamma_{ks}g^{ik}D_l\gamma^{sr}(D_r\tilde{u}/\tilde{v}), \qquad i, l = 1,\ldots, n, \tag{4.22}$$

where (γ^{ij}) is any continuously differentiable $n \times n$ matrix on G_δ satisfying

$$\begin{aligned} &\gamma^{ij} = \gamma^{ji}, \qquad \gamma^{si}\gamma^{sj} = e^{ij}, \qquad i, j = 1,\ldots, n,\\ &\gamma^{ni} = 0, \qquad i = 1,\ldots, n-1, \qquad \gamma^{nn} = 1, \end{aligned} \tag{4.23}$$

and where $(\gamma_{ij}) = (\gamma^{ij})^{-1}$.

In view of (4.11) we may choose

$$(\gamma^{ij}) = (e^{ij})^{1/2}. \tag{4.24}$$

The coefficients γ^{ij} will then be of class C^2 in G_δ and their derivatives up to order 2 will be bounded in terms of n and Λ.

The relation (4.22) can be easily derived from the following identities

$$D_l(\gamma^{mj}D_j\tilde{u}/\tilde{v}) = \tilde{v}^{-1} \cdot \{\delta^{ms} - \gamma^{mj}\gamma^{sr} \cdot (D_j\tilde{u}/\tilde{v}) \cdot (D_r\tilde{u}/\tilde{v})\} \cdot D_l(\gamma^{ss'}D_{s'}\tilde{u}), \qquad l, m = 1,\ldots, n, \tag{4.25}$$

and

$$\gamma_{km}g^{kj} = \gamma^{mj} - \gamma^{mr}(D_r\tilde{u}/\tilde{v})\,\tilde{\nu}^j, \qquad m, j = 1,\ldots, n. \tag{4.26}$$

δ^{ms} denotes the Kronecker symbol.

Due to the fact that $D_k\gamma^{ij} = 0$ if i or j are equal to n, we derive from (4.22)

$$| D_\sigma\tilde{\nu}^i | \leqq c \cdot (\mathscr{C} + \chi^{1/2}) \qquad i = 1,\ldots, n, \quad \sigma = 1,\ldots, n-1, \tag{4.27}$$

where the non-negative function $\mathscr{C}$ is defined on G_δ by

$$\mathscr{C}^2 = \tilde{v}^{-2} \cdot g^{ij}D_\sigma D_i\tilde{u} \cdot D_\sigma D_j\tilde{u}, \tag{4.28}$$

and where the constant c depends on n and Λ. In (4.28) and in the following we use the summation convention to sum over Greek indices from 1 to $n - 1$.

Moreover, since

$$\begin{aligned} D_n\tilde{\nu}^n &= D_n(\mu^{-1}\mu\tilde{\nu}^n) = (D_n\mu^{-1}) \cdot \mu \cdot \tilde{\nu}^n + \mu^{-1}D_n(\mu\tilde{\nu}^n)\\ &= (D_n\mu^{-1})\,\mu\tilde{\nu} - \mu^{-1}D_\sigma(\mu\tilde{\nu}^\sigma) + \tilde{H}(y, \tilde{u}) + \dot{\tilde{u}}, \end{aligned} \tag{4.29}$$

we obtain from (4.27)

$$| D_n\tilde{\nu}^n | \leqq c \cdot (\mathscr{C} + 1), \tag{4.30}$$

where c depends on n, M, and Λ.

The quantity $\mathscr{C}^2$ defined in (4.28) satisfies

$$\tilde{v}^{-1} \cdot g^{ij} D_i \hat{v}_T \cdot D_j \hat{v}_T \leqq \tilde{v} \cdot \mathscr{C}^2 \tag{4.31}$$

as is easily calculated.

We are going to prove that $\hat{v}_T$ is uniformly bounded in $G_\delta \times (0, T_0)$ (or equivalently that $(Du)_T$ is uniformly bounded in $\Omega_\delta \times (0, T_0)$). We obtain the crucial equation to start with as follows: First replace $\tilde{\zeta}$ in (4.21) by $\mu^{-1} \cdot \zeta$, $\zeta \in C_c^{0,1}(U_\delta)$, to get

$$\int_{G_\delta} \dot{\tilde{u}} \zeta \, dy + \int_{G_\delta} \tilde{\nu}^i D_i \zeta \, dy + \int_{G_\delta} \mu \cdot D_i \mu^{-1} \cdot \tilde{\nu}^i \cdot \zeta \, dy$$
$$+ \int_{G_\delta} \tilde{H}(y, \tilde{u}) \zeta \, dy + \int_\Gamma \tilde{\beta}(y, \tilde{u} - \tilde{\varphi}) \cdot \zeta \, d\mathscr{H}_{n-1} = 0. \tag{4.32}$$

Then replacing ζ by $-D_\sigma(\zeta \cdot D_\sigma \tilde{u})$, $\zeta \in C_c^{0,1}(U_\delta)$, we conclude via integrating by parts

$$\int_{G_\delta} 1/2 \, \frac{d}{dt} \, | D_\sigma \tilde{u} |^2 \cdot \zeta \, dy + \int_{G_\delta} \Big\{ D_\sigma \tilde{\nu}^i \cdot D_i (\zeta \cdot D_\sigma \tilde{\mu})$$
$$+ D_\sigma(\mu D_i \mu^{-1}) \cdot \tilde{\nu}^i \cdot D_\sigma \tilde{u} \cdot \zeta + \mu D_i \mu^{-1} \cdot D_\sigma \tilde{\nu}^i \cdot D_\sigma \tilde{u} \cdot \zeta$$
$$+ \frac{\partial \tilde{H}}{\partial y^\sigma} \cdot D_\sigma \tilde{u} \cdot \zeta + \frac{\partial \tilde{H}}{\partial \tau} \, | D_\sigma \tilde{u} |^2 \cdot \zeta \Big\} \, dy$$
$$= - \int_\Gamma \Big\{ \frac{\partial \tilde{\beta}}{\partial y^\sigma} \cdot D_\sigma \tilde{u} \cdot \zeta + \frac{\partial \tilde{\beta}}{\partial \tau} \cdot D_\sigma(\tilde{u} - \tilde{\varphi}) \cdot D_\sigma \tilde{u} \cdot \zeta \Big\} \, d\mathscr{H}_{n-1} . \tag{4.33}$$

Replace now ζ by $\zeta \cdot \hat{v}_T^{-1}$, where $\zeta \in C_c^{0,1}(U_\delta)$ is non-negative and has support in $\{x \in U_\delta : | D_\sigma \tilde{u} | > | D_\sigma \tilde{\varphi} |\}$, and take (4.16), (4.22), and (4.28) into account to get

$$\int_{G_\delta} \dot{\hat{v}}_T \cdot \zeta \, dy + \int_{G_\delta} \tilde{v} \cdot \mathscr{C}^2 \cdot \hat{v}_T^{-1} \cdot \zeta \, dy + \int_{G_\delta} \tilde{v}^{-1} g^{ij} D_j \hat{v}_T \cdot D_i \zeta \, dy$$
$$- \int_{G_\delta} \tilde{v}^{-1} g^{ij} D_j \hat{v}_T \cdot D_i \hat{v}_T \cdot \hat{v}_T^{-1} \cdot \zeta \, dy \leqslant - \int_{G_\delta} \{ D_\sigma \gamma^{mi} \gamma^{mj}$$
$$\cdot (D_j \tilde{u} / \tilde{v}) + \gamma_{ks} g^{ik} D_\sigma \gamma^{sr} (D_r \tilde{u} / \tilde{v}) \} \cdot D_i (D_\sigma \tilde{u} \cdot \hat{v}_T^{-1} \cdot \zeta) \, dy$$
$$+ c \cdot \int_{G_\delta} (\mathscr{C} + 1) \cdot \zeta \, dy + c \cdot \int_\Gamma \zeta \, d\mathscr{H}_{n-1} , \tag{4.34}$$

where we used the Cauchy inequality for positive semi-definite matrices once. The integrals involving $\partial \tilde{H} / \partial \tau$ and $\partial \tilde{\beta} / \partial \tau$ can be neglected in view of the monotonicity of $\tilde{H}$ and $\tilde{\beta}$.

Let us estimate the integral

$$
\begin{aligned}
I = & - \int_{G_\delta} D_\sigma \gamma^{mi} \cdot \gamma^{mj} \cdot (D_j \tilde{u}/\tilde{v}) \cdot D_i(D_\sigma \tilde{u} \cdot \hat{v}_T^{-1} \cdot \zeta)\, dy \\
& - \int_{G_\delta} \gamma_{ks} g^{ik} \cdot D_\sigma \gamma^{sr}(D_r \tilde{u}/\tilde{v}) \cdot D_i(D_\sigma \tilde{u} \cdot \hat{v}_T^{-1} \cdot \zeta)\, dy \equiv I_1 + I_2 . \qquad (4.35)
\end{aligned}
$$

To estimate I_1, we use (4.25), (4.26), and

$$
D_\sigma \gamma^{ij} = 0, \qquad \text{if } i \text{ or } j \text{ are equal to } n, \qquad (4.36)
$$

and transform it via integrating by parts as follows

$$
\begin{aligned}
I_1 = \sum_{i=1}^{n-1} \int_{G_\delta} & \{ D_i D_\sigma \gamma^{mi} \gamma^{mi} \cdot (D_j \tilde{u}/\tilde{v}) \cdot D_\sigma \tilde{u} \cdot \hat{v}_T^{-1} \cdot \zeta \\
& + \tilde{v}^{-1} \cdot D_\sigma \gamma^{mi} [\delta^{ms} - \gamma^{mj} \gamma^{sr} \cdot (D_j \tilde{u}/\tilde{v}) \cdot (D_r \tilde{u}/\tilde{v})] . \\
& \cdot D_i \gamma^{ss'} \cdot D_{s'} \tilde{u} \cdot D_\sigma \tilde{u} \cdot \hat{v}_T^{-1} \cdot \zeta \\
& + \tilde{v}^{-1} \cdot D_\sigma \gamma^{mi} \cdot \gamma_{km} \cdot g^{ks'} \cdot D_i D_{s'} \tilde{u} \cdot D_\sigma \tilde{u} \cdot \hat{v}_T^{-1} \cdot \zeta \}\, dy. \qquad (4.37)
\end{aligned}
$$

Hence we deduce

$$
I_1 \leq c \cdot \int_{G_\delta} \zeta\, dy + c \cdot \int_{G_\delta} \mathscr{C} \cdot \zeta\, dy. \qquad (4.38)
$$

I_2 can be estimated by direct calculations yielding

$$
I_2 \leq c \cdot \int_{G_\delta} \mathscr{C} \zeta\, dy + \int_{G_\delta} g^{ik} D_i \zeta \cdot b_k\, dy, \qquad (4.39)
$$

where

$$
b_k = -\gamma_k \cdot D_\sigma \gamma^{sr}(D_r \tilde{u}/\tilde{v}) \cdot D_\sigma \tilde{u} \cdot \hat{v}_T^{-1}, \qquad (4.40)
$$

i.e.

$$
| b_k | \leq c \cdot \chi^{1/2} \qquad (4.41)
$$

in view of (4.36).

Combining (4.34), (4.38), (4.39), and applying Young's inequality we obtain

$$
\begin{aligned}
& \int_{G_\delta} \dot{\hat{v}}_T \zeta\, dy + \int_{G_\delta} \tilde{v} \mathscr{C}^2 \cdot \hat{v}_T^{-1} \cdot \zeta\, dy \\
& \quad + \int_{G_\delta} \tilde{v}^{-1} \cdot g^{ij} D_j \hat{v}_T \cdot D_i \zeta\, dy - \int_{G_\delta} \tilde{v}^{-1} g^{ij} D_i \hat{v}_T D_j \hat{v}_T \cdot \hat{v}_T^{-1} \zeta\, dy \\
& \leqslant c \cdot \int_{G_\delta} \zeta\, dy + \int_{G_\delta} g^{ik} D_i \zeta \cdot b_k\, dy + c \cdot \int_{\Gamma} \zeta\, d\mathscr{H}_{n-1} \qquad (4.42)
\end{aligned}
$$

for all non-negative $\zeta \in C_c^{0,1}(U_\delta)$ with support in $\{x \in U_\delta : |D_\sigma \tilde{u}| > |D_\sigma \tilde{\varphi}|\}$, where we note that the boundary integral vanishes if β is independent of x.

To estimate the boundary integral in the general case $\beta = \beta(x, \tau)$, we use (4.2) and the inequality

$$\int_\Gamma \chi^{1/2} \cdot f \cdot \tilde{v}\, d\mathscr{H}_{n-1} \leqslant c \cdot \int_{G_\delta} \{\chi \cdot f + \chi(g^{ij} D_i f D_j f)^{1/2} + f \cdot \chi^{1/2} \cdot \mathscr{C}\} \cdot \tilde{v}\, dy \tag{4.43}$$

valid for all non-negative functions $f \in C_c^{0,1}(U_\delta)$, where the constant c depends on n, A, M, and a. (cf. [14; formula (2.13)]).

We use (4.43) with $f = \zeta \cdot \hat{v}_T^{-1}$ and conclude

$$\begin{aligned}\int_\Gamma \zeta\, d\mathscr{H}_{n-1} &\leqslant c \cdot \int_\Gamma \chi^{1/2} \cdot \zeta \cdot \hat{v}_T^{-1} \cdot \tilde{v}\, d\mathscr{H}_{n-1} \\ &\leqq c \cdot \int_{G_\delta} \zeta\, dy + c \cdot \int_{G_\delta} \mathscr{C} \cdot \zeta\, dy + c \cdot \int_{G_\delta} \chi^{1/2} \cdot (g^{ij} D_i \zeta D_j \zeta)^{1/2}\, dy \\ &\quad + c \cdot \int_{G_\delta} \chi^{1/2} (g^{ij} D_i \hat{v}_T \cdot D_j \hat{v}_T)^{1/2} \cdot \hat{v}_T^{-1} \cdot \zeta\, dy,\end{aligned} \tag{4.44}$$

where the last integral can be estimated by

$$\epsilon \cdot \int_{G_\delta} \tilde{v}\mathscr{C}^2 \cdot \hat{v}_T^{-1} \cdot \zeta\, dy + c_\epsilon \cdot \int_{G_\delta} \zeta\, dy. \tag{4.45}$$

Thus, we finally obtain the crucial inequality

$$\begin{aligned}&\int_{G_\delta} \dot{\hat{v}}_T \zeta\, dy + 1/2 \cdot \int_{G_\delta} \tilde{v}\mathscr{C}^2 \cdot \hat{v}_T^{-1} \cdot \zeta\, dy \\ &\quad + \int_{G_\delta} \tilde{v}^{-1} g^{ij} D_j \hat{v}_T \cdot D_i \zeta\, dy - \int_{G_\delta} \tilde{v}^{-1} g^{ij} D_i \hat{v}_T \cdot D_j \hat{v}_T \cdot \hat{v}_T^{-1} \cdot \zeta\, dy \\ &\leqslant c \cdot \int_{G_\delta} \zeta\, dy + c \cdot \int_{G_\delta} \chi^{1/2} (g^{ij} D_i \zeta D_j \zeta)^{1/2}\, dy\end{aligned} \tag{4.46}$$

(here we also used (4.41)).

This is the fundamental inequality to start with. As in the proof of the interior gradient bounds we first show

4.1. Lemma. *Let $\tilde{u}$ be a solution of* (4.18). *Then, we have the estimate*

$$\| \hat{v}_T \|_{L^p(\tilde{Q}_{\delta/2, T_0})} \leqq \text{const} \tag{4.47}$$

for any p, $1 \leqq p < \infty$, where the constant depends on a, p, δ, T_0, L, M, N, and is independent of a if $\beta = \beta(\tau)$.

Proof. *For* $p = 1$ (4.47) is valid due to the fact that

$$\tilde{v} \in L^{\infty}(0, T_0; L^1(G_\delta)) \tag{4.48}$$

in view of our assumptions (cf. Lemma 1.3).

For $2 \leqslant p < \infty$ insert $\zeta = \max(\hat{v}_T - k, 0)^{p-1} \cdot \eta^2 = w^{p-1} \cdot \eta^2$ in (4.46), where k is a positive constant satisfying

$$k \geqslant k_0 = \sup_{G_\delta} | D\tilde{u}_0 | + \sup_{\Gamma \times (0, T_0)} | D_\sigma \tilde{\varphi} | + 1, \tag{4.49}$$

and where η, $0 \leqslant \eta \leqslant 1$, is a cut-off function with support in U_δ such that η is equal to 1 in $U_{\delta/2}$. Then, integrating over $\tilde{Q}_{\delta,t}$ we obtain

$$\begin{aligned}
&\int_{G_\delta} | w(t)|^p \eta^2 \, dy + \int_0^t \int_{G_\delta} \tilde{v}\mathscr{C}^2 \cdot \hat{v}_T^{-1} w^{p-1} \cdot \eta^2 \, dy \, d\tau \\
&\quad + k \cdot \int_0^t \int_{G_\delta} \tilde{v}^{-1} g^{ij} D_i w D_j w w^{p-2} \cdot \eta^2 \hat{v}_T^{-1} \, dy \, d\tau \\
&\quad \leqslant c \cdot \int_0^t \int_{G_\delta} w^{p-1} \eta^2 \, dy \, d\tau + c \cdot \int_0^t \int_{G_\delta} \hat{v}_T{}^2 \cdot w^{p-2} \eta^2 \chi^{1/2} \, dy \, d\tau \\
&\quad + \int_0^t \int_{G_\delta} (| D\eta | + | D\eta |^2) \, w^{p-1} \chi^{1/2} \, dy \, d\tau,
\end{aligned} \tag{4.50}$$

where we used the following estimates

$$\hat{v}^{-1} g^{ij} D_j \hat{v}_T D_i \eta \eta w^{p-1} \leqslant \epsilon \cdot \tilde{v}\mathscr{C}^2 \hat{v}_T^{-1} w^{p-1} \eta^2 + c_\epsilon | D\eta |^2 \, w^{p-1} \chi^{1/2} \tag{4.51}$$

and

$$\begin{aligned}
&\chi^{1/2} (g^{ij} D_i \zeta D_j \zeta)^{1/2} \\
&\quad \leqslant \chi^{1/2} (g^{ij} D_i w D_j w)^{1/2} w^{p-2} (p - 1) \, \eta^2 \\
&\quad + \chi^{1/2} 2 (g^{ij} D_i \eta D_j \eta)^{1/2} \eta w^{p-1} \leqslant \epsilon \cdot \tilde{v}^{-1} g^{ij} D_i w D_j w \cdot \\
&\quad \cdot w^{p-2} \eta^2 \hat{v}_T^{-1} + c_\epsilon \hat{v}_T{}^2 w^{p-2} \eta^2 \chi^{1/2} + c \cdot | D\eta | \, w^{p-1} \chi^{1/2}.
\end{aligned} \tag{4.52}$$

Using Gronwall's lemma we can now draw from (4.50) the following conclusion:

Suppose

$$\int_0^{T_0} \int_{G'} | \hat{v}_T |^{p-1} \, dy \, d\tau \leqslant c(G') \tag{4.53}$$

$\forall G' \Subset G_\delta$, and for $2 \leqslant p < \infty$, then

$$\int_0^{T_0} \int_{G''} | \hat{v}_T |^p \, dy \, d\tau \leqslant c(G'', G') \tag{4.54}$$

$\forall G'' \Subset G'$, which proves the lemma.

We are now ready to prove

4.2. THEOREM. *Let $\tilde{u}$ be a solution of* (4.18). *Then*

$$\| \hat{v}_T \|_{L^\infty(\tilde{Q}_{\delta/4,T_0})} \leqslant \text{const} \cdot \| \tilde{v} \|^{1/2}_{L^\infty(\tilde{Q}_{\delta,T_0})} \tag{4.55}$$

where the constant depends on δ, T_0, M, N, and a.

4.3. *Remark.* We note that the constants in Lemma 4.1 and in the preceding theorem are independent of a, if $\beta = \beta(\tau)$.

Proof of Theorem 4.2. For the proof we need some kind of Sobolev inequality

4.4. LEMMA. *For each non-negative function $f \in C_c^{0,1}(U_\delta)$ we have*

$$\begin{aligned} &\left(\int_{G_\delta} f^{2\alpha} \cdot \chi^{2\alpha-1} \cdot \chi\tilde{v}\, dy\right)^{1/\alpha} \\ &\quad\leqslant c \cdot \int_{G_\delta} f^2\chi\tilde{v}\, dy + c \cdot \int_{G_\delta} \chi^{1/2}(g^{ij}D_ifD_jf)^{1/2} f\chi^{1/2}\tilde{v}\, dy \\ &\qquad + c \cdot \int_{G_\delta} \mathscr{C} f^2 \chi^{1/2}\tilde{v}\, dy, \end{aligned} \tag{4.56}$$

where $\alpha = n/n - 1$, and where c depends on n, Λ, and M.

Lemma 4.4 is proved in [14; cf. the formula following (2.16)] taking the boundedness of $\dot{\tilde{u}}$ into account.

Applying (4.56) with $w\eta\hat{v}_T^{-1/2\alpha}$ in place of f, noting that $1 < \alpha \leqslant 2$, we deduce

$$\begin{aligned} &\left(\int_{G_\delta} w^{2\alpha}\eta^{2\alpha}\chi^{1/2}\, dy\right)^{1/\alpha} \\ &\quad\leqslant \epsilon \int_{G_\delta} \tilde{v}^{-1}g^{ij}D_iwD_jw\eta^2\hat{v}_T^{-1}\, dy + \epsilon \cdot \int_{G_\delta} \tilde{v}\mathscr{C}^2 w\eta^2\hat{v}_T^{-1}\, dy \\ &\qquad + c_\epsilon \int_{G_\delta} w^2 \cdot \hat{v}_T{}^4(|\eta|^2 + |D\eta|^2) \cdot \chi^{1/2}\, dy. \end{aligned} \tag{4.57}$$

In view of (4.50) (use this inequality with $p = 2$) we then obtain

$$\begin{aligned} &\sup_{0\leqslant\tau\leqslant t} \int_{G_\delta} w^2\eta^2\, dy + \int_0^t \left(\int_{G_\delta} w^{2\alpha}\eta^{2\alpha} \cdot \chi^{2\alpha-1}\chi^{1/2}\, dy\right)^{1/2} d\tau \\ &\quad\leqslant c \cdot \int_0^t \int_{G_\delta} w\eta^2\, dy\, d\tau + c \cdot \int_0^t \int_{B(k)} (1 + w^2)\, \hat{v}_T{}^4 \\ &\qquad\qquad (|\eta|^2 + |D\eta| + |D\eta|^2)\, \chi^{1/2}\, dy\, d\tau, \end{aligned} \tag{4.58}$$

where

$$B(k) = \{x \in G_\delta : \hat{v}_T > k\}.$$

Now, let us estimate the left-hand side of this inequality from below. Let us introduce the measure $d\mu = \chi^{1/2}\, dy$ and the abbreviation $\zeta = w \cdot \eta$; the left-hand side of this inequality looks like

$$\sup_{0 \leqslant \tau \leqslant t} \int_{G_\delta} \zeta^2 \cdot \chi^{-1/2}\, d\mu + \int_0^t \left(\int_{G_\delta} \zeta^{2\alpha} \cdot \chi^{2\alpha-1}\, d\mu \right)^{1/\alpha} d\tau. \tag{4.59}$$

On the other hand, we have with $q = (3n+2)/(3n+1)$

$$\zeta^{2q} = \zeta^{4(n+1)/(3n+1)} \cdot \chi^{-(n+1)/(3n+1)} \cdot \zeta^{2n/(3n+1)} \cdot \chi^{(n+1)/(3n+1)} \tag{4.60}$$

from which we deduce

$$\int_{G_\delta} \zeta^{2q}\, d\mu \leqq \left(\int_{G_\delta} \zeta^2 \cdot \chi^{-1/2}\, d\mu \right)^{2(n+1)/(3n+1)} \cdot \left(\int_{G_\delta} \zeta^{2\alpha} \chi^{2\alpha-1}\, d\mu \right)^{(n-1)/(3n+1)}, \tag{4.61}$$

where we used the Hölder inequality.

Moreover, noting that

$$\frac{n-1}{3n+1} = 1/\alpha \cdot \frac{n}{3n+1} \tag{4.62}$$

and that

$$q = \frac{2(n+1)}{3n+1} + \frac{n}{3n+1} \tag{4.63}$$

we derive from (4.61) and (4.58)

$$\left(\int_0^t \int_{G_\delta} w^{2q} \eta^{2q}\, d\mu\, d\tau \right)^{1/q}$$
$$\leqslant c \sup_{\tilde{Q}_{\delta,T_0}} \tilde{v} \cdot \int_0^t \int_{B(k)} \hat{v}_T{}^6 (|\eta|^2 + |D\eta| + |D\eta|^2)\, d\mu\, d\tau. \tag{4.64}$$

The factor $\sup_{\tilde{Q}_{\delta,T_0}} \tilde{v}$ is due to the first integral on the right-hand side of (4.58) where the measure "dy" has to be replaced by "$d\mu$".

We are now in the same situation as in the proof of Lemma 3.3 (cf. formula (3.29)), and we deduce

$$\sup_{\tilde{Q}_{\delta/4,T_0}} \hat{v}_T \leqslant k_0 + c \cdot (\sup_{\tilde{Q}_{\delta,T_0}} \tilde{v})^{1/2}. \tag{4.65}$$

The Theorem is therefore proved.

Since Ω is compact we conclude that the tangential derivatives of u are bounded by the normal derivative; hence the whole gradient of u is bounded on $\partial\Omega \times (0, T_0)$ in view of the assumption (4.2). To deduce a global bound for the gradient of u we apply the interior gradient estimates up to the boundary:

we have only to replace the Sobolev imbedding theorem (Lemma 3.6), valid for solutions of the perturbed equation, which is only applicable in convex domains, by the usual Sobolev inequality valid in arbitrary domains (cf. [11]), for details we refer to [7; Appendix] where this has been proved for stationary surfaces of prescribed mean curvature.

The further results in [4] are also valid in the present situation, mutatis mutandis. We shall summarize some of the possible conclusion in the following theorem.

4.5. THEOREM. *Under the assumption of Theorem* 4.2 *the evolutionary boundary value problem has a unique solution* $u \in H^{1,\infty}(Q_T)$ *for any finite* T. *Moreover, if* φ *is the trace of a function* $\varphi \in L^2(0, T; H^{2,2}(\Omega))$, *then*

$$u \in L^2(0, T; H^{2,2}(\Omega)). \tag{4.66}$$

5. ATTAINIBILITY OF STATIONARY SOLUTIONS

Let $\bar{u} \in C^{0,1}(\Omega) \cap L^\infty(\Omega)$ be a solution of the stationary variational problem

$$\Phi(\bar{u}) + \int_{\partial\Omega} j(x, \bar{u} - \varphi)\, dH_{n-1} \leqslant \Phi(v) + \int_{\partial\Omega} j(x, v - \varphi)\, d\mathscr{H}_{n-1} \tag{5.1}$$
$$\forall v \in BV(\Omega),$$

where we have used the notation of Section 2. We note that $j = j(x, \tau)$ is non-expansive and convex in $\tau \cdot \varphi = \varphi(x)$ is bounded. Sufficient conditions for the existence of a bounded solution $\bar{u}$ can be found in [5, 7]. $\bar{u}$ is uniquely determined up to an additive constant. Ω is assumed to satisfy an internal sphere condition.

Let $u = u(x, t)$ be a solution of the corresponding evolutionary problem (2.37) with initial value u_0. Evidently, $\bar{u}$ is also a time-independent solution of (2.37) with initial value $\bar{u}$. Then, by a weak maximum principle (cf. Lemma 1.4 for motivation and [5; Lemma 3.3] for justification) we conclude

$$| u - \bar{u} | \leqq \sup_{\Omega} | \bar{u} - u_0 | \qquad \text{in } Q_t, \tag{5.2}$$

for arbitrary $0 \leqq t < \infty$. u is therefore uniformly bounded in Q_∞, and thus satisfies the estimate

$$\int_0^\infty \int_\Omega | \dot{u} |^2\, dx\, dt + \sup_{0 \leqslant t \leqslant \infty} \int_\Omega (1 + | Du |^2)^{1/2}\, dx \leqq \text{const}, \tag{5.3}$$

cf. Lemma 1.3.

We therefore conclude that there exists a sequence (t_k), $t_k \to \infty$, such that

$$\int_\Omega | \dot{u}(x, t_k)|^2\, dx \to 0. \tag{5.4}$$

From (2.37), (5.3) and from the lower semicontinuity of the convex functional $\Phi + \int_{\partial\Omega} j(x, \cdot - \varphi)\, d\mathscr{H}_{n-1}$ we conclude that a subsequence of the $u(\cdot, t_k)$'s converges to a solution $u^* \in BV(\Omega) \cap L^\infty(\Omega)$ of the variational problem (5.1); hence u^* is equal to $\bar{u}$ modulo an additive constant (cf. [8]).

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