

THE INVERSE MEAN CURVATURE FLOW IN ARW SPACES—TRANSITION FROM BIG CRUNCH TO BIG BANG

CLAUS GERHARDT

ABSTRACT. We consider spacetimes N satisfying some structural conditions, which are still fairly general, and prove convergence results for the leaves of an inverse mean curvature flow.

Moreover, we define a new spacetime \hat{N} by switching the light cone and using reflection to define a new time function, such that the two spacetimes N and \hat{N} can be pasted together to yield a smooth manifold having a metric singularity, which, when viewed from the region N is a big crunch, and when viewed from \hat{N} is a big bang.

The inverse mean curvature flows in N resp. \hat{N} correspond to each other via reflection. Furthermore, the properly rescaled flow in N has a natural smooth extension of class C^3 across the singularity into \hat{N} . With respect to this natural, globally defined diffeomorphism we speak of a transition from big crunch to big bang.

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0. INTRODUCTION

In [3] we considered the inverse mean curvature flow (IMCF) in cosmological spacetimes having a future mean curvature barrier and showed that the IMCF exists for all time and runs directly into the future singularity, if and only if N satisfies a *strong volume decay condition*.

Apart from the fact that the leaves run straight into the future singularity no further convergence results could be derived due to the weak assumptions on the spacetime.

In the present paper we consider spacetimes N satisfying some structural conditions, which are still fairly general, and prove convergence results for the leaves of the IMCF.

Moreover, we define a new spacetime \hat{N} by switching the light cone and using reflection to define a new time function, such that the two spacetimes N and \hat{N} can be pasted together to yield a smooth manifold having a metric singularity, which, when viewed from the region N is a big crunch, and when viewed from \hat{N} is a big bang.

The inverse mean curvature flows in N resp. \hat{N} correspond to each other via reflection. Furthermore, the properly rescaled flow in N has a natural smooth extension of class C^3 across the singularity into \hat{N} . With respect to this natural diffeomorphism we speak of a transition from big crunch to big bang.

0.1. Definition. A cosmological spacetime N , $\dim N = n + 1$, is said to be *asymptotically Robertson-Walker* (ARW) with respect to the future, if a future end of N , N_+ , can be written as a product $N_+ = [a, b) \times \mathcal{S}_0$, where \mathcal{S}_0 is a compact Riemannian space, and there exists a future directed time function $\tau = x^0$ such that the metric in N_+ can be written as

$$(0.1) \quad d\tilde{s}^2 = e^{2\tilde{\psi}} \{ -(dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j \},$$

where \mathcal{S}_0 corresponds to $x^0 = a$, $\tilde{\psi}$ is of the form

$$(0.2) \quad \tilde{\psi}(x^0, x) = f(x^0) + \psi(x^0, x),$$

and we assume that there exists a positive constant c_0 and a smooth Riemannian metric $\bar{\sigma}_{ij}$ on \mathcal{S}_0 such that

$$(0.3) \quad \lim_{\tau \rightarrow b} e^{\psi} = c_0 \quad \wedge \quad \lim_{\tau \rightarrow b} \sigma_{ij}(\tau, x) = \bar{\sigma}_{ij}(x),$$

and

$$(0.4) \quad \lim_{\tau \rightarrow b} f(\tau) = -\infty.$$

Without loss of generality we shall assume $c_0 = 1$. Then N is ARW with respect to the future, if the metric is close to the Robertson-Walker metric

$$(0.5) \quad d\bar{s}^2 = e^{2f} \{ -dx^{02} + \bar{\sigma}_{ij}(x) dx^i dx^j \}$$

near the singularity $\tau = b$. By *close* we mean that the derivatives of arbitrary order with respect to space and time of the conformal metric $e^{-2f}\check{g}_{\alpha\beta}$ in (0.1) should converge to the corresponding derivatives of the conformal limit metric in (0.5) when x^0 tends to b . We emphasize that in our terminology Robertson-Walker metric does not imply that $(\bar{\sigma}_{ij})$ is a metric of constant curvature, it is only the spatial metric of a warped product.

We assume, furthermore, that f satisfies the following five conditions

$$(0.6) \quad -f' > 0,$$

there exists $\omega \in \mathbb{R}$ such that

$$(0.7) \quad n + \omega - 2 > 0 \quad \wedge \quad \lim_{\tau \rightarrow b} |f'|^2 e^{(n+\omega-2)f} = m > 0.$$

Set $\tilde{\gamma} = \frac{1}{2}(n + \omega - 2)$, then there exists the limit

$$(0.8) \quad \lim_{\tau \rightarrow b} (f'' + \tilde{\gamma}|f'|^2)$$

and

$$(0.9) \quad |D_\tau^m (f'' + \tilde{\gamma}|f'|^2)| \leq c_m |f'|^m \quad \forall m \geq 1,$$

as well as

$$(0.10) \quad |D_\tau^m f| \leq c_m |f'|^m \quad \forall m \geq 1.$$

We call N a *normalized* ARW spacetime, if

$$(0.11) \quad \int_{S_0} \sqrt{\det \bar{\sigma}_{ij}} = |S^n|.$$

0.2. Remark. (i) If these assumptions are satisfied, then we shall show that the range of τ is finite, hence, we may—and shall—assume w.l.o.g. that $b = 0$, i.e.,

$$(0.12) \quad a < \tau < 0.$$

(ii) Any ARW spacetime can be normalized as one easily checks. For normalized ARW spaces the constant m in (0.7) is defined uniquely and can be identified with the mass of N , cf. [4].

(iii) In view of the assumptions on f the mean curvature of the coordinate slices $M_\tau = \{x^0 = \tau\}$ tends to ∞ , if τ goes to zero.

(iv) ARW spaces satisfy a strong volume decay condition, cf. [3, Definition 0.1].

(v) Similarly one can define N to be ARW with respect to the past. In this case the singularity would lie in the past, correspond to $\tau = 0$, and the mean curvature of the coordinate slices would tend to $-\infty$.

We assume that N satisfies the timelike convergence condition. Consider the future end N_+ of N and let $M_0 \subset N_+$ be a spacelike hypersurface with positive mean curvature $\check{H}|_{M_0} > 0$ with respect to the past directed normal vector $\check{\nu}$ —we shall explain in Section 2 why we use the symbols \check{H} and $\check{\nu}$ and not the usual ones H and ν . Then, as we have proved in [3], the inverse mean curvature flow

$$(0.13) \quad \dot{x} = -\check{H}^{-1}\check{\nu}$$

with initial hypersurface M_0 exists for all time, is smooth, and runs straight into the future singularity.

If we express the flow hypersurfaces $M(t)$ as graphs over \mathcal{S}_0

$$(0.14) \quad M(t) = \text{graph } u(t, \cdot),$$

then our main results can be formulated as

0.3. Theorem. (i) *Let N satisfy the above assumptions, then the range of the time function x^0 is finite, i.e., we may assume that $b = 0$. Set*

$$(0.15) \quad \tilde{u} = ue^{\gamma t},$$

where $\gamma = \frac{1}{n}\tilde{\gamma}$, then there are positive constants c_1, c_2 such that

$$(0.16) \quad -c_2 \leq \tilde{u} \leq -c_1 < 0,$$

and \tilde{u} converges in $C^\infty(\mathcal{S}_0)$ to a smooth function, if t goes to infinity. We shall also denote the limit function by \tilde{u} .

(ii) *Let \check{g}_{ij} be the induced metric of the leaves $M(t)$, then the rescaled metric*

$$(0.17) \quad e^{\frac{2}{n}t}\check{g}_{ij}$$

converges in $C^\infty(\mathcal{S}_0)$ to

$$(0.18) \quad (\tilde{\gamma}m)^{\frac{1}{\tilde{\gamma}}}(-\tilde{u})^{\frac{2}{\tilde{\gamma}}}\bar{\sigma}_{ij}.$$

(iii) *The leaves $M(t)$ get more umbilical, if t tends to infinity, namely, there holds*

$$(0.19) \quad \check{H}^{-1}|\check{h}_i^j - \frac{1}{n}\check{H}\delta_i^j| \leq ce^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate

$$(0.20) \quad |\check{h}_i^j - \frac{1}{n}\check{H}\delta_i^j| \leq ce^{-\frac{1}{2n}(n+\omega-4)t}.$$

For a description of the results related to the transition from big crunch to big bang we refer to Section 8.

1. NOTATIONS AND DEFINITIONS

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces M in a $(n+1)$ -dimensional Lorentzian manifold N . Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc.. Greek indices range from 0 to n and Latin from 1 to n ; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i) . Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e., for a function u in N , (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$(1.1) \quad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a *spacelike* hypersurface, i.e., the induced metric is Riemannian, with a differentiable normal ν which is time-like.

In local coordinates, (x^α) and (ξ^i) , the geometric quantities of the space-like hypersurface M are connected through the following equations

$$(1.2) \quad x_{ij}^\alpha = h_{ij} \nu^\alpha$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$(1.3) \quad x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma.$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to ν .

The second equation is the *Weingarten equation*

$$(1.4) \quad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$(1.5) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauß equation*

$$(1.6) \quad R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. N is then a topological product $I \times \mathcal{S}_0$, where I is an open interval, \mathcal{S}_0 is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α) , such that the metric in N has the form

$$(1.7) \quad d\bar{s}_N^2 = e^{2\psi} \{-dx^{0^2} + \sigma_{ij}(x^0, x) dx^i dx^j\},$$

where σ_{ij} is a Riemannian metric, ψ a function on N , and x an abbreviation for the spacelike components (x^i) . We also assume that the coordinate system is

future oriented, i.e., the time coordinate x^0 increases on future directed curves. Hence, the *contravariant* time-like vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \text{graph } u|_{\mathcal{S}_0}$ be a space-like hypersurface

$$(1.8) \quad M = \{ (x^0, x) : x^0 = u(x), x \in \mathcal{S}_0 \},$$

then the induced metric has the form

$$(1.9) \quad g_{ij} = e^{2\psi} \{-u_i u_j + \sigma_{ij}\}$$

where σ_{ij} is evaluated at (u, x) , and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$(1.10) \quad g^{ij} = e^{-2\psi} \left\{ \sigma^{ij} + \frac{u^i u^j}{v} \right\},$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$(1.11) \quad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2. \end{aligned}$$

Hence, graph u is space-like if and only if $|Du| < 1$.

The covariant form of a normal vector of a graph looks like

$$(1.12) \quad (\nu_\alpha) = \pm v^{-1} e^\psi (1, -u_i).$$

and the contravariant version is

$$(1.13) \quad (\nu^\alpha) = \mp v^{-1} e^{-\psi} (1, u^i).$$

Thus, we have

1.1. Remark. Let M be space-like graph in a future oriented coordinate system. Then the contravariant future directed normal vector has the form

$$(1.14) \quad (\nu^\alpha) = v^{-1} e^{-\psi} (1, u^i)$$

and the past directed

$$(1.15) \quad (\nu^\alpha) = -v^{-1} e^{-\psi} (1, u^i).$$

In the Gauß formula (1.2) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal for reasons that we have explained in [2, Section 2].

Look at the component $\alpha = 0$ in (1.2) and obtain in view of (1.15)

$$(1.16) \quad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{I}_{00}^0 u_i u_j - \bar{I}_{0j}^0 u_i - \bar{I}_{0i}^0 u_j - \bar{I}_{ij}^0.$$

Here, the covariant derivatives are taken with respect to the induced metric of M , and

$$(1.17) \quad -\bar{I}_{ij}^0 = e^{-\psi} \bar{h}_{ij},$$

where (\bar{h}_{ij}) is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$(1.18) \quad \bar{h}_{ij}e^{-\psi} = -\frac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij},$$

where the dot indicates differentiation with respect to x^0 .

2. THE EVOLUTION PROBLEM

When proving the convergence results for the inverse mean curvature flow, we shall consider the flow hypersurfaces to be embedded in N equipped with the conformal metric

$$(2.1) \quad d\bar{s}^2 = -(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j.$$

Though, formally, we have a different ambient space we still denote it by the same symbol N and distinguish only the metrics $\check{g}_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$

$$(2.2) \quad \check{g}_{\alpha\beta} = e^{2\check{\psi}}\bar{g}_{\alpha\beta}$$

and the corresponding geometric quantities of the hypersurfaces $\check{h}_{ij}, \check{g}_{ij}, \check{\nu}$ resp. h_{ij}, g_{ij}, ν , etc., i.e., the notations of the preceding section now apply to the case when N is equipped with the metric in (2.1).

The second fundamental forms \check{h}_i^j and h_i^j are related by

$$(2.3) \quad e^{\check{\psi}}\check{h}_i^j = h_i^j + \check{\psi}_\alpha \nu^\alpha \delta_i^j$$

and, if we define F by

$$(2.4) \quad F = e^{\check{\psi}}\check{H},$$

then

$$(2.5) \quad F = H - n\tilde{v}f' + n\psi_\alpha \nu^\alpha,$$

where

$$(2.6) \quad \tilde{v} = v^{-1},$$

and the evolution equation can be written as

$$(2.7) \quad \dot{x} = -F^{-1}\nu,$$

since

$$(2.8) \quad \check{\nu} = e^{-\check{\psi}}\nu.$$

The flow exists for all time and is smooth.

Next, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives. We refer to [2] for more general results and to [1, Section 3], where proofs are given in a Riemannian setting, but these proofs are also valid in a Lorentzian environment.

2.1. **Lemma.** *The metric, the normal vector, and the second fundamental form of $M(t)$ satisfy the evolution equations*

$$(2.9) \quad \dot{g}_{ij} = -2F^{-1}h_{ij},$$

$$(2.10) \quad \dot{\nu} = \nabla_M(-F^{-1}) = g^{ij}(-F^{-1})_i x_j,$$

and

$$(2.11) \quad \dot{h}_i^j = (-F^{-1})_i^j + F^{-1}h_i^k h_k^j + F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj}$$

$$(2.12) \quad \dot{h}_{ij} = (-F^{-1})_{ij} - F^{-1}h_i^k h_{kj} + F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta.$$

Since the initial hypersurface is a graph over \mathcal{S}_0 , we can write

$$(2.13) \quad M(t) = \text{graph } u(t)|_{\mathcal{S}_0} \quad \forall t \in I,$$

where u is defined in the cylinder $\mathbb{R}_+ \times \mathcal{S}_0$. We then deduce from (2.7), looking at the component $\alpha = 0$, that u satisfies a parabolic equation of the form

$$(2.14) \quad \dot{u} = \frac{\tilde{v}}{F},$$

where we use the notations in Section 1, and where we emphasize that the time derivative is a total derivative, i.e.

$$(2.15) \quad \dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i.$$

Since the past directed normal can be expressed as

$$(2.16) \quad (\nu^\alpha) = -e^{-\psi} v^{-1}(1, u^i),$$

we conclude from (2.14)

$$(2.17) \quad \frac{\partial u}{\partial t} = \frac{v}{F}.$$

Sometimes, we need a Riemannian reference metric, e.g., if we want to estimate tensors. Since the Lorentzian metric can be expressed as

$$(2.18) \quad \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = -(dx^0)^2 + \sigma_{ij} dx^i dx^j,$$

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

$$(2.19) \quad \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = (dx^0)^2 + \sigma_{ij} dx^i dx^j$$

and we abbreviate the corresponding norm of a vectorfield η by

$$(2.20) \quad \|\eta\| = (\tilde{g}_{\alpha\beta} \eta^\alpha \eta^\beta)^{1/2},$$

with similar notations for higher order tensors.

3. LOWER ORDER ESTIMATES

We first draw a few immediate conclusions from our assumptions on f .

3.1. Lemma. *Let $f \in C^2([a, b])$ satisfy the conditions*

$$(3.1) \quad \lim_{\tau \rightarrow b} f(\tau) = -\infty$$

and

$$(3.2) \quad \lim_{\tau \rightarrow b} |f'|^2 e^{2\tilde{\gamma}f} = m,$$

where $\tilde{\gamma}, m$ are positive, then b is finite.

Proof. From (3.2) we deduce that f' tends to $-\infty$ and

$$(3.3) \quad \lim(-f'e^{\tilde{\gamma}f}) = \sqrt{m}.$$

Moreover,

$$(3.4) \quad e^{\tilde{\gamma}\tau} - e^{\tilde{\gamma}\tau_0} = \int_{\tau_0}^{\tau} \tilde{\gamma}f'e^{\tilde{\gamma}f} \leq -\tilde{\gamma}\frac{\sqrt{m}}{2}(\tau - \tau_0),$$

if τ_0 is close to b in the topology of $\overline{\mathbb{R}}$ and $\tau > \tau_0$. Hence b has to be finite. \square

3.2. Corollary. *We may—and shall—therefore assume that $b = 0$, i.e., the time interval I is given by $I = [a, 0)$.*

A simple application of de L'Hospital's rule then yields

$$(3.5) \quad \lim_{\tau \rightarrow 0} \frac{e^{\tilde{\gamma}f}}{\tau} = -\tilde{\gamma}\sqrt{m}$$

From this relation and (0.8) we conclude

3.3. Lemma. *There holds*

$$(3.6) \quad f'e^{\tilde{\gamma}f} + \sqrt{m} \sim c\tau^2,$$

where c is a constant, and where the relation

$$(3.7) \quad \varphi \sim c\tau^2$$

means

$$(3.8) \quad \lim_{\tau \rightarrow 0} \frac{\varphi(\tau)}{\tau^2} = c.$$

Proof. Applying de L'Hospital's rule we get

$$(3.9) \quad \lim_{\tau \rightarrow 0} \frac{f'e^{\tilde{\gamma}f} + \sqrt{m}}{\frac{1}{2}\tau^2} = \lim_{\tau \rightarrow 0} \frac{(f'' + \tilde{\gamma}|f'|^2)e^{\tilde{\gamma}f}}{\tau} = -c\tilde{\gamma}\sqrt{m}.$$

\square

3.4. Lemma. *The asymptotic relation*

$$(3.10) \quad \tilde{\gamma}f'\tau - 1 \sim c\tau^2$$

is valid.

Proof. The relation (3.6) yields

$$(3.11) \quad \tilde{\gamma}f'\tau e^{\tilde{\gamma}f} + \sqrt{m}\tilde{\gamma}\tau \sim c_1\tau^3,$$

or equivalently,

$$(3.12) \quad (\tilde{\gamma}f'\tau - 1)e^{\tilde{\gamma}f} + \sqrt{m}\tilde{\gamma}\tau + e^{\tilde{\gamma}f} \sim c_1\tau^3.$$

Dividing by τ^3 and applying de L'Hospital's rule we infer

$$(3.13) \quad \lim \frac{\tilde{\gamma}f'\tau - 1}{\tau^2} \cdot \lim \frac{e^{\tilde{\gamma}f}}{\tau} + \lim \frac{\sqrt{m}\tilde{\gamma} + \tilde{\gamma}f'e^{\tilde{\gamma}f}}{3\tau^2} = c_1,$$

hence the result in view of (3.5) and (3.6). \square

After these preliminary results we now want to prove that there are positive constants c_1, c_2 such that

$$(3.14) \quad -c_1 \leq \tilde{u} \equiv ue^{\gamma t} \leq -c_2 < 0 \quad \forall t \in \mathbb{R}_+,$$

where u is the solution of the scalar version of the inverse mean curvature flow, i.e., u is the solution of equation (2.14).

We shall proceed in two steps, first we shall derive

$$(3.15) \quad |ue^{\lambda t}| \leq c(\lambda) \quad \forall 0 < \lambda < \gamma,$$

and then the final result in the limiting case $\lambda = \gamma$.

This procedure will also be typical for higher order estimates in the next sections.

3.5. Lemma. *For any $0 < \lambda < \gamma$, there exists a constant $c(\lambda)$ such that the estimate (3.15) is valid.*

Proof. Define $\varphi = \varphi(t)$ by

$$(3.16) \quad \varphi(t) = \inf_{x \in \mathcal{S}_0} u(t, x).$$

Then φ is Lipschitz continuous and

$$(3.17) \quad \dot{\varphi}(t) = \frac{\partial u}{\partial t}(t, x_t) \quad \text{for a.e. } t,$$

where $x_t \in \mathcal{S}_0$ is such that the infimum of $u(t, \cdot)$ is attained. This is a well-known result, for a simple proof see e.g., [3, Lemma 3.2].

Let

$$(3.18) \quad w = \log(-\varphi) + \lambda t,$$

then, for a.e. t , we have

$$(3.19) \quad \dot{w} = \varphi^{-1}\dot{\varphi} + \lambda = u^{-1}\frac{\partial u}{\partial t} + \lambda,$$

where u is evaluated at (t, x_t) . In x_t $u(t, \cdot)$ attains its infimum, i.e., $Du = 0$ and $-\Delta u \leq 0$.

From the parabolic equation (2.17), we obtain in x_t

$$(3.20) \quad \frac{\partial u}{\partial t} = \frac{1}{F} = \frac{1}{H - nf' - n\dot{\psi}}.$$

The mean curvature H can be expressed as

$$(3.21) \quad H = -\Delta u + \bar{H} = -\Delta u + \sigma^{ij} \bar{h}_{ij} = -\Delta u - \frac{1}{2} \sigma^{ij} \dot{\sigma}_{ij}.$$

Thus we deduce

$$(3.22) \quad \frac{\partial u}{\partial t} \geq \frac{1}{-nf' - n\dot{\psi} - \frac{1}{2} \sigma^{ij} \dot{\sigma}_{ij}}$$

and

$$(3.23) \quad \begin{aligned} \dot{w} &\leq \frac{1}{-nf'u - (n\dot{\psi} - \frac{1}{2} \sigma^{ij} \dot{\sigma}_{ij})u} + \lambda \\ &= \frac{1 - nf'u\lambda - (n\dot{\psi} - \frac{1}{2} \sigma^{ij} \dot{\sigma}_{ij})\lambda u}{-nf'u - (n\dot{\psi} - \frac{1}{2} \sigma^{ij} \dot{\sigma}_{ij})u}. \end{aligned}$$

Now, we observe that the argument of f' is u and

$$(3.24) \quad \lim_{t \rightarrow \infty} \inf_{x \in \mathcal{S}_0} u(t, x) = 0,$$

cf. [3, Lemma 3.1]. Hence

$$(3.25) \quad \lim_{t \rightarrow \infty} f'u = \tilde{\gamma}^{-1},$$

in view of Lemma 3.4, and we infer that the right-hand side of inequality (3.23) is negative for large t , $t \geq t_\lambda$, and therefore

$$(3.26) \quad w \leq w(t_\lambda) \quad \forall t \geq t_\lambda,$$

or equivalently,

$$(3.27) \quad -ue^{\lambda t} \leq c(\lambda) \quad \forall t \in \mathbb{R}_+. \quad \square$$

3.6. Theorem. *Let u be a solution of the evolution equation (2.14), where f satisfies the assumptions (0.7) and (0.8), then there are positive constants c_1, c_2 such that*

$$(3.28) \quad -c_1 \leq \tilde{u} \equiv ue^{\gamma t} \leq -c_2 < 0.$$

Proof. We only prove the estimate from above. Define

$$(3.29) \quad \varphi(t) = \sup_{x \in \mathcal{S}_0} u(t, x)$$

and

$$(3.30) \quad w = \log(-\varphi) + \gamma t.$$

Arguing similar as in the proof of the previous lemma, we obtain for a.e. t

$$(3.31) \quad \dot{w} \geq \frac{1 - nf'u\gamma - (n\dot{\psi} - \frac{1}{2}\sigma^{ij}\dot{\sigma}_{ij})\gamma u}{-nf'u - (n\dot{\psi} - \frac{1}{2}\sigma^{ij}\dot{\sigma}_{ij})u}.$$

Since $\tilde{\gamma} = n\gamma$, we deduce from Lemma 3.4 that the right-hand side can be estimated from below by cu , i.e.,

$$(3.32) \quad \dot{w} \geq cu \geq -cc_\lambda e^{-\lambda t}$$

for any $0 < \lambda < \gamma$. Hence w is bounded from below, or equivalently,

$$(3.33) \quad \tilde{u} \leq -c_2 < 0. \quad \square$$

3.7. Corollary. *For any $k \in \mathbb{N}^*$ there exists c_k such that*

$$(3.34) \quad |f^{(k)}| \leq c_k e^{k\gamma t},$$

where $f^{(k)}$ is evaluated at u .

Proof. In view of the assumption (0.10) there holds

$$(3.35) \quad |f^{(k)}| \leq c_k |f'|^k = c_k |f'|^k u^k \tilde{u}^{-k} e^{k\gamma t}.$$

Then use Lemma 3.4 and the preceding theorem. \square

4. C^1 -ESTIMATES

We want to prove estimates for \tilde{v} and $\|D\tilde{u}\|$, where we recall that

$$(4.1) \quad \tilde{u} = ue^{\gamma t}.$$

Our final goal is to show that $\|D\tilde{u}\|$ is uniformly bounded, but this estimate has to be deferred to Section 5. At the moment we only prove an exponential decay for any $0 < \lambda < \gamma$, i.e., we shall estimate $\|Du\|e^{\lambda t}$.

The starting point is the evolution equation satisfied by \tilde{v} .

4.1. Lemma (Evolution of \tilde{v}). *Consider the flow (2.7). Then \tilde{v} satisfies the evolution equation*

$$(4.2) \quad \begin{aligned} \dot{\tilde{v}} - F^{-2}\Delta\tilde{v} &= -F^{-2}\|A\|^2\tilde{v} + F^{-2}\bar{R}_{\alpha\beta}\nu^\alpha x_i^\beta u^i \\ &\quad - F^{-2}(2H - nf'\tilde{v} + n\psi_\alpha\nu^\alpha)\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &\quad - F^{-2}(\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma g^{ij} + \eta_{\alpha\beta}x_i^\alpha x_j^\beta h^{ij}) \\ &\quad - F^{-2}(-nf''\|Du\|^2\tilde{v} - nf'\tilde{v}_k u^k + n\psi_{\alpha\beta}\nu^\alpha x_i^\beta u^i + n\psi_\alpha x_k^\alpha h_i^k u^i), \end{aligned}$$

where $\eta = (\eta_\alpha) = (-1, 0, \dots, 0)$ is a covariant unit vectorfield.

Proof. We have

$$(4.3) \quad \tilde{v} = \eta_\alpha \nu^\alpha.$$

Let (ξ^i) be local coordinates for $M(t)$; differentiating \tilde{v} covariantly we deduce

$$(4.4) \quad \tilde{v}_i = \eta_{\alpha\beta} x_i^\beta \nu^\alpha + \eta_\alpha \nu_i^\alpha,$$

and

$$(4.5) \quad \tilde{v}_{ij} = \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha + \eta_{\alpha\beta} \nu_j^\alpha x_i^\beta + \eta_{\alpha\beta} \nu^\alpha \nu^\beta h_{ij} + \eta_{\alpha} \nu_{ij}^\alpha.$$

The time derivative of \tilde{v} is equal to

$$(4.6) \quad \begin{aligned} \dot{\tilde{v}} &= \eta_{\alpha\beta} \nu^\alpha \dot{x}^\beta + \eta_{\alpha} \dot{\nu}^\alpha \\ &= -\eta_{\alpha\beta} \nu^\alpha \nu^\beta F^{-1} + F^{-2} \eta_{\alpha} F^k x_k^\alpha. \end{aligned}$$

From these relations the evolution equation for \tilde{v} follows immediately with the help of the Weingarten and Codazzi equations, the Gauß formula, and the definition of F . \square

4.2. Lemma. *The following estimates are valid*

$$(4.7) \quad |\eta_{\alpha\beta} \nu^\alpha \nu^\beta| \leq c \tilde{v}^2 \|\eta_{\alpha\beta}\|,$$

$$(4.8) \quad |\eta_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_j^\gamma g^{ij}| \leq c \tilde{v}^3 \|\eta_{\alpha\beta\gamma}\|,$$

$$(4.9) \quad |\eta_{\alpha\beta} \nu^\alpha x_k^\beta u^k| \leq c \|\eta_{\alpha\beta}\| \tilde{v}^3,$$

$$(4.10) \quad |\psi_{\alpha} x_k^\alpha h_i^k u^i| \leq c \|D\psi\| \|A\| \tilde{v}^2,$$

$$(4.11) \quad |\eta_{\alpha\beta} x_i^\alpha x_j^\beta h^{ij}| \leq c \|\eta_{\alpha\beta}\| \|A\| \tilde{v}^2,$$

and

$$(4.12) \quad \begin{aligned} |\bar{R}_{\alpha\beta} \nu^\alpha x_k^\beta u^k| &\leq c \tilde{v}^3 |\bar{R}_{0k} \check{u}^k| + c \tilde{v} |\bar{R}_{00}| \|Du\|^2 \\ &\quad + c \tilde{v}^3 |\bar{R}_{ij} \check{u}^i \check{u}^j|, \end{aligned}$$

where

$$(4.13) \quad \check{u}^i = \sigma^{ij} u_j.$$

Proof. Easy exercise.

We can now prove that \tilde{v} is uniformly bounded.

4.3. Lemma. *The quantity \tilde{v} is uniformly bounded*

$$(4.14) \quad \tilde{v} \leq c.$$

Proof. For large T , $0 < T < \infty$, assume that

$$(4.15) \quad \sup_{[0, T]} \sup_{M(t)} \tilde{v} = \tilde{v}(t_0, x_0).$$

Applying the maximum principle we shall deduce that either $\tilde{v} \leq 2$ or that t_0 is a priori bounded

$$(4.16) \quad t_0 \leq T_0.$$

In (t_0, x_0) the left-hand side of equation (4.2) is non-negative, assuming $t_0 \neq 0$. Multiplying the resulting inequality by F^2 and using the estimates in Lemma 4.2 we conclude

$$(4.17) \quad 0 \leq -\|A\|^2 \tilde{v} - n f'' \|Du\|^2 \tilde{v} + c(1 + |f'|) \tilde{v}^3 + c\|A\| \tilde{v}^2.$$

If $\tilde{v} \geq 2$, then

$$(4.18) \quad \|Du\|^2 \geq \epsilon_0 \tilde{v}^2$$

with a positive constant ϵ_0 , and if t_0 would be large, then $-f''$ would be very large; recall that $\lim_{\tau \rightarrow 0} (-f'') = \infty$.

In view of (0.8), $-f''$ is also dominating $|f'|$, hence \tilde{v} is a priori bounded independent of T . \square

Before we can show that $\|Du\|$ decays exponentially, we need the following lemma

4.4. Lemma. *For any $k \in \mathbb{N}$ there exists c_k such that*

$$(4.19) \quad \|\eta_{\alpha\beta}\| \leq c_k |\tau|^k.$$

Corresponding estimates also hold for $\|\eta_{\alpha\beta\gamma}\|$, $\|D\psi\|$, $\|\bar{R}_{\alpha\beta}\eta^\alpha\|$, or more generally, for any tensor that would vanish identically, if it would have been formed with respect to the product metric

$$(4.20) \quad -(dx^0)^2 + \bar{\sigma}_{ij} dx^i dx^j.$$

Proof. We only prove the estimate (4.19) in detail. The remaining claims can easily be deduced with the help of the arguments that will follow; in case of $\|D\psi\|$ we use in addition the assumption that all derivatives of ψ of arbitrary order vanish if τ tends to 0.

Let $(\xi^\alpha), (\chi^\alpha)$ be arbitrary smooth contravariant vectorfields and set

$$(4.21) \quad \varphi = \eta_{\alpha\beta} \xi^\alpha \chi^\beta.$$

Let us evaluate φ in (x^0, x) , $x \in \mathcal{S}_0$ fixed. Then we have

$$(4.22) \quad \frac{\partial \varphi}{\partial x^0} = \eta_{\alpha\beta\gamma} \xi^\alpha \chi^\beta \eta^\gamma + \eta_{\alpha\beta} \xi^\alpha{}_{;\gamma} \eta^\gamma \chi^\beta + \eta_{\alpha\beta} \xi^\alpha \chi^\beta{}_{;\gamma} \eta^\gamma.$$

Since $(\eta_{\alpha\beta})$ is a tensor that vanishes identically in the product metric, we conclude that $\frac{\partial \varphi}{\partial x^0}$ vanishes identically in the product metric, and by induction we further deduce

$$(4.23) \quad \lim_{x^0 \rightarrow 0} D_{x^0}^k \varphi = 0 \quad \forall k \in \mathbb{N}$$

and

$$(4.24) \quad |D_{x^0}^k \varphi| \leq c_k \quad \forall k \in \mathbb{N}.$$

The mean value theorem then yields

$$(4.25) \quad |\varphi(\tau, x) - \varphi(\tau_0, x)| \leq \sup_{[\tau, \tau_0]} |D_{x^0} \varphi| |\tau - \tau_0|,$$

and, by letting τ_0 tend to 0, we conclude

$$(4.26) \quad |\varphi(\tau, x)| \leq \sup_{[\tau, 0)} |D_{x^0} \varphi| |\tau|.$$

Applying now induction to $|D_{x^0} \varphi|$ yields the result because of the arbitrariness of $(\xi^\alpha), (\chi^\alpha)$. \square

4.5. Lemma. *There exists $\epsilon > 0$ and a constant c_ϵ such that*

$$(4.27) \quad \|Du\|e^{\epsilon t} \leq c_\epsilon \quad \forall t \in \mathbb{R}_+.$$

Proof. We employ the relation

$$(4.28) \quad \tilde{v}^2 = 1 + \|Du\|^2$$

and the fact that \tilde{v} is uniformly bounded to conclude that for small $\|Du\|$

$$(4.29) \quad 2 \log \tilde{v} \sim \|Du\|^2,$$

i.e., we can equivalently prove that $\log \tilde{v} e^{2\epsilon t}$ is uniformly bounded.

Let $\epsilon > 0$ be small and set

$$(4.30) \quad \varphi = \log \tilde{v} e^{2\epsilon t},$$

then φ satisfies

$$(4.31) \quad \dot{\varphi} - F^{-2} \Delta \varphi = \tilde{v}^{-1} (\dot{\tilde{v}} - F^{-2} \Delta \tilde{v}) e^{2\epsilon t} + F^{-2} \|D\varphi\|^2 + 2\epsilon \varphi.$$

To get an a priori estimate for φ we shall proceed as in the proof of Lemma 4.3. For large T , $0 < T < \infty$, assume that

$$(4.32) \quad \sup_{[0, T]} \sup_{M(t)} \varphi = \varphi(t_0, x_0).$$

Applying the maximum principle we infer from (4.31), (4.2), Lemma 4.2, and Lemma 4.3, after multiplying by F^2 ,

$$(4.33) \quad 0 \leq -\|A\|^2 e^{2\epsilon t} + c\|A\|\|u\|e^{2\epsilon t} + c|u|^2 e^{2\epsilon t} + n f'' \|Du\|^2 e^{2\epsilon t} \tilde{v} \\ + c|u|\|Du\|e^{2\epsilon t} + c\|A\|\|Du\|e^{2\epsilon t} + c\|Du\|^2 e^{2\epsilon t} + 2\epsilon F^2 \varphi.$$

Now, we have

$$(4.34) \quad F^2 = H^2 + n^2 |f'|^2 \tilde{v}^2 + n^2 |\psi_\alpha \nu^\alpha|^2 \\ - 2nHf'\tilde{v} + 2nH\psi_\alpha \nu^\alpha - 2n^2 f'\tilde{v}\psi_\alpha \nu^\alpha,$$

hence φ is a priori bounded, if ϵ is small enough, $0 < \epsilon \ll \tilde{\gamma}$.

Here we also used the boundedness of \tilde{v} so that

$$(4.35) \quad \varphi \leq c e^{2\epsilon t},$$

as well as the boundedness of $\tilde{u} = u e^{\gamma t}$.

To control the term

$$(4.36) \quad \epsilon n^2 |f'|^2 \tilde{v}^2 \varphi$$

we employed the assumption (0.8) yielding

$$(4.37) \quad -c \leq f'' + \tilde{\gamma} |f'|^2 \leq c$$

as well as the estimate

$$(4.38) \quad |\log \tilde{v} - \frac{1}{2}\|Du\|^2| \leq c\|Du\|^4$$

because of (4.28). \square

After having established the exponential decay of $\|Du\|$, we can improve the decay rate.

4.6. Lemma. *For any $0 < \lambda < \gamma$ there exists c_λ such that*

$$(4.39) \quad \|Du\|e^{\lambda t} \leq c_\lambda.$$

Proof. As in the proof of the preceding lemma set

$$(4.40) \quad \varphi = \log \tilde{v} e^{2\lambda t}.$$

Let T , $0 < T < \infty$, be large and (t_0, x_0) be such that

$$(4.41) \quad \sup_{[0, T]} \sup_{M(t)} \varphi = \varphi(t_0, x_0).$$

Applying the maximum principle we then obtain an inequality as in (4.33), where ϵ has to be replaced by λ .

The bad terms which need further consideration are part of

$$(4.42) \quad 2\lambda F^2 \varphi,$$

especially

$$(4.43) \quad 2\lambda H^2 \varphi$$

and

$$(4.44) \quad 2\lambda n^2 |f'|^2 \tilde{v}^2 \varphi.$$

The quantity in (4.43) can be absorbed by

$$(4.45) \quad -\|A\|^2 e^{2\lambda t},$$

since $\varphi = \log \tilde{v} e^{2\lambda t}$ and $\log \tilde{v}$ decays exponentially.

The second term is dominated by

$$(4.46) \quad -nf''\|Du\|^2 e^{2\lambda t} \tilde{v},$$

because of (4.28), (4.37), (4.38), the exponential decay of $\|Du\|$, and the assumption that $\lambda < \gamma$.

Thus we see that φ is a priori bounded independent of T . \square

5. C^2 -ESTIMATES

The ultimate goal is to show that $\|A\|e^{\gamma t}$ is uniformly bounded. However, this result can only be derived by first establishing some preliminary estimates.

Let us start by proving that F grows exponentially fast. From the evolution equation (2.11) we deduce

$$(5.1) \quad \dot{H} - F^{-2}\Delta F = -2F^{-3}\|DF\|^2 + F^{-2}(\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)F,$$

where we have used that

$$(5.2) \quad \dot{H} = \delta_j^i \dot{h}_i^j.$$

Replacing \dot{H} by \dot{F} in the evolution equation (5.1) and observing that

$$(5.3) \quad \begin{aligned} \dot{F} &= \dot{H} - nf''\tilde{v}^2F^{-1} + nf'\eta_{\alpha\beta}\nu^\alpha\nu^\beta F^{-1} \\ &\quad + nf'u^iF_iF^{-2} - n\psi_{\alpha\beta}\nu^\alpha\nu^\beta F^{-1} + n\psi_\alpha x_i^\alpha F^iF^{-2} \end{aligned}$$

we obtain

$$(5.4) \quad \begin{aligned} \dot{F} - F^{-2}\Delta F &= -2F^{-3}\|DF\|^2 + F^{-2}(\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)F \\ &\quad + F^{-2}(-nf''\tilde{v}^2 + nf'\eta_{\alpha\beta}\nu^\alpha\nu^\beta - n\psi_{\alpha\beta}\nu^\alpha\nu^\beta)F \\ &\quad + F^{-2}(nf'u_i + n\psi_\alpha x_i^\alpha)F^i. \end{aligned}$$

5.1. Lemma. *There exist positive constants δ and c_δ such that*

$$(5.5) \quad c_\delta e^{\delta t} \leq F \quad \forall t \in \mathbb{R}_+.$$

Proof. Define

$$(5.6) \quad \varphi = Fe^{-\delta t}.$$

Let $T, 0 < T < \infty$, be large and (t_0, x_0) be such that

$$(5.7) \quad \inf_{[0, T]} \inf_{M(t)} \varphi = \varphi(t_0, x_0).$$

Applying the maximum principle we deduce from (5.4)

$$(5.8) \quad 0 \geq \|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta + nf'\eta_{\alpha\beta}\nu^\alpha\nu^\beta - nf''\tilde{v} - n\psi_{\alpha\beta}\nu^\alpha\nu^\beta - \delta F^2,$$

and we further conclude that, for small δ , t_0 cannot exceed a certain value in view of the relations (4.34) and (4.37), hence the result. \square

Replacing in (5.1) F by H we obtain an evolution equation for H

$$(5.9) \quad \begin{aligned} \dot{H} - F^{-2}\Delta H &= -2F^{-3}\|DF\|^2 + F^{-2}(\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)F \\ &\quad + F^{-2}(nf''\tilde{v}^2H - nf''\tilde{v}g^{ij}\bar{h}_{ij} - nf''\|Du\|^2\tilde{v} - 2nf''\eta_{\alpha\beta}\nu^\alpha x_i^\beta u^i \\ &\quad + 4nf''h_{ij}u^i u^j - nf'\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma g^{ij} - 2nf'h^{ij}\eta_{\alpha\beta}x_i^\alpha x_j^\beta \\ &\quad - n\eta_{\alpha\beta}\nu^\alpha\nu^\beta f'H - nf'\|A\|^2\tilde{v} + nf'H_k u^k + nf'\bar{R}_{\alpha\beta}\nu^\alpha x_k^\beta u^k) \\ &\quad + nF^{-2}(\psi_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma g^{ij} + \psi_{\alpha\beta}\nu^\alpha\nu^\beta H + 2\psi_{\alpha\beta}x_i^\alpha x_j^\beta h^{ij} \\ &\quad + \|A\|^2\psi_\alpha\nu^\alpha + \psi_\alpha x_k^\alpha H^k + \bar{R}_{\alpha\beta}\nu^\alpha x_k^\beta \psi_\gamma x_l^\gamma g^{kl}). \end{aligned}$$

In deriving this equation we used the Weingarten and Codazzi equations, the definition of F and the relation

$$(5.10) \quad \tilde{v}H = -\Delta u + g^{ij}\bar{h}_{ij},$$

where \bar{h}_{ij} is the second fundamental form of the slices $\{x^0 = \text{const}\}$.

5.2. Lemma. *H is uniformly bounded from below during the evolution.*

Proof. Let T , $0 < T < \infty$, be large and $x_0 = x(t_0, \xi_0)$ be such that

$$(5.11) \quad \inf_{[0, T]} \inf_{M(t)} H = H(x_0).$$

Applying the maximum principle and some trivial estimates we deduce from (5.9)

$$(5.12) \quad \begin{aligned} 0 \geq & -2F^{-3}\|DF\|^2 + F^{-2}(\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)F \\ & + F^{-2}(nf''\tilde{v}H - c|f'|^{\frac{3}{2}} - \frac{n}{2}f'\|A\|^2\tilde{v} - c(1 + \|A\|^2)), \end{aligned}$$

where we have used Corollary 3.7, Lemma 4.4, Lemma 4.6 and assumed that $H(x_0) \leq -1$.

To estimate the term involving $\|DF\|^2$ we note that

$$(5.13) \quad \begin{aligned} \|DF\|^2 = & \|DH\|^2 + n^2|f''|^2\|Du\|^2\tilde{v}^2 + n^2|f'|^2\|D\tilde{v}\|^2 \\ & + n^2\|D(\psi_\alpha\nu^\alpha)\|^2 - 2nf''H_k u^k \tilde{v} - 2nf'H_k \tilde{v}^k \\ & + 2nH^k(\psi_\alpha\nu^\alpha)_k + 2n^2f'f''\tilde{v}_k u^k \tilde{v} \\ & - 2n^2f''\tilde{v}(\psi_\alpha\nu^\alpha)_k u^k - 2n^2f'(\psi_\alpha\nu^\alpha)_k \tilde{v}^k. \end{aligned}$$

DH vanishes in x_0 , and because of (4.4), Lemma 4.4 and Lemma 4.6 we have

$$(5.14) \quad \|D\tilde{v}\| \leq c\|\eta_{\alpha\beta}\| + \|A\|\|Du\| \leq c_\lambda(1 + \|A\|)e^{-\lambda t} \quad \forall 0 < \lambda < \gamma.$$

Combining these estimates with the exponential growth of F we conclude

$$(5.15) \quad F^{-1}\|DF\|^2 \leq c(1 + |f''| + \|A\|^2),$$

hence the a priori bound from below for H . \square

Next we shall show that the principal curvatures of $M(t)$ are uniformly bounded from above, i.e., we want to estimate h_i^j from above.

Let us first derive a parabolic equation satisfied by h_i^j from the evolution equation (2.11).

Using the definition of F we immediately obtain

$$(5.16) \quad \begin{aligned} \dot{h}_i^j - F^{-2}H_i^j = & -2F^{-3}F_i F^j + F^{-1}h_{ik}h^{kj} \\ & + F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \\ & + F^{-2}(-n(f'\tilde{v})_i^j + n(\psi_\alpha\nu^\alpha)_i^j) \end{aligned}$$

and conclude further

$$\begin{aligned}
(5.17) \quad & \dot{h}_i^j - F^{-2} \Delta h_i^j = \\
& - 2F^{-3} F_i F^j + F^{-1} h_{ik} h^{kj} + F^{-1} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \\
& - F^{-2} \|A\|^2 h_i^j + F^{-2} H h_{ik} h^{kj} + 2F^{-2} h^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_k^\alpha x_i^\beta x_l^\gamma x_r^\delta g^{rj} \\
& - F^{-2} (g^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} + g^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^m j \\
& \quad + \bar{R}_{\alpha\beta} \nu^\alpha \nu^\beta h_i^j - H \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj}) \\
& + F^{-2} g^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} (\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}) \\
& + F^{-2} (n f'' h_i^j \tilde{v}^2 + n f'' \tilde{v} \eta_{\alpha\beta} x_i^\alpha x_k^\beta g^{kj} - n f''' u_i u^j \tilde{v} \\
& \quad - n f'' (\tilde{v}_i u^j + \tilde{v}^j u_i) - n f' [\eta_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_k^\gamma g^{kj} + \eta_{\alpha\beta} x_k^\alpha x_i^\beta h^{kj} \\
& \quad + \eta_{\alpha\beta} \nu^\alpha \nu^\beta h_i^j + h_i^k h_k^j \tilde{v} - h_{i;k}^j u^k + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_l^\delta g^{lj} u_k]) \\
& + n F^{-2} (\psi_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_k^\gamma g^{kj} + \psi_{\alpha\beta} \nu^\alpha \nu^\beta h_i^j + \psi_{\alpha\beta} x_k^\alpha x_i^\beta h^{kj} \\
& \quad + \psi_{\alpha\beta} x_l^\beta x_k^\alpha h_i^k g^{lj} + \psi_{\alpha\beta} \nu^\alpha h_{ki} h^{kj} + \psi_{\alpha\beta} x_k^\alpha h_{i;l}^k g^{lj}),
\end{aligned}$$

where we used the relation

$$(5.18) \quad h_{ij} \tilde{v} = -u_{ij} + \bar{h}_{ij} = -u_{ij} - \eta_{\alpha\beta} x_i^\alpha x_j^\beta,$$

equation (4.4) as well as the Weingarten and Codazzi equations.

5.3. Lemma. *The principal curvatures κ_i of $M(t)$ are uniformly bounded during the evolution.*

Proof. Since we already know that $H \geq -c$, it suffices to prove an uniform estimate from above.

Let φ be defined by

$$(5.19) \quad \varphi = \sup \{ h_{ij} \eta^i \eta^j : \|\eta\| = 1 \}.$$

We shall prove that

$$(5.20) \quad w = \log \varphi + \lambda \tilde{v}$$

is uniformly bounded from above, if λ is large enough.

Let $0 < T < \infty$ be large, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(5.21) \quad \sup_{M_0} w < \sup \{ \sup_{M(t)} w : 0 < t \leq T \} = w(x_0).$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$(5.22) \quad g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h_n^n.$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$(5.23) \quad \tilde{\eta} = (0, \dots, 0, 1),$$

and set

$$(5.24) \quad \tilde{\varphi} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j} \quad \wedge \quad \tilde{w} = \log \tilde{\varphi} + \lambda\tilde{v}.$$

\tilde{w} is well defined in neighbourhood of (t_0, ξ_0) , and \tilde{w} assumes its maximum at (t_0, ξ_0) . Moreover, at (t_0, ξ_0) we have

$$(5.25) \quad \dot{\tilde{\varphi}} = \dot{h}_n^n,$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde{\varphi}$ satisfies the same differential equation (5.17) as h_n^n . For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that w is defined by

$$(5.26) \quad w = \log h_n^n + \lambda\tilde{v}.$$

At (t_0, ξ_0) we have $\dot{w} \geq 0$, and, in view of the maximum principle, we deduce from (5.17) and (4.2)

$$(5.27) \quad \begin{aligned} 0 &\leq -\lambda\|A\|^2\tilde{v} + c\lambda(1 + \|A\| + |f'|e^{-\frac{3}{2}\gamma t}\|A\|) \\ &\quad + c(|H|h_n^n + |f'|h_n^n + |f'|) + nf''\tilde{v}^2 \\ &\quad + c|f'|\|D \log h_n^n\|\|Du\| + \|D \log h_n^n\|^2 + c\|D \log h_n^n\|, \end{aligned}$$

where we assumed $h_n^n \geq 1$, and in addition used (4.4) and the known exponential decay estimates for $\|Du\|$.

Since $Dw = 0$ in x_0 , we have

$$(5.28) \quad \|D \log h_n^n\| = \lambda\|D\tilde{v}\| \leq \lambda c(1 + \|A\|\|Du\|).$$

Hence, if λ is chosen large enough, we obtain an a priori bound for h_n^n from above. \square

An immediate corollary is

5.4. Corollary. *There exist positive constants c_1, c_2 such that*

$$(5.29) \quad c_1 \leq Fe^{-\gamma t} \leq c_2.$$

Proof. Since H is uniformly bounded we conclude

$$(5.30) \quad Fe^{-\gamma t} \sim -nf'e^{-\gamma t}\tilde{v} = -nf'u(ue^{\gamma t})^{-1}\tilde{v}$$

and the result follows from Lemma 3.4 and Theorem 3.6. \square

We can now prove an exponential decay for $\|A\|$.

5.5. Lemma. *For any $0 < \lambda < \gamma$ there exists c_λ such that*

$$(5.31) \quad \|A\|e^{\lambda t} \leq c_\lambda \quad \forall t \in \mathbb{R}_+.$$

Proof. Let $\varphi = \frac{1}{2}\|A\|^2$, then

$$(5.32) \quad \dot{\varphi} - F^{-2}\Delta\varphi = -F^{-2}\|DA\|^2 + (\dot{h}_i^j - F^{-2}\Delta h_i^j)h_j^i,$$

where

$$(5.33) \quad \|DA\|^2 = h_{ij;k}h^{ij}{}_{;l}g^{kl}.$$

Define $w = \varphi e^{2\lambda t}$ with $0 < \lambda < \gamma$. Let $0 < T < \infty$ be large, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(5.34) \quad \sup_{M_0} w < \sup\{ \sup_{M(t)} w : 0 < t \leq T \} = w(x_0).$$

Applying the maximum principle we deduce from (5.32) and (5.17)

$$(5.35) \quad \begin{aligned} 0 \leq & -\|DA\|^2 e^{2\lambda t} - 2F^{-1}h^{ij}F_iF_j e^{2\lambda t} + 2nf''\tilde{v}^2w \\ & + c_\epsilon e^{-\epsilon t}|f''|\|A\|e^{\lambda t} + c|f'|(w+1) + 2\lambda F^2w, \end{aligned}$$

with some small positive $\epsilon = \epsilon(\lambda)$; here we used Lemma 4.6 and Corollary 3.7.

It remains to estimate the second and the last term in the preceding inequality. The only relevant term in $2\lambda F^2w$ is

$$(5.36) \quad 2\lambda n^2|f'|^2\tilde{v}^2w;$$

combining it with $2nf''\tilde{v}^2w$ gives

$$(5.37) \quad 2nf''\tilde{v}^2w + 2\lambda n^2|f'|^2\tilde{v}^2w \leq -2n^2(\gamma - \lambda)|f'|^2\tilde{v}^2w + cw,$$

in view of (4.37).

The remaining term can be estimated

$$(5.38) \quad -F^{-1}h^{ij}F_iF_j e^{2\lambda t} \leq ce^{-\gamma t}\|DA\|^2 e^{2\lambda t} + c_\epsilon|f'|^2 e^{-\epsilon t}\|A\|e^{\lambda t} + c(1+w),$$

with some positive $\epsilon = \epsilon(\lambda)$.

Inserting these estimates in (5.35) we obtain an a priori bound for w . \square

Though we now could prove an a priori estimate for $\|A\|e^{\gamma t}$, let us first derive a corresponding estimate for $\|Du\|e^{\gamma t}$. The estimate for the second fundamental form is then slightly easier to prove.

5.6. Theorem. *Let $\tilde{u} = u^{\gamma t}$, then $\|D\tilde{u}\|$ is uniformly bounded during the evolution.*

Proof. Let $\varphi = \varphi(t)$ be defined by

$$(5.39) \quad \varphi = \sup_{M(t)} \log \tilde{v} e^{2\gamma t}.$$

Then, in view of the maximum principle, we deduce from equation (4.2)

$$(5.40) \quad \dot{\varphi} \leq ce^{-\epsilon t} + F^{-2}(nf''\|D\tilde{u}\|^2\tilde{v} + 2\gamma F^2\varphi)$$

for some positive ϵ , where we have used the known exponential decay of $\|A\|$ and $\|Du\|$ as well as Lemma 4.2, Lemma 4.4, Corollary 5.4 and the inequalities (4.37) and (4.38); the inequality is valid for a.e. t .

The second term on the right-hand side of (5.40) can be estimated from above by

$$(5.41) \quad ce^{-\epsilon t}(1 + \varphi),$$

in view of (4.37), (4.38) and the known decay of $\|A\|$, $\|Du\|$ as well as the result in Corollary 5.4. Hence we conclude

$$(5.42) \quad \dot{\varphi} \leq ce^{-\epsilon t}(1 + \varphi),$$

i.e., φ is uniformly bounded. \square

5.7. Theorem. *The quantity $w = \frac{1}{2}\|A\|^2 e^{2\gamma t}$ is uniformly bounded during the evolution.*

Proof. Define $\varphi = \varphi(t)$ by

$$(5.43) \quad \varphi = \sup_{M(t)} w.$$

Applying the maximum principle we deduce from (5.32) that for a.e. t

$$(5.44) \quad \begin{aligned} \dot{\varphi} &\leq -F^{-2}\|DA\|^2 e^{2\gamma t} + F^{-3}(-2h^{ij}F_i F_j e^{2\gamma t} - nFf'''h^{ij}\tilde{u}_i\tilde{u}_j\tilde{v}) \\ &\quad + F^{-2}(nf''\tilde{v}^2\varphi + \gamma F^2\varphi) + ce^{-\epsilon t}(1 + \varphi) \\ &\quad + F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h^{ij} e^{2\gamma t} \end{aligned}$$

The last terms on the right-hand side of this inequality can be estimated as follows

$$(5.45) \quad \begin{aligned} &F^{-3}(-2h^{ij}F_i F_j e^{2\gamma t} - nFf'''h^{ij}\tilde{u}_i\tilde{u}_j\tilde{v}) \leq \\ &F^{-3}(-2|f''|^2 + f'f''')h^{ij}\tilde{u}_i\tilde{u}_j\tilde{v}^2 n^2 + cF^{-3}\|DA\|^2 e^{2\gamma t} \\ &\quad + ce^{-\epsilon t}(1 + \varphi). \end{aligned}$$

Now, we observe that

$$(5.46) \quad (f'' + \tilde{\gamma}|f'|^2)' = f''' + 2\tilde{\gamma}f'f'' = \mathcal{C}f',$$

where \mathcal{C} is a bounded function in view of assumption (0.9), and hence

$$(5.47) \quad 2|f''|^2 - f'f''' = 2|f''|^2 + 2\tilde{\gamma}|f'|^2 f'' - \mathcal{C}|f'|^2,$$

i.e.,

$$(5.48) \quad |2|f''|^2 - f'f'''| \leq c|f'|^2,$$

and we conclude that the left-hand side of (5.45) can be estimated from above by

$$(5.49) \quad ce^{-\epsilon t}(1 + \varphi) + cF^{-2}\|DA\|^2 e^{\gamma t}$$

Next, we estimate

$$(5.50) \quad F^{-2}(nf''\tilde{v}^2 + \gamma F^2)\varphi \leq ce^{-\epsilon t}\varphi,$$

and finally

$$(5.51) \quad F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h^{ij} e^{2\gamma t} \leq ce^{-\epsilon t}(1 + \varphi) + F^{-1}\bar{R}_{0i0j}h^{ij} e^{2\gamma t}\tilde{v}^2,$$

but

$$(5.52) \quad |\bar{R}_{0i0j}| \leq c|u|,$$

cf. Lemma 4.4.

Hence, we deduce

$$(5.53) \quad \dot{\varphi} \leq ce^{-\epsilon t}(1 + \varphi)$$

for some positive ϵ and for a.e. t , i.e., φ is bounded. \square

6. HIGHER ORDER ESTIMATES

After having established the boundedness of

$$(6.1) \quad \|A\|^2 e^{2\gamma t}$$

corresponding estimates for the derivatives of the second fundamental form will be proved recursively.

Our starting point is the equation (5.17). It contains two very bad terms

$$(6.2) \quad -nF^{-2}f'''u_iu^j\tilde{v},$$

and another one which is hidden in the expression

$$(6.3) \quad -2F^{-3}F_iF^j.$$

To handle these terms we proceed as in the proof of Theorem 5.7 by combining the two crucial terms in

$$(6.4) \quad F^{-3}(-2F_iF^j - nFf'''u_iu^j\tilde{v})$$

to

$$(6.5) \quad F^{-3}(-2|f''|^2 + f'f''')u_iu^jn^2\tilde{v}^2$$

and observing that

$$(6.6) \quad \varphi = -2|f''|^2 + f'f''' = (f'' + \tilde{\gamma}|f'|^2)'f' - 2f''(f'' + \tilde{\gamma}|f'|^2).$$

In view of our assumption (0.9) and Corollary 3.7 we conclude that the spatial derivatives of φ can be estimated by

$$(6.7) \quad \|D^m\varphi\| \leq c_m(1 + \|\tilde{u}\|_{m-1})^{p_{m-1}}(1 + \|D^m\tilde{u}\|)e^{2\gamma t} \quad \forall m \in \mathbb{N},$$

for some suitable $p_{m-1} \in \mathbb{N}$.

Let us introduce the following abbreviations

6.1. Definition. (i) For arbitrary tensors S, T denote by $S \star T$ any linear combination of tensors formed by contracting over S and T . The result can be a tensor or a function. Note that we do not distinguish between $S \star T$ and $cS \star T$, c a constant.

(ii) The symbol A represents the second fundamental form of the hypersurfaces $M(t)$ in N , $\tilde{A} = Ae^{\gamma t}$ is the scaled version, and $D^m A$ resp. $D^m \tilde{A}$ represent the covariant derivatives of order m .

(iii) For $m \in \mathbb{N}$ denote by \mathcal{O}_m a tensor expression defined on $M(t)$ that satisfies the pointwise estimates

$$(6.8) \quad \|\mathcal{O}_m\| \leq c_m(1 + \|\tilde{A}\|_m)^{p_m},$$

where c_m, p_m are positive constants, and

$$(6.9) \quad \|\tilde{A}\|_m = \sum_{|\alpha| \leq m} \|D^\alpha \tilde{A}\|.$$

Moreover, the derivative of \mathcal{O}_m is of class \mathcal{O}_{m+1} and can be estimated by

$$(6.10) \quad \|D\mathcal{O}_m\| \leq c_m(1 + \|\tilde{A}\|_m)^{p_m}(1 + \|D^{m+1}\tilde{A}\|)$$

with (different) constants c_m, p_m .

(iv) The symbol \mathcal{O} represents a tensor such that $D\mathcal{O}$ is of class \mathcal{O}_0 .

6.2. Remark. We emphasize the following relations

$$(6.11) \quad D^m \mathcal{O}_0 = \mathcal{O}_m \quad \forall m \in \mathbb{N},$$

$$(6.12) \quad F^{-1}DF = F^{-1}DA + \mathcal{O},$$

$$(6.13) \quad DFe^{-\gamma t} = e^{-\gamma t}DA + \mathcal{O},$$

$$(6.14) \quad F^{-1}\mathcal{O}_m = \mathcal{O}_m \quad \forall m \in \mathbb{N},$$

and

$$(6.15) \quad |\bar{R}_{0i0j}| \leq c_m |u|^m \quad \forall m \in \mathbb{N},$$

cf. Lemma 4.4.

With these definitions and the relations (6.5) and (6.7) in mind we can write the evolution equation for \tilde{h}_i^j in the form

$$(6.16) \quad \begin{aligned} \dot{\tilde{h}}_i^j - F^{-2}\Delta\tilde{h}_i^j &= F^{-3}D\tilde{A} \star DA + F^{-2}\mathcal{O} \star D\tilde{A} \\ &\quad + F^{-3}\mathcal{O}_0 \star D\tilde{A} + F^{-2}\mathcal{O}_0 + F^{-1}\mathcal{O}, \end{aligned}$$

where the right-hand side is considered to be a mixed tensor of order two though we omitted the indices.

Using the fact that

$$(6.17) \quad \dot{g}_{ij} = -2F^{-1}h_{ij} = -2F^{-1}e^{-\gamma t}\tilde{h}_{ij} = F^{-2}\mathcal{O}_0$$

we can rewrite (6.16) in the form

$$(6.18) \quad \begin{aligned} \dot{\tilde{A}} - F^{-2}\Delta\tilde{A} &= F^{-3}D\tilde{A} \star DA + F^{-2}\mathcal{O} \star D\tilde{A} \\ &\quad + F^{-3}\mathcal{O}_0 \star D\tilde{A} + F^{-2}\mathcal{O}_0 + F^{-1}\mathcal{O} \end{aligned}$$

regardless of representing \tilde{A} as a covariant, contravariant or mixed tensor.

Differentiating this equation covariantly with respect to a spatial variable we deduce

$$(6.19) \quad \begin{aligned} \frac{D}{dt}(D\tilde{A}) - F^{-2}\Delta D\tilde{A} &= F^{-1}\mathcal{O}_0 + F^{-3}D^2\tilde{A} \star DA + F^{-2}\mathcal{O} \star D^2\tilde{A} \\ &+ F^{-4}D\tilde{A} \star DA \star DA + F^{-3}\mathcal{O} \star D\tilde{A} \star DA + F^{-2}D\tilde{A} \star \mathcal{O}_0 \\ &+ F^{-4}D\tilde{A} \star DA \star \mathcal{O}_0 + F^{-3}D\tilde{A} \star D\mathcal{O}_0 + F^{-3}D^2\tilde{A} \star \mathcal{O}_0, \end{aligned}$$

where we used the Ricci identities to commute the second derivatives of a tensor.

Finally, using induction, we conclude

$$(6.20) \quad \begin{aligned} \frac{D}{dt}(D^{m+1}\tilde{A}) - F^{-2}\Delta D^{m+1}\tilde{A} &= F^{-1}\mathcal{O}_m + F^{-3}D^{m+2}\tilde{A} \star DA \\ &+ F^{-2}D^{m+1}A \star \mathcal{O}_m + F^{-3}D^{m+2}\tilde{A} \star \mathcal{O}_0 \\ &+ \Theta F^{-3}D^{m+1}\tilde{A} \star D^{m+1}A, \end{aligned}$$

for any $m \in \mathbb{N}^*$, where $\Theta = 1$, if $m = 1$, and $\Theta = 0$ otherwise.

We are now going to prove uniform bounds for $\frac{1}{2}\|D^{m+1}\tilde{A}\|^2$ for all $m \in \mathbb{N}$.

First we observe that

$$(6.21) \quad \begin{aligned} \frac{D}{dt}(\frac{1}{2}\|D^{m+1}\tilde{A}\|^2) - F^{-2}\Delta\frac{1}{2}\|D^{m+1}\tilde{A}\|^2 &= -F^{-2}\|D^{m+2}\tilde{A}\|^2 \\ &+ F^{-1}\mathcal{O}_m \star D^{m+1}\tilde{A} + F^{-3}D^{m+2}\tilde{A} \star DA \star D^{m+1}\tilde{A} \\ &+ F^{-2}D^{m+1}A \star \mathcal{O}_m \star D^{m+1}\tilde{A} + F^{-3}D^{m+2}\tilde{A} \star \mathcal{O}_0 \star D^{m+1}\tilde{A} \\ &+ \Theta F^{-3}D^{m+1}\tilde{A} \star D^{m+1}A \star D^{m+1}\tilde{A}, \end{aligned}$$

if $m \in \mathbb{N}^*$, in view of (6.20), where similar equations are also valid for $\frac{1}{2}\|\tilde{A}\|^2$ and $\frac{1}{2}\|D\tilde{A}\|^2$, cf. (6.18) and (6.19).

6.3. Theorem. *The quantities $\frac{1}{2}\|D^m\tilde{A}\|^2$ are uniformly bounded during the evolution for all $m \in \mathbb{N}^*$.*

Proof. We proof the theorem recursively by estimating

$$(6.22) \quad \varphi = \log \frac{1}{2}\|D^{m+1}\tilde{A}\|^2 + \mu \frac{1}{2}\|D^m\tilde{A}\|^2 + \lambda e^{-\gamma t},$$

where μ is a small positive constant

$$(6.23) \quad 0 < \mu = \mu(m) \ll 1,$$

and λ large, $\lambda = \lambda(m) \gg 1$.

We shall only treat the case $m = 0$, since then the structure of the right-hand side is worst, at least formally, cf. (6.19).

Fix $0 < T < \infty$, T very large, and suppose that

$$(6.24) \quad 2 \sup \|\tilde{A}\|^2 < \sup_{[0, T]} \sup_{M(t)} \varphi = \varphi(x(t_0, \xi_0))$$

for $0 < t_0 \leq T$, where $e^{-\gamma t_0}$ should be small compared with μ , i.e., t_0 has to be large.

Applying the maximum principle we deduce

$$(6.25) \quad \begin{aligned} 0 \leq & \mu^2 F^{-2} \|D_{\frac{1}{2}} \tilde{A}\|^2 - \frac{1}{2} F^{-2} \|D^2 \tilde{A}\|^2 \|D\tilde{A}\|^{-2} - \frac{\lambda}{2} \gamma e^{-\gamma t} \\ & - \frac{\mu}{2} F^{-2} \|D\tilde{A}\|^2 + cF^{-4} \|D\tilde{A}\|^2. \end{aligned}$$

Now, we observe that

$$(6.26) \quad \|D_{\frac{1}{2}} \tilde{A}\|^2 \leq c \|D\tilde{A}\|^2 \|\tilde{A}\|^2 \leq c \|D\tilde{A}\|^2$$

and hence the right-hand side of inequality (6.25) would be negative, if μ is small, λ large and t_0 large.

Thus φ is a priori bounded.

The proof for $m \geq 1$ is similar. \square

7. CONVERGENCE OF \tilde{u} AND THE BEHAVIOUR OF DERIVATIVES IN t

Let us first prove that \tilde{u} converges when t tends to infinity.

7.1. Lemma. *\tilde{u} converges in $C^m(\mathcal{S}_0)$ for any $m \in \mathbb{N}$, if t tends to infinity, and hence $D^m \tilde{A}$ converges.*

Proof. \tilde{u} satisfies the evolution equation

$$(7.1) \quad \dot{\tilde{u}} = \frac{\tilde{v}e^{\gamma t}}{F} + \gamma\tilde{u} = \frac{\tilde{v}e^{\gamma t}}{F} (1 - \tilde{\gamma}f'u + v\gamma He^{-\gamma t} + v\gamma m\psi_\alpha \nu^\alpha e^{-\gamma t}),$$

hence we deduce

$$(7.2) \quad |\dot{\tilde{u}}| \leq ce^{-2\gamma t},$$

in view of Lemma 3.4 and the known estimates for H, F and ψ , i.e., \tilde{u} converges uniformly. Due to Theorem 6.3, $D^m \tilde{u}$ is uniformly bounded, hence \tilde{u} converges in $C^m(\mathcal{S}_0)$.

The convergence of $D^m \tilde{A}$ follows from Theorem 6.3 and the convergence of \tilde{h}_{ij} , which in turn can be deduced from equation (5.18). \square

Combining the equations (6.18), (6.19), (6.20), and Theorem 6.3 we immediately conclude

7.2. Lemma. *$\|\frac{D}{dt} D^m \tilde{A}\|$ and $\|\frac{D}{dt} D^m A\|$ decay by the order $e^{-\gamma t}$ for any $m \in \mathbb{N}$.*

7.3. Corollary. *$\frac{D}{dt} D^m A e^{\gamma t}$ converges, if t tends to infinity.*

Proof. Applying the product rule we obtain

$$(7.3) \quad \frac{D}{dt} D^m \tilde{A} = \frac{D}{dt} D^m A e^{\gamma t} + \gamma D^m \tilde{A},$$

hence the result, since the left-hand side converges to zero and $D^m \tilde{A}$ converges. \square

In view of Lemma 3.4 $f'u$ converges to $\tilde{\gamma}^{-1}$, if t tends to infinity, moreover, because of the condition (0.10) and the estimates for u resp. \tilde{u} , we further deduce

7.4. Lemma. *For any $m \in \mathbb{N}$ we have*

$$(7.4) \quad \|D^m(f'u)\| \leq c_m.$$

Proof. We only consider the case $m = 1$. Differentiating $f'u$ we get

$$(7.5) \quad (f'u)_k = f''uu_k + f'u_k = f''u^2u^{-1}u_k + f'uu^{-1}u_k,$$

but

$$(7.6) \quad u^{-1}u_k = \tilde{u}^{-1}\tilde{u}_k$$

and hence uniformly bounded in view of Theorem 3.6 and Theorem 5.6. \square

7.5. Corollary. *We have*

$$(7.7) \quad \|D^m F^{-1}\| \leq c_m F^{-1} \quad \forall m \in \mathbb{N}.$$

Proof. Recall that

$$(7.8) \quad F = H - n\tilde{v}f' + n\psi_\alpha\nu^\alpha$$

and hence

$$(7.9) \quad (F^{-1})_k = -F^{-2}(H_k - n\tilde{v}_k f' - n\tilde{v}f''u_k + n(\psi_\alpha\nu^\alpha)_k).$$

Now, writing

$$(7.10) \quad \begin{aligned} F^{-1}(H_k - n\tilde{v}_k f' - n\tilde{v}f''u_k + n(\psi_\alpha\nu^\alpha)_k) = \\ (Fu)^{-1}(uH_k - n\tilde{v}_k f'u - n\tilde{v}f''uu_k + n(\psi_\alpha\nu^\alpha)_k u) \end{aligned}$$

we conclude that the expression is smooth in x with uniformly bounded C^m - norms.

The estimate (7.7) follows by induction. \square

7.6. Lemma. *The following estimates are valid*

$$(7.11) \quad \|D\dot{u}\| \leq ce^{-\gamma t},$$

$$(7.12) \quad \left\| \frac{d}{dt} F^{-1} \right\| \leq cF^{-1},$$

and

$$(7.13) \quad |\dot{\tilde{v}}| + |\ddot{\tilde{v}}| + \|D\dot{\tilde{v}}\| \leq ce^{-2\gamma t}.$$

Moreover, $\dot{\tilde{v}}e^{2\gamma t}$ and $\ddot{\tilde{v}}e^{2\gamma t}$ converge, if t goes to infinity.

Proof. „(7.11)“ The estimate follows immediately from

$$(7.14) \quad \dot{u} = \frac{\tilde{v}}{F},$$

in view of Corollary 7.5.

„(7.12)“ Differentiating with respect to t we obtain

$$(7.15) \quad \frac{d}{dt}F^{-1} = -F^{-2}(\dot{H} - n\dot{v}f' - n\tilde{v}f''\dot{u} + n\frac{d}{dt}(\psi_\alpha\nu^\alpha))$$

and the result follows from (7.13) and the known estimates for $|\dot{u}|$ and F .

„(7.13)“ We differentiate the relation $\tilde{v} = \eta_\alpha\nu^\alpha$ to get

$$(7.16) \quad \begin{aligned} \dot{\tilde{v}} &= \eta_{\alpha\beta}\nu^\alpha\dot{x}^\beta + \eta_\alpha\dot{\nu}^\alpha \\ &= -\eta_{\alpha\beta}\nu^\alpha\nu^\beta F^{-1} + (F^{-1})_k u^k \end{aligned}$$

yielding the estimate for $|\dot{\tilde{v}}|$, in view of Corollary 7.5 and the decay of $\eta_{\alpha\beta}$.

Differentiating (7.16) covariantly with respect to x we infer the estimate for $\|D\dot{\tilde{v}}\|$, while the estimate for $\|\ddot{\tilde{v}}\|$ can be deduced after differentiating (7.16) covariantly with respect to t , in view of (7.11).

The convergence of $\dot{\tilde{v}}e^{2\gamma t}$ and $\ddot{\tilde{v}}e^{2\gamma t}$ can be easily verified. \square

Finally, let us estimate \ddot{h}_i^j and \check{h}_i^j .

7.7. Lemma. \ddot{h}_i^j and \check{h}_i^j decay like $e^{-\gamma t}$.

Proof. The estimate for \check{h}_i^j follows immediately by differentiating equation (5.17) covariantly with respect to t and by applying the above lemmata as well as Theorem 6.3.

Observing the remarks at the beginning of Section 6 about rearranging crucial terms in (5.17), cf. equations (6.4) and (6.5), we further conclude

$$(7.17) \quad \|\check{h}_i^j\| \leq ce^{-\gamma t}. \quad \square$$

Using the same argument as in the proof of Corollary 7.3 we infer

7.8. Corollary. The tensor $\check{h}_i^j e^{\gamma t}$ converges, if t tends to infinity.

The claims in Theorem 0.3 are now almost all proved with the exception of two. In order to prove the remaining claims we need

7.9. Lemma. The function $\varphi = e^{\tilde{\gamma}f}u^{-1}$ converges to $-\tilde{\gamma}\sqrt{m}$ in $C^\infty(\mathcal{S}_0)$, if t tends to infinity.

Proof. φ converges to $-\tilde{\gamma}\sqrt{m}$ in view of (3.5). Hence, we only have to show that

$$(7.18) \quad \|D^m \varphi\| \leq c_m \quad \forall m \in \mathbb{N}^*,$$

which will be achieved by induction.

We have

$$(7.19) \quad \begin{aligned} \varphi_i &= \tilde{\gamma} e^{\tilde{\gamma} f} f' u_i u^{-1} - e^{\tilde{\gamma} f} u^{-2} u_i \\ &= \varphi(\tilde{\gamma} f' u - 1) u^{-1} u_i. \end{aligned}$$

Now, we observe that

$$(7.20) \quad u^{-1} u = \tilde{u}^{-1} \tilde{u}_i$$

and $f' u$ have uniformly bounded C^m -norms in view of Theorem 3.6, Lemma 7.1 and Lemma 7.4.

The proof of the lemma is then completed by a simple induction argument.

7.10. Lemma. *Let (\check{g}_{ij}) be the induced metric of the leaves of the inverse mean curvature flow, then the rescaled metric*

$$(7.21) \quad e^{\frac{2}{n}t} \check{g}_{ij}$$

converges in $C^\infty(\mathcal{S}_0)$ to

$$(7.22) \quad (\tilde{\gamma} m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}} \bar{\sigma}_{ij},$$

where we are slightly ambiguous by using the same symbol to denote $\tilde{u}(t, \cdot)$ and $\lim \tilde{u}(t, \cdot)$.

Proof. There holds

$$(7.23) \quad \check{g}_{ij} = e^{2f} e^{2\psi} (-u_i u_j + \sigma_{ij}(u, x)).$$

Thus, it suffices to prove that

$$(7.24) \quad e^{2f} e^{\frac{2}{n}t} \rightarrow (\tilde{\gamma} m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}}$$

in $C^\infty(\mathcal{S}_0)$. But this is evident in view of the preceding lemma, since

$$(7.25) \quad e^{2f} e^{\frac{2}{n}t} = (-e^{\tilde{\gamma} f} u^{-1})^{\frac{2}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}}. \quad \square$$

Finally, let us prove that the leaves $M(t)$ of the IMCF get more umbilical, if t tends to infinity. Denote by $\check{h}_{ij}, \check{\nu}$, etc., the geometric quantities of the hypersurfaces $M(t)$ with respect to the original metric $(\check{g}_{\alpha\beta})$ in N , then

$$(7.26) \quad e^{\check{\psi}} \check{h}_i^j = h_i^j + \check{\psi}_\alpha \nu^\alpha \delta_i^j,$$

and hence,

$$(7.27) \quad \check{H}^{-1} |\check{h}_i^j - \frac{1}{n} \check{H} \delta_i^j| = F^{-1} |h_i^j - \frac{1}{n} H \delta_i^j| \leq c e^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely,

$$(7.28) \quad \begin{aligned} |\check{h}_i^j - \frac{1}{n}\check{H}\delta_i^j| &= e^{-\psi}e^{-f}e^{-\frac{1}{n}t}|h_i^j - \frac{1}{n}H\delta_i^j|e^{\gamma t}e^{(\frac{1}{n}-\gamma)t} \\ &\leq ce^{-\frac{1}{2n}(n+\omega-4)t}, \end{aligned}$$

in view of (7.24).

8. TRANSITION FROM BIG CRUNCH TO BIG BANG

We shall define a new spacetime \hat{N} by reflection and time reversal such that the IMCF in the old spacetime transforms to an IMCF in the new one.

By switching the light cone we obtain a new spacetime \hat{N} . The flow equation in N is independent of the time orientation, and we can write it as

$$(8.1) \quad \dot{x} = -\check{H}^{-1}\check{\nu} = -(-\check{H})^{-1}(-\check{\nu}) \equiv -\hat{H}^{-1}\hat{\nu},$$

where the normal vector $\hat{\nu} = -\check{\nu}$ is past directed in \hat{N} and the mean curvature $\hat{H} = -\check{H}$ negative.

Introducing a new time function $\hat{x}^0 = -x^0$ and formally new coordinates (\hat{x}^α) by setting

$$(8.2) \quad \hat{x}^0 = -x^0, \quad \hat{x}^i = x^i,$$

we define a spacetime \hat{N} having the same metric as N —only expressed in the new coordinate system—such that the flow equation has the form

$$(8.3) \quad \dot{\hat{x}} = -\hat{H}^{-1}\hat{\nu},$$

where $M(t) = \text{graph } \hat{u}(t)$, $\hat{u} = -u$, and

$$(8.4) \quad (\hat{\nu}^\alpha) = -\tilde{\nu}e^{-\tilde{\psi}}(1, \hat{u}^i)$$

in the new coordinates, since

$$(8.5) \quad \hat{\nu}^0 = -\check{\nu}^0 \frac{\partial \hat{x}^0}{\partial x^0} = \nu^0$$

and

$$(8.6) \quad \hat{\nu}^i = -\check{\nu}^i.$$

The singularity in $\hat{x}^0 = 0$ is now a past singularity, and can be referred to as a big bang singularity.

The union $N \cup \hat{N}$ is a smooth manifold, topologically a product $(-a, a) \times \mathcal{S}_0$ —we are well aware that formally the singularity $\{0\} \times \mathcal{S}_0$ is not part of the union; equipped with the respective metrics and time orientation it is a spacetime which has a (metric) singularity in $x^0 = 0$. The time function

$$(8.7) \quad \hat{x}^0 = \begin{cases} x^0, & \text{in } N, \\ -x^0, & \text{in } \hat{N}, \end{cases}$$

is smooth across the singularity and future directed.

$N \cup \hat{N}$ can be regarded as a *cyclic universe* with a contracting part $N = \{\hat{x}^0 < 0\}$ and an expanding part $\hat{N} = \{\hat{x}^0 > 0\}$ which are joined at the singularity $\{\hat{x}^0 = 0\}$, cf. [5, 6] for similar ideas.

We shall show that the inverse mean curvature flow, properly rescaled, defines a natural C^3 -diffeomorphism across the singularity and with respect to this diffeomorphism we speak of a transition from big crunch to big bang.

Using the time function in (8.7) the inverse mean curvature flows in N and \hat{N} can be uniformly expressed in the form

$$(8.8) \quad \dot{\hat{x}} = -\hat{H}^{-1}\hat{\nu},$$

where (8.8) represents the original flow in N , if $\hat{x}^0 < 0$, and the flow in (8.3), if $\hat{x}^0 > 0$.

Let us now introduce a new flow parameter

$$(8.9) \quad s = \begin{cases} -\gamma^{-1}e^{-\gamma t}, & \text{for the flow in } N, \\ \gamma^{-1}e^{-\gamma t}, & \text{for the flow in } \hat{N}, \end{cases}$$

and define the flow $y = y(s)$ by $y(s) = \hat{x}(t)$. $y = y(s, \xi)$ is then defined in $[-\gamma^{-1}, \gamma^{-1}] \times \mathcal{S}_0$, smooth in $\{s \neq 0\}$, and satisfies the evolution equation

$$(8.10) \quad y' \equiv \frac{d}{ds}y = \begin{cases} -\hat{H}^{-1}\hat{\nu}e^{\gamma t}, & s < 0, \\ \hat{H}^{-1}\hat{\nu}e^{\gamma t}, & s > 0. \end{cases}$$

8.1. Theorem. *The flow $y = y(s, \xi)$ is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times \mathcal{S}_0$ and defines a natural diffeomorphism across the singularity. The flow parameter s can be used as a new time function.*

The flow y is certainly continuous across the singularity, and also future directed, i.e., it runs into the singularity, if $s < 0$, and moves away from it, if $s > 0$.

The continuous differentiability of $y = y(s, \xi)$ with respect s and ξ up to order three will be proved in a series of lemmata.

As in the previous sections we again view the hypersurfaces as embeddings with respect to the ambient metric

$$(8.11) \quad d\bar{s}^2 = -(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j.$$

The flow equation for $s < 0$ can therefore be written as

$$(8.12) \quad y' = -F^{-1}\nu e^{\gamma t}.$$

8.2. Lemma. *y is of class C^1 in $(-\gamma^{-1}, \gamma^{-1}) \times \mathcal{S}_0$.*

Proof. Here, as in the proofs to come, we have to show that y' and y_i are continuous in $\{0\} \times \mathcal{S}_0$.

Now, we have

$$(8.13) \quad y^0(s) = x^0(t), \quad y^i(s) = x^i(t) \quad \forall s < 0,$$

and

$$(8.14) \quad y^0(s) = -x^0(t), \quad y^i(s) = x^i(t) \quad \forall s > 0,$$

hence y' is continuous across the singularity if and only if

$$(8.15) \quad \lim_{s \uparrow 0} \frac{d}{ds} y^0 = \lim_{s \downarrow 0} \frac{d}{ds} y^0,$$

and

$$(8.16) \quad \lim_{s \uparrow 0} \frac{d}{ds} y^i = - \lim_{s \downarrow 0} \frac{d}{ds} y^i.$$

Furthermore, we have to show that

$$(8.17) \quad \lim_{s \uparrow 0} y_i^0 = 0$$

and

$$(8.18) \quad \lim_{s \uparrow 0} y_i^j = \lim_{s \downarrow 0} y_i^j.$$

The last two relations are obviously valid.

To verify (8.15) and (8.16) we observe

8.3. Remark. The limit relations for $\langle D^m y, \frac{\partial}{\partial x^0} \rangle$ and $\langle D^m y, \frac{\partial}{\partial x^i} \rangle$, where $D^m y$ stands for covariant derivatives of order m of y with respect to s or ξ^i , are identical to those for $\langle D^m y, \nu \rangle$ and $\langle D^m y, x_i \rangle$, because ν converges to $-\frac{\partial}{\partial x^0}$, if $s \uparrow 0$.

Thus, in view of (8.10) and (8.12), it suffices to prove the convergence of $F e^{-\gamma t}$, if t goes to infinity. But this has already been shown in the proof of Corollary 5.4, cf. equation (5.30). \square

Let us examine the second derivatives.

8.4. Lemma. y is of class C^2 in $(-\gamma^{-1}, \gamma^{-1}) \times \mathcal{S}_0$.

Proof. „ y'_i “ The normal component of y'_i has to converge and the tangential components have to converge to zero.

We may only consider the behaviour for $s < 0$. Then

$$(8.19) \quad y' = -F^{-1} e^{\gamma t} \nu$$

and

$$(8.20) \quad y'_i = F^{-2} F_i e^{\gamma t} \nu - F^{-1} e^{\gamma t} \nu_i$$

The normal component is therefore equal to

$$(8.21) \quad F^{-2} e^{\gamma t} (H_i - n \tilde{\nu}_i f' - n \tilde{\nu} f'' u_i + n \psi_{\alpha\beta} \nu^\alpha x_i^\beta, + n \psi_\alpha \nu_i^\alpha)$$

which converges to

$$(8.22) \quad \lim -F^{-2} e^{2\gamma t} n f'' u^2 \tilde{u}_i \tilde{u}^{-1} = \gamma \tilde{u} \tilde{u}_i.$$

The tangential components are equal to

$$(8.23) \quad -F^{-1}e^{\gamma t}h_i^k,$$

which converge to zero.

„ y_{ij} “ The Gauß formula yields

$$(8.24) \quad y_{ij} = h_{ij}\nu,$$

which converges to zero as it should.

„ y'' “ Here, the normal component has to converge to zero, while the tangential ones have to converge.

We get for $s < 0$

$$(8.25) \quad \begin{aligned} y'' &= -\frac{D}{dt}(F^{-1}\nu)e^{2\gamma t} - F^{-1}\nu\gamma e^{2\gamma t} \\ &= -F^{-1}\dot{\nu}e^{2\gamma t} + F^{-2}\nu\dot{F}e^{2\gamma t} - F^{-1}\nu\gamma e^{2\gamma t}. \end{aligned}$$

The normal component is equal to

$$(8.26) \quad F^{-2}e^{2\gamma t}(\dot{H} - n\dot{\tilde{v}}f' - n\tilde{v}f''\dot{u} + n\psi_{\alpha\beta}\nu^\alpha\dot{x}^\beta + n\psi_\alpha\dot{\nu}^\alpha - \gamma F).$$

$F^{-2}e^{2\gamma t}$ converges, all other terms converge to zero with the possible exception of

$$(8.27) \quad -n\tilde{v}f''\dot{u} - \gamma F = -F^{-1}n(\tilde{v}^2f'' + \frac{1}{n}\gamma F^2)$$

which however converges to zero too, in view of (4.37) and the estimate for $|H|$.

The tangential components are equal to

$$(8.28) \quad \begin{aligned} F^{-1}D_i(F^{-1})e^{2\gamma t} &= -F^{-3}e^{2\gamma t}(H_i - n\tilde{v}_if' - n\tilde{v}f''u_i \\ &\quad + n\psi_{\alpha\beta}\nu^\alpha x_i^\beta + n\psi_\alpha\nu_i^\alpha), \end{aligned}$$

which converge to

$$(8.29) \quad \lim F^{-3}e^{3\gamma t}n\tilde{v}f''u_i^2\tilde{u}_i\tilde{u}^{-2}. \quad \square$$

8.5. **Lemma.** y is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times \mathcal{S}_0$.

Proof. „ y_{ijk} “ Now, the normal component has to converge to zero, while the tangential ones have to converge. Again we look at $s < 0$ and get

$$(8.30) \quad y_{ij} = h_{ij}\nu,$$

$$(8.31) \quad y_{ijk} = h_{ijk}\nu + h_{ij}\nu_k.$$

Hence, y_{ijk} converges to zero.

„ y'_{ij} “ The normal component has to converge, while the tangential ones should converge to zero.

Using the Ricci identities it can be easily checked that, instead of y'_{ij} , we may look at $\frac{D}{ds}(y_{ij})$, since

$$(8.32) \quad \bar{R}_{0i0j} \rightarrow 0,$$

cf. Lemma 4.4.

From (8.30) we deduce

$$(8.33) \quad \frac{D}{ds}y_{ij} = \dot{h}_{ij}\nu e^{\gamma t} + h_{ij}\dot{\nu}e^{\gamma t},$$

and conclude further that the normal component converges, in view of Corollary 7.3, and the tangential ones converge to zero, since $\dot{\nu}$ vanishes in the limit.

„ y_i'' “ The normal component has to converge to zero and the tangential ones have to converge.

From (8.25) we infer

$$(8.34) \quad \begin{aligned} y'' &= -F^{-3}e^{2\gamma t}(H^k - n\tilde{v}^k f' - n f'' u^k + n(\psi_\alpha \nu^\alpha)^k)x_k \\ &\quad + F^{-2}e^{2\gamma t}(\dot{H} - n\dot{v}f' + n\frac{D}{dt}(\psi_\alpha \nu^\alpha))\nu \\ &\quad + F^{-3}e^{2\gamma t}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + n^2(\psi_\alpha \nu^\alpha)^2 - 2nHf'\tilde{v} \\ &\quad \quad + 2nH\psi_\alpha \nu^\alpha - 2n^2f'\tilde{v}\psi_\alpha \nu^\alpha])\nu \end{aligned}$$

and thus

$$(8.35) \quad \begin{aligned} y_i'' &= -(F^{-3}e^{2\gamma t}(H^k - n\tilde{v}^k f' - n f'' u^k + (n\psi_\alpha \nu^\alpha)^k))_i x_k \\ &\quad - F^{-3}e^{2\gamma t}(H^k - n\tilde{v}^k f' - n f'' u^k + (n\psi_\alpha \nu^\alpha)^k)h_{ik}\nu \\ &\quad + (F^{-2}e^{2\gamma t}(\dot{H} - n\dot{v}f' + \frac{D}{dt}(n\psi_\alpha \nu^\alpha)))_i \nu \\ &\quad + F^{-2}e^{2\gamma t}(\dot{H} - n\dot{v}f' + \frac{D}{dt}(n\psi_\alpha \nu^\alpha))\nu_i \\ &\quad + (F^{-3}e^{2\gamma t}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + (n\psi_\alpha \nu^\alpha)^2 - 2nHf'\tilde{v} \\ &\quad \quad + 2Hn\psi_\alpha \nu^\alpha - 2nf'\tilde{v}n\psi_\alpha \nu^\alpha]))_i \nu \\ &\quad + F^{-3}e^{2\gamma t}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + (n\psi_\alpha \nu^\alpha)^2 - 2nHf'\tilde{v} \\ &\quad \quad + 2Hn\psi_\alpha \nu^\alpha - 2nf'\tilde{v}n\psi_\alpha \nu^\alpha])\nu_i. \end{aligned}$$

Therefore, the normal component converges to zero, while the tangential ones converge.

„ y''' “ The normal component has to converge, while the tangential ones have to converge to zero.

Differentiating the equation (8.34) we get

$$\begin{aligned}
(8.36) \quad y''' &= 3F^{-4}e^{3\gamma t}\dot{F}(H^k - n\tilde{v}^k f' - nf''u^k + (n\psi_\alpha\nu^\alpha)^k)x_k \\
&\quad - 2\gamma F^{-3}e^{3\gamma t}(H^k - n\tilde{v}^k f' - nf''u^k + (n\psi_\alpha\nu^\alpha)^k)x_k \\
&\quad - F^{-3}e^{3\gamma t}\frac{D}{dt}(H^k - n\tilde{v}^k f' - nf''u^k + (n\psi_\alpha\nu^\alpha)^k)x_k \\
&\quad - F^{-3}e^{3\gamma t}(H^k - n\tilde{v}^k f' - nf''u^k + (n\psi_\alpha\nu^\alpha)^k)\dot{x}_k \\
&\quad - 2F^{-3}e^{3\gamma t}\dot{F}(\dot{H} - n\dot{\tilde{v}}f' + \frac{D}{dt}(n\psi_\alpha\nu^\alpha))\nu \\
&\quad + 2\gamma F^{-2}e^{3\gamma t}(\dot{H} - n\dot{\tilde{v}}f' + \frac{D}{dt}(n\psi_\alpha\nu^\alpha))\nu \\
&\quad + F^{-2}e^{3\gamma t}\frac{D}{dt}(\dot{H} - n\dot{\tilde{v}}f' + \frac{D}{dt}(n\psi_\alpha\nu^\alpha))\nu \\
&\quad + F^{-2}e^{3\gamma t}(\dot{H} - n\dot{\tilde{v}}f' + \frac{D}{dt}(n\psi_\alpha\nu^\alpha))\dot{\nu} \\
&\quad - 3F^{-4}e^{3\gamma t}\dot{F}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + (n\psi_\alpha\nu^\alpha)^2 - 2nHf'\tilde{v} \\
&\quad \quad + 2Hn\psi_\alpha\nu^\alpha - 2nf'\tilde{v}n\psi_\alpha\nu^\alpha])\nu \\
&\quad + 2\gamma F^{-3}e^{3\gamma t}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + (n\psi_\alpha\nu^\alpha)^2 - 2nHf'\tilde{v} \\
&\quad \quad + 2Hn\psi_\alpha\nu^\alpha - 2nf'\tilde{v}n\psi_\alpha\nu^\alpha])\nu \\
&\quad + F^{-3}e^{3\gamma t}\frac{D}{dt}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + (n\psi_\alpha\nu^\alpha)^2 - 2nHf'\tilde{v} \\
&\quad \quad + 2Hn\psi_\alpha\nu^\alpha - 2nf'\tilde{v}n\psi_\alpha\nu^\alpha])\nu \\
&\quad + F^{-3}e^{3\gamma t}(-n\tilde{v}^2[f'' + \tilde{\gamma}|f'|^2] - \gamma[H^2 + (n\psi_\alpha\nu^\alpha)^2 - 2nHf'\tilde{v} \\
&\quad \quad + 2Hn\psi_\alpha\nu^\alpha - 2nf'\tilde{v}n\psi_\alpha\nu^\alpha])\dot{\nu}
\end{aligned}$$

Observing that

$$(8.37) \quad \dot{x}_k = F^{-2}F_k\nu - F^{-1}\nu_k$$

and

$$(8.38) \quad \dot{u}_k = F^{-1}\tilde{v}_k - F^{-2}\tilde{v}F_k$$

and taking the results of Lemma 7.6, Lemma 7.7, and Corollary 7.8 into account we conclude that the normal component converges.

The tangential components contain the following crucial terms

$$\begin{aligned}
(8.39) \quad &3F^{-4}e^{3\gamma t}n^2\tilde{v}^2|f''|^2u^k\dot{u} + 2\gamma F^{-3}e^{3\gamma t}n\tilde{v}f''u^k \\
&+ F^{-3}e^{3\gamma t}n\tilde{v}f'''u^k\dot{u} + F^{-5}e^{3\gamma t}n^2\tilde{v}^3|f''|^2u^k,
\end{aligned}$$

which can be rearranged to yield

$$(8.40) \quad F^{-5}e^{3\gamma t}n\tilde{v}u^k(4f''(f'' + \tilde{\gamma}|f'|^2) - f'(f'' + \tilde{\gamma}|f'|^2)).$$

Hence, the tangential components tend to zero.

The remaining mixed derivatives of y , which are obtained by commuting the order of differentiation in the derivatives we already treated, are also continuous across the singularity in view of the Ricci identities and (8.32). \square

9. ARW SPACES AND THE EINSTEIN EQUATIONS

Let N be a cosmological spacetime such that the metric has the form as specified in Definition 0.1, though, with regard to f , we only assume at the moment that f is smooth and satisfies

$$(9.1) \quad \lim_{\tau \rightarrow b} f(\tau) = -\infty$$

and

$$(9.2) \quad \lim_{\tau \rightarrow b} f' = -\infty.$$

The conformal metric

$$(9.3) \quad d\bar{s}^2 = e^{2\psi}(-(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j)$$

should satisfy the conditions in Definition 0.1, and, in addition, the partial derivatives of ψ as well as the second fundamental form of the coordinate slices $\{x^0 = \text{const}\}$ and its derivatives should be integrable over the range $[a, b)$ of x^0 .

In contrast to the previous sections we suppose that the Einstein equations are valid

$$(9.4) \quad G_{\alpha\beta} = \kappa T_{\alpha\beta},$$

where κ is a positive constant, and the stress-energy tensor is asymptotically equal to that of a perfect fluid.

9.1. Definition. Let x^0 be a time function such the preceding assumptions are satisfied. A symmetric, divergent free tensor $(T_{\alpha\beta})$ is said to be asymptotically equal to that of a perfect fluid with respect to the future, if the mixed tensor (T_{β}^{α}) splits in the form

$$(9.5) \quad T_{\beta}^{\alpha} = \bar{T}_{\beta}^{\alpha} + \hat{T}_{\beta}^{\alpha},$$

where $(\bar{T}_{\beta}^{\alpha})$ is the stress-energy tensor of a perfect fluid, i.e.,

$$(9.6) \quad \bar{T}_0^0 = -\rho, \quad T_i^{\alpha} = \delta_i^{\alpha} p;$$

$0 \leq \rho$ is the *density* and p is the *pressure*, and $(\hat{T}_{\beta}^{\alpha})$ as well as its partial derivatives of arbitrary order are supposed to vanish, if x^0 tends to b , and they should be integrable over the range $[a, b)$ of x^0 . Moreover, $\hat{T}_{\beta}^{\alpha} f'$ should vanish and be integrable as well.

Let us assume an *equation of state*

$$(9.7) \quad p = \frac{\omega}{n} \rho$$

holds, where $\omega \in \mathbb{R}$ is a constant such that

$$(9.8) \quad n + \omega - 2 > 0.$$

We shall now show that, because of the Einstein equations, f has to satisfy the conditions stated in Definition 0.1, even slightly stronger ones.

First, we prove

9.2. Lemma. *There exist τ_0 and $c > 0$ such that*

$$(9.9) \quad \rho(\tau, x) \geq c > 0 \quad \forall \tau \geq \tau_0, \forall x \in \mathcal{S}_0.$$

Moreover,

$$(9.10) \quad \lim_{\tau \rightarrow b} \rho = \infty.$$

Proof. We use the Einstein equations

$$(9.11) \quad G_{00} = \kappa T_{00}$$

to conclude

$$(9.12) \quad \frac{1}{2}n(n-1)|f'|^2 + \frac{1}{2}\bar{R} + \epsilon = \kappa\rho e^{2\bar{\psi}} + \kappa\hat{T}_{00},$$

where we recall that \bar{R} is the scalar curvature of the metric in (9.3), and where ϵ represents terms that converge to zero, if τ tends to b , or equivalently,

$$(9.13) \quad \frac{1}{2}n(n-1)|f'|^2 e^{-2\bar{\psi}} + \frac{1}{2}\bar{R}e^{-2\bar{\psi}} + \epsilon e^{-2\bar{\psi}} = \kappa\rho + \kappa\hat{T}_0^0.$$

Hence, we have

$$(9.14) \quad \kappa\rho \sim \frac{1}{2}n(n-1)|f'|^2 e^{-2f},$$

which proves the result, in view of (9.1) and (9.2). \square

9.3. Lemma. *Let $\tilde{\gamma} = \frac{1}{2}(n + \omega - 2)$, then there exists a constant $m > 0$ such that*

$$(9.15) \quad \lim_{\tau \rightarrow b} |f'|^2 e^{2\tilde{\gamma}f} = m$$

and

$$(9.16) \quad |D^m f| \leq c_m |f'|^m \quad \forall m \in \mathbb{N}.$$

Furthermore, the limit metric $(\bar{\sigma}_{ij})$ must have constant scalar curvature R . The function

$$(9.17) \quad \varphi = f'' + \tilde{\gamma}|f'|^2,$$

converges to

$$(9.18) \quad \lim_{\tau \rightarrow b} \varphi = -\frac{\gamma}{n-1}R,$$

where $\gamma = \frac{1}{n}\tilde{\gamma}$, and in addition

$$(9.19) \quad \lim_{\tau \rightarrow b} D^m \varphi = 0 \quad \forall m \in \mathbb{N}^*.$$

Proof. „(9.15)“ Since $(T_{\alpha\beta})$ is divergent free, we deduce

$$(9.20) \quad \begin{aligned} 0 &= T_{0;\gamma}^\gamma = T_{0,\gamma}^\gamma + \check{I}_{\gamma\alpha}^\gamma T_0^\alpha - \check{I}_{0\gamma}^\alpha T_\alpha^\gamma \\ &= -\dot{\rho} - (n + \omega)\dot{\check{\psi}}\rho - \frac{1}{2}\dot{\sigma}_i^i(1 + \frac{\omega}{n})\rho + \mathcal{C}, \end{aligned}$$

where \mathcal{C} tends to zero and is integrable over the range $[a, b]$ of x^0 .

In view of Lemma 9.2 we deduce

$$(9.21) \quad \frac{d}{dt} \log \rho = -(n + \omega)\dot{\check{\psi}} + \mathcal{C},$$

where we still use the same symbol \mathcal{C} , and hence, for fixed x ,

$$(9.22) \quad \rho(\tau, x)e^{(n+\omega)\check{\psi}(\tau, x)} = \rho(\tau', x)e^{(n+\omega)\check{\psi}(\tau', x)}e^{\int_{\tau'}^\tau \mathcal{C}}.$$

Thus, we conclude, first, that $\rho(\tau, x)e^{(n+\omega)\check{\psi}(\tau, x)}$ is uniformly bounded, and then, that it converges to a positive function, if τ tends to b .

At the moment the limit can depend on the spatial variables x , but we shall see immediately that it is a constant.

Now, multiplying equation (9.13) with $e^{(n+\omega)\check{\psi}}$ we deduce

$$(9.23) \quad \lim |f'|^2 e^{(n+\omega-2)f} = \frac{2}{n(n-1)}\kappa \lim \rho e^{(n+\omega)f},$$

i.e., the limit on the left-hand side exists, and the limit on the right-hand side is a constant.

„(9.16)“ We consider the contracted version of the Einstein equations

$$(9.24) \quad G_\alpha^\alpha = \kappa T_\alpha^\alpha$$

and infer with the help of equation (9.13)

$$(9.25) \quad \begin{aligned} \check{R} &= \frac{2}{n-1}\kappa\rho(1 - \omega) + \mathcal{C} \\ &= n|f'|^2 e^{-2\check{\psi}}(1 - \omega) + \frac{1}{n-1}\bar{R}e^{-2\check{\psi}}(1 - \omega) + \epsilon e^{-2\check{\psi}} + \mathcal{C}, \end{aligned}$$

and we further conclude

$$(9.26) \quad \frac{\gamma}{n-1}\bar{R} + f'' + \frac{1}{2}(n + \omega - 2)|f'|^2 = \epsilon + \mathcal{C}e^{2\check{\psi}}.$$

The estimate in (9.16) now follows immediately by induction.

„(9.18) and (9.19)“ One easily checks that

$$(9.27) \quad \lim_{\tau \rightarrow b} \bar{R} = R,$$

where R is the scalar curvature of $(\bar{\sigma}_{ij})$.

The relation (9.26) implies that φ is uniformly Lipschitz continuous and bounded, hence there exists a sequence $\tau_k \rightarrow b$ such that $\varphi(\tau_k)$ converges, from which we deduce that R has to be constant. Therefore, $\varphi = f'' + \tilde{\gamma}|f'|^2$ converges.

Moreover, after having established the relation (9.15), we can apply the result of Lemma 3.1, i.e., b is finite, and without loss of generality we may assume that $b = 0$, which in turn allows us to conclude that derivatives of

arbitrary order of the right-hand side of (9.26) tend to zero in the limit, cf. Lemma 4.4.

This completes the proof of the lemma. \square

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RUPRECHT-KARLS-UNIVERSITÄT, INSTITUT FÜR ANGEWANDTE MATHEMATIK, IM NEUENHEIMER FELD 294, 69120 HEIDELBERG, GERMANY

E-mail address: gerhardt@math.uni-heidelberg.de

URL: <http://www.math.uni-heidelberg.de/studinfo/gerhardt/>