

AN ENERGY GAP FOR YANG-MILLS CONNECTIONS

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ABSTRACT. Consider a Yang-Mills connection over a Riemann manifold $M = M^n$, $n \geq 3$, where M may be compact or complete. Then its energy must be bounded from below by some positive constant, if M satisfies certain conditions, unless the connection is flat.

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1. INTRODUCTION

We consider the problem: When is a Yang-Mills connection non-flat? Of course, the trivial answer $F_{\mu\lambda} \not\equiv 0$ is unsatisfactory. Bourguignon and Lawson proved in [3, Theorem C], among other results, that any Yang-Mills connection over S^n , $n \geq 3$, the field strength of which satisfies the pointwise estimate

$$(1.1) \quad F^2 = -\operatorname{tr}(F_{\mu\lambda}F^{\mu\lambda}) < \binom{n}{2}$$

is flat.

We want to prove that under certain assumptions on the base space M , which is supposed to be a Riemannian manifold of dimension $n \geq 3$, the energy of a Yang-Mills connection has to satisfy

$$(1.2) \quad \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \geq \kappa_0 > 0,$$

where κ_0 depends only on the Sobolev constants of M , n and the dimension of the Lie group \mathcal{G} , unless the connection is flat.

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Here,

$$(1.3) \quad |F| = \sqrt{F^2},$$

and we also call the left-hand side of (1.2) energy though this label is only correct when $n = 4$. However, this norm is also the crucial norm, which has to be (locally) small, used to prove regularity of a connection, cf. [4, Theorem 1.3].

The exponent $\frac{n}{2}$ naturally pops up when Sobolev inequalities are applied to solutions of differential equations satisfied by the field strength or the energy density of a connection in the adjoint bundle.

We distinguish two cases: M compact and M complete and non-compact. When M is compact, we require

$$(1.4) \quad \bar{R}_{\alpha\beta} \Lambda_\lambda^\alpha \Lambda^{\beta\lambda} - \frac{1}{2} \bar{R}_{\alpha\beta\mu\lambda} \Lambda^{\alpha\beta} \Lambda^{\mu\lambda} \geq c_0 \Lambda_{\alpha\beta} \Lambda^{\alpha\beta}$$

for all skew-symmetric $\Lambda_{\alpha\beta} \in T^{0,2}(M)$, where $0 < c_0$, while for non-compact M the weaker assumption

$$(1.5) \quad \bar{R}_{\alpha\beta} \Lambda_\lambda^\alpha \Lambda^{\beta\lambda} - \frac{1}{2} \bar{R}_{\alpha\beta\mu\lambda} \Lambda^{\alpha\beta} \Lambda^{\mu\lambda} \geq 0$$

and in addition

$$(1.6) \quad \left(\int_M u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c_1 \int_M |Du|^2 \quad \forall u \in H^{1,2}(M)$$

should be satisfied.

1.1. **Remark.** (i) If M is a space of constant curvature

$$(1.7) \quad \bar{R}_{\alpha\beta\mu\lambda} = K_M (\bar{g}_{\alpha\mu} \bar{g}_{\beta\lambda} - \bar{g}_{\alpha\lambda} \bar{g}_{\beta\mu}),$$

then

$$(1.8) \quad \bar{R}_{\alpha\beta} \Lambda_\lambda^\alpha \Lambda^{\beta\lambda} - \frac{1}{2} \bar{R}_{\alpha\beta\mu\lambda} \Lambda^{\alpha\beta} \Lambda^{\mu\lambda} = (n-2) K_M \Lambda_{\alpha\beta} \Lambda^{\alpha\beta}.$$

In case $n = 2$ the curvature term therefore vanishes, and this result is also valid for an arbitrary two-dimensional Riemannian manifold, since the curvature tensor then has the same structure as in (1.7) though K_M is not necessarily constant.

(ii) If $M = \mathbb{R}^n$, $n \geq 3$, the conditions (1.5) and (1.6) are always valid.

1.2. **Theorem.** *Let $M = M^n$, $n \geq 3$, be a compact Riemannian manifold for which the condition (1.4) with $c_0 > 0$ holds. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or satisfies (1.2) for some constant $\kappa_0 > 0$ depending on the Sobolev constants of M , n, c_0 , and the dimension of the Lie group.*

1.3. **Theorem.** *Let $M = M^n$, $n \geq 3$, be complete, non-compact and assume that the conditions (1.5) and (1.6) hold. Then any Yang-Mills connection over M with compact, semi-simple Lie group is either flat or the estimate (1.2) is valid. The constant $\kappa_0 > 0$ in (1.2) depends on the constant c_1 in (1.6), n , and the dimension of the Lie group.*

2. THE COMPACT CASE

Let $(P, M, \mathcal{G}, \mathcal{G})$ be a principal fiber bundle where $M = M^n$, $n \geq 3$ is a compact Riemannian manifold with metric $\bar{g}_{\alpha\beta}$ and \mathcal{G} a compact, semi-simple Lie group with Lie algebra \mathfrak{g} . Let $f_c = (f_{cb}^a)$ be a basis of $\text{ad } \mathfrak{g}$ and

$$(2.1) \quad A_\mu = f_c A_\mu^c$$

a Yang-Mills connection in the adjoint bundle $(E, M, \mathfrak{g}, \text{Ad}(\mathcal{G}))$.

The curvature tensor of the connection is given by

$$(2.2) \quad R_{b\mu\lambda}^a = f_{cb}^a F_{\mu\lambda}^c,$$

where

$$(2.3) \quad F_{\mu\lambda} = f_c F_{\mu\lambda}^c$$

is the field strength of the connection, and

$$(2.4) \quad F^2 \equiv \gamma_{ab} F_{\mu\lambda}^a F^{b\mu\lambda} = R_{ab\mu\lambda} R^{ab\mu\lambda}$$

the energy density of the connection—at least up to a factor $\frac{1}{4}$.

Here, γ_{ab} is the Cartan-Killing metric acting on elements of the fiber \mathfrak{g} , and Latin indices are raised or lowered with respect to the inverse γ^{ab} or γ_{ab} , and Greek indices with respect to the metric of M .

2.1. Definition. The adjoint bundle E is vector bundle; let E^* be the dual bundle, then we denote by

$$(2.5) \quad T^{r,s}(E) = \Gamma(\underbrace{E \otimes \cdots \otimes E}_r \otimes \underbrace{E^* \otimes \cdots \otimes E^*}_s)$$

the sections of the corresponding tensor bundle.

Thus, we have

$$(2.6) \quad F_{\mu\lambda}^a \in T^{1,0}(E) \otimes T^{0,2}(M).$$

Since A_μ is a Yang-Mills connection it solves the Yang-Mills equation

$$(2.7) \quad F^{a\alpha}_{\lambda;\alpha} = 0,$$

where we use Einstein's summation convention, a semi-colon indicates covariant differentiation, and where we stipulate that a covariant derivative is always a *full* tensor, i.e.,

$$(2.8) \quad F_{\mu\lambda;\alpha}^a = F_{\mu\lambda,\alpha}^a + f_{bc}^a A_\alpha^b F_{\mu\lambda}^c - \bar{\Gamma}_{\alpha\mu}^\gamma F_{\gamma\lambda}^a - \bar{\Gamma}_{\alpha\lambda}^\gamma F_{\mu\gamma}^a,$$

where $\bar{\Gamma}_{\alpha\beta}^\gamma$ are the Christoffel symbols of the Riemannian connection; a comma indicates partial differentiation.

Before we formulate the crucial lemma let us note that $\bar{R}_{\alpha\beta\gamma\delta}$ resp. $\bar{R}_{\alpha\beta}$ symbolize the Riemann curvature tensor resp. the Ricci tensor of $\bar{g}_{\alpha\beta}$.

2.2. Lemma. *Let A_μ be a Yang-Mills connection, then its energy density F^2 solves the equation*

$$(2.9) \quad \begin{aligned} & -\frac{1}{4}\Delta F^2 + \frac{1}{2}F_{a\mu\lambda;\alpha}F^{a\mu\lambda\alpha} + \bar{R}_{\beta\mu}F^{a\beta}{}_\lambda F_a{}^{\mu\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}F_a{}^{\alpha\beta}F^{a\mu\lambda} \\ & = -f_{cb}^a F_{\alpha\mu}^c F^{b\alpha}{}_\lambda F_a{}^{\mu\lambda}. \end{aligned}$$

Proof. Differentiating (2.7) covariantly with respect to x^μ and using the Ricci identities we obtain

$$(2.10) \quad \begin{aligned} 0 & = -F^{a\alpha}{}_{\lambda;\alpha\mu} \\ & = -F^{a\alpha}{}_{\lambda;\mu\alpha} + R^a{}_{b\alpha\mu}F^{b\alpha}{}_\lambda + \bar{R}^\alpha{}_{\beta\alpha\mu}F^{a\beta}{}_\lambda + \bar{R}^\beta{}_{\lambda\mu\alpha}F^{a\alpha}{}_\beta. \end{aligned}$$

On the other hand, differentiating the second Bianchi identities

$$(2.11) \quad 0 = F_{\alpha\lambda;\mu}^a + F_{\mu\alpha;\lambda}^a + F_{\lambda\mu;\alpha}^a$$

we infer

$$(2.12) \quad 0 = F^{a\alpha}{}_{\lambda;\mu\alpha} + F^{a\alpha}{}_{\mu;\lambda\alpha} + \Delta F_{\lambda\mu}^a,$$

and we deduce further

$$(2.13) \quad -\Delta F_{\mu\lambda}^a F_a{}^{\mu\lambda} = -2F^{a\alpha}{}_{\lambda;\mu\alpha}F_a{}^{\mu\lambda}.$$

In view of (2.10) we then conclude

$$(2.14) \quad \begin{aligned} 0 & = -\frac{1}{2}\Delta F_{\mu\lambda}^a F_a{}^{\mu\lambda} + R^a{}_{b\alpha\mu}F^{b\alpha}{}_\lambda F_a{}^{\mu\lambda} + \bar{R}_{\beta\mu}F^{a\beta}{}_\lambda F_a{}^{\mu\lambda} \\ & \quad + \bar{R}^\beta{}_{\lambda\mu\alpha}F^{a\alpha}{}_\beta F_a{}^{\mu\lambda}, \end{aligned}$$

which is equivalent to

$$(2.15) \quad \begin{aligned} 0 & = -\frac{1}{2}\Delta F_{\mu\lambda}^a F_a{}^{\mu\lambda} + f_{cb}^a F_{\alpha\mu}^c F^{b\alpha}{}_\lambda F_a{}^{\mu\lambda} + \bar{R}_{\beta\mu}F^{a\beta}{}_\lambda F_a{}^{\mu\lambda} \\ & \quad - \bar{R}_{\alpha\mu\beta\lambda}F^{a\alpha\beta}F_a{}^{\mu\lambda}, \end{aligned}$$

in view of (2.2).

Finally, using the first Bianchi identities,

$$(2.16) \quad \bar{R}_{\alpha\beta\mu\lambda} + \bar{R}_{\alpha\mu\lambda\beta} + \bar{R}_{\alpha\lambda\beta\mu} = 0,$$

we deduce

$$(2.17) \quad \bar{R}_{\alpha\beta\mu\lambda}F^{a\alpha\beta}F_a{}^{\mu\lambda} + \bar{R}_{\alpha\mu\lambda\beta}F^{a\alpha\beta}F_a{}^{\mu\lambda} + \bar{R}_{\alpha\lambda\beta\mu}F^{a\alpha\beta}F_a{}^{\mu\lambda} = 0,$$

and hence

$$(2.18) \quad \bar{R}_{\alpha\beta\mu\lambda}F^{a\alpha\beta}F_a{}^{\mu\lambda} = 2\bar{R}_{\alpha\mu\beta\lambda}F^{a\alpha\beta}F_a{}^{\mu\lambda},$$

from which the equation (2.9) immediately follows. \square

Proof of Theorem 1.2 on page 2. Define

$$(2.19) \quad u = F^2,$$

then

$$(2.20) \quad \bar{R}_{\beta\mu}F^{a\beta}{}_\lambda F_a{}^{\mu\lambda} - \frac{1}{2}\bar{R}_{\alpha\beta\mu\lambda}F_a{}^{\alpha\beta}F^{a\mu\lambda} \geq c_0 u,$$

where $c_0 > 0$, in view of the assumption (1.4) on page 2.

Multiplying (2.9) with u and integrating by part we obtain

$$(2.21) \quad \frac{3}{8} \int_M |Du|^2 + c_0 \int_M u^2 \leq c \int_M \sqrt{u} u^2,$$

where we used the simple estimate

$$(2.22) \quad |Du|^2 \leq 4F_{a\mu\lambda;\alpha} F^{a\mu\lambda;\alpha} u$$

and where c depends on n and the dimension of \mathfrak{g} ; note that

$$(2.23) \quad f_c \in \text{SO}(\mathfrak{g}, \gamma_{ab}).$$

The integral on the right-hand side of (2.21) is estimated by

$$(2.24) \quad \int_M \sqrt{u} u^2 \leq \left(\int_M u^{\frac{n}{4}} \right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}},$$

where

$$(2.25) \quad \left(\int_M u^{\frac{n}{4}} \right)^{\frac{2}{n}} = \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

Applying then the Sobolev inequality

$$(2.26) \quad \left(\int_M u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c_1 \int_M |Du|^2 + c_2 \int_M u^2,$$

cf. [1], we obtain

$$(2.27) \quad \left(\int_M u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c_3 \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}},$$

where c_3 depends on c_1, c_2, c_0 and c . Hence, we deduce $u \equiv 0$ or

$$(2.28) \quad c_3^{-1} \leq \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

Setting

$$(2.29) \quad \kappa_0 = c_3^{-1}$$

finishes the proof. \square

3. THE NON-COMPACT CASE

We now suppose that $M = M^n$ is a complete, non-compact Riemannian manifold. Then there holds

$$(3.1) \quad H^{1,2}(M) = H_0^{1,2}(M),$$

i.e., the test functions $C_c^\infty(M)$ are dense in the Sobolev space $H^{1,2}(M)$, see [1, Lemme 4] or [2, Theorem 2.6].

Since we do not a priori know

$$(3.2) \quad F^2 \in H^{1,2}(M),$$

but only

$$(3.3) \quad F^2 \in H_{\text{loc}}^{1,2}(M),$$

the preceding proof has to be modified.

Let $\eta = \eta(t)$ be defined through

$$(3.4) \quad \eta(t) = \begin{cases} 1, & t \leq 1, \\ (2-t)^q, & 1 \leq t \leq 2, \\ 0, & t \geq 2, \end{cases}$$

where

$$(3.5) \quad q = \max(1, \frac{8}{n}).$$

Fix a point $x_0 \in M$ and let r be the Riemannian distance function with center in x_0

$$(3.6) \quad r(x) = d(x_0, x).$$

Then r is Lipschitz such that

$$(3.7) \quad |Dr| = 1$$

almost everywhere.

For $k \geq 1$ define

$$(3.8) \quad \eta_k(x) = \eta(k^{-1}r).$$

The functions

$$(3.9) \quad u^{p-1}\eta_k^p,$$

where

$$(3.10) \quad p = \frac{n}{4},$$

then have compact support, and multiplying (2.9) on page 4 with $u^{p-1}\eta_k^p$ yields

$$(3.11) \quad \begin{aligned} (\frac{p}{4} + \frac{1}{8} - \epsilon) \int_M |Du|^2 u^{p-2} \eta_k^p &\leq c \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M (u\eta_k)^{\frac{n}{n-2p}} \right)^{\frac{n-2}{n}} \\ &\quad + c_\epsilon \int_M |D\eta_k|^2 \eta_k^{p-2} u^p, \end{aligned}$$

where $0 < \epsilon$ is supposed to be small.

Furthermore, there holds

$$(3.12) \quad \begin{aligned} \int_M |D(u\eta_k)^{\frac{p}{2}}|^2 &= \frac{p^2}{4} \int_M |Du\eta_k + uD\eta_k|^2 (u\eta_k)^{p-2} \\ &\leq (1+\epsilon) \frac{p^2}{4} \int_M |Du|^2 u^{p-2} \eta_k^p + c_\epsilon \frac{p^2}{4} \int_M |D\eta_k|^2 \eta_k^{p-2} u^p. \end{aligned}$$

Now, choosing ϵ so small such that

$$(3.13) \quad (1+\epsilon) \frac{p^2}{4} \leq p(\frac{p}{4} + \frac{1}{8} - \epsilon)$$

and setting

$$(3.14) \quad \varphi = (u\eta_k)^{\frac{p}{2}}$$

we obtain

$$(3.15) \quad \int_M |D\varphi|^2 \leq pc \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M \varphi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + c_\epsilon \int_M |D\eta_k|^2 \eta_k^{p-2} u^p,$$

where c_ϵ is a new constant.

We furthermore observe that

$$(3.16) \quad |D\eta_k|^2 \eta_k^{p-2} \leq q^2 k^{-2} (2 - k^{-1}r)^{qp-2},$$

subject to

$$(3.17) \quad 1 \leq k^{-1}r \leq 2.$$

In view of (3.5) and (3.10)

$$(3.18) \quad qp - 2 \geq 0$$

and hence

$$(3.19) \quad |D\eta_k|^2 \eta_k^{p-2} \leq q^2 k^{-2}.$$

Applying now the Sobolev inequality (1.6) on page 2 to φ and choosing

$$(3.20) \quad \kappa_0 = (c_1 c p)^{-1}$$

we conclude $|F| \equiv 0$, if

$$(3.21) \quad \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} < \kappa_0.$$

Indeed, if the preceding inequality is valid, then we deduce from (3.15)

$$(3.22) \quad \left(1 - \kappa_0^{-1} \left(\int_M |F|^{\frac{n}{2}} \right)^{\frac{2}{n}} \right) \left(\int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c_\epsilon q^2 k^{-2} \int_M |F|^{\frac{n}{2}}.$$

In the limit $k \rightarrow \infty$ we obtain

$$(3.23) \quad \left(\int_M |u|^{\frac{pn}{n-2}} \right)^{\frac{n-2}{n}} \leq 0.$$

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