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Curvature Problems

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Preface

Applying analytic methods to geometric problems has proved to be extremely fruitful in the last decades. Among the new techniques, with the help of which many problems have been solved, curvature flows and a priori estimates for fully non-linear elliptic partial differential equations are especially important.

The use of curvature flows started with the groundbreaking paper of Hamilton [**41**] in which he considered the Ricci flow which is driven by the Ricci curvature. Huisken [**43**] then studied the mean curvature flow. These fundamental papers created a new analytical tool for solving problems in geometry and physics.

In the present book we consider curvature problems in Riemannian and Lorentzian geometry which have in common that either the extrinsic curvature of closed hypersurfaces is prescribed or that curvature flows driven by the extrinsic curvature are studied and used to obtain some insight in the nature of possible singularities.

The first chapter provides some background material from differential geometry and some sections might even be interesting for those working in this field.

Chapter 1 gives a thorough introduction to the theory of curvature functions and extrinsic curvature flows with detailed proofs and also offers a complete proof of the short time existence and existence in a maximal time interval.

After this very general treatment of curvature flows, we consider specific geometrical problems: Either finding closed hypersurfaces of prescribed curvature, where the right-hand side is defined in the ambient space or in the tangent bundle of the ambient space, or studying the inverse mean curvature flow in Lorentzian manifolds having a future singularity in order to obtain some insight in the nature of this singularity like finding a sufficiently smooth transition from big crunch to big bang under certain circumstances.

This book is supposed to be an advanced textbook for graduate students and researchers interested in geometry and general relativity.

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Heidelberg, July 2006 Claus Gerhardt

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