TRACES AND EXTENSIONS OF BV-FUNCTIONS

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Abstract. We give a simple proof of Gagliardo's trace and extension theorem for BV-functions.

Contents

1. Introduction

Definition 1.1. We say a sequence $u_k \in BV(\Omega)$ converges in the BV sense to $u \in BV(\Omega)$, if

$$\int_\Omega |u - u_k| \to 0 \tag{1.1}$$

and

$$\int_\Omega |Du_k| \to \int_\Omega |Du|. \tag{1.2}$$

Lemma 1.2. *Let $\Omega = B_1^+(0)$ be the upper half ball and let $u \in BV(\Omega)$ have compact support with respect to the spherical boundary. Define for $0 < h$*

$$u_h(\hat{x}, x^n) = u(\hat{x}, x^n + h), \tag{1.3}$$

then u_h is defined in $\{x^n > -h\}$ and the mollification $u_{h,\epsilon}$ with an even mollifier is well defined in Ω if $0 < \epsilon < \epsilon_0(h)$. There are sequences (h_k, ϵ_k) converging to zero such that u_{h_k,ϵ_k} converge in the BV sense to u.

Proof. Let $\eta^i \in C_c^\infty(\Omega)$, $\|(\eta^i)\| \le 1$, then

$$\int_\Omega u_{h,\epsilon} D_i \eta^i = \int_\Omega u D_i \eta^i_{-h,\epsilon} \le \int_\Omega |Du|, \tag{1.4}$$

if $\epsilon \le \epsilon_0(h)$, since

$$\eta^i_{-h,\epsilon} \in C_c^\infty(\Omega) \tag{1.5}$$

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for those ϵ, where we may consider only η_i the support of which is uniformly compact with respect to the spherical boundary in view of the assumption on u. This proves the lemma. □

Lemma 1.3. *Let u, Ω be as above, then the approximating functions u_{h_k,ϵ_k}, or any other sequence $u_k \in C^1(\bar\Omega)$ converging in the BV sense to u, define a unique trace of u, $\operatorname{tr} u$, on the hyperplane such that the divergence theorem holds*

$$\int_\Omega D_i u\eta = -\int_\Omega u D_i\eta + \int_{\partial\Omega} u\eta\nu_i \tag{1.6}$$

for all $\eta \in C^1_c(\Omega_\epsilon \cup \partial\Omega)$, cf. Remark 1.6.

Extending u as an even function to the lower half space

$$\tilde u(\hat x, x^n) = \begin{cases} u(\hat x, x^n) & x^n > 0, \\ u(\hat x, -x^n) & x^n < 0, \end{cases} \tag{1.7}$$

there holds

$$\int_{B_1(0)} |D\tilde u| \le 2\int_\Omega |Du| \tag{1.8}$$

and

$$\int_{\partial\Omega} |D\tilde u| = 0. \tag{1.9}$$

Proof. Let $\eta^i \in C^1(B_1(0))$, then

$$\begin{aligned} \int_{B_1(0)} D_i\tilde u\eta^i &= \int_{B_1^+} D_i u\eta^i + \int_{B_1^-} D_i\tilde u\eta^i + \int_\Gamma D_i\tilde u\eta^i \\ &= -\int_{B_1(0)} \tilde u D_i\eta^i + \int_\Gamma (u^+ - u^+)\eta^i\nu_i + \int_\Gamma D_i\tilde u\eta^i \end{aligned} \tag{1.10}$$

hence

$$\int_\Gamma D_i\tilde u\eta^i = 0. \tag{1.11}$$

Moreover, if $\|(\eta^i)\| \le 1$, we conclude from [2, Theorem 10.10.3] and (1.10) the estimate (1.8). □

Theorem 1.4. *Let Ω be as in the theorem below, then $u \in BV(\Omega)$ can be extended to $\mathbb{R}^n$ with compact support such that, if the extended function is called $\tilde u$,*

$$\int_{\mathbb{R}^n} |D\tilde u| \le c\int_\Omega (|Du| + |u|) \tag{1.12}$$

and

$$\int_{\partial\Omega} |D\tilde u| = 0. \tag{1.13}$$

Proof. Let U_k be a finite open covering of $\bar{\Omega}$ with the usual conditions for the boundary neighbourhoods and let η_k be a subordinate finite partition of unity, then each function $u\eta_k$ can be extended to $\mathbb{R}^n$ with compact support such that, if $\tilde{u}_k$ is the extended function,

$$\int_{\mathbb{R}^n} |D\tilde{u}_k| \le c\int_{\Omega}(|Du|+|u|). \tag{1.14}$$

The function

$$\tilde{u} = \sum_k \tilde{u}_k \tag{1.15}$$

is then an extension of u with the required properties. □

The above theorem is due to [1].

Theorem 1.5. *Let $u \in BV(\Omega)$, where $\partial\Omega \in C^{0,1}$ and compact, and let $u_k \in C^1(\bar{\Omega})$ converging in the BV sense to u, at least in a boundary neighbourhood, then* $\operatorname{tr} u_k$ *converges in $L^1(\partial\Omega)$ to a uniquely defined function depending only u and not on the approximating functions; we call this limit* $\operatorname{tr} u$*. If $u \in BV(\Omega)\cap C^0(\bar{\Omega})$, then*

$$\operatorname{tr} u = u_{|\partial\Omega} \tag{1.16}$$

Proof. It suffices to prove the uniqueness of the trace, since u can be extended to $\mathbb{R}^n$, and hence also be approximated by mollifications.

Let u_k and $\tilde{u}_k$ be $C^1(\bar{\Omega})$ approximations of u and let $v_k = u_k - \tilde{u}_k$, then

$$\lim \int_{\Omega_\epsilon} |v_k| = 0, \tag{1.17}$$

where Ω_ϵ is a boundary strip of width ϵ such that

$$\int_{\{d=\epsilon\}} |Du| = 0 \tag{1.18}$$

and

$$\limsup \int_{\Omega_\epsilon} |Dv_k| \le 2\int_{\Omega_\epsilon} |Du|. \tag{1.19}$$

Now, there holds for any $\epsilon > 0$

$$\int_{\partial\Omega} |v_k| \le c\int_{\Omega_\epsilon} |Dv_k| + c_\epsilon \int_{\Omega_\epsilon} |v_k|, \tag{1.20}$$

hence

$$\limsup \int_{\partial\Omega} |v_k| \le 2c \limsup \int_{\Omega_\epsilon} |Du| = 0, \tag{1.21}$$

where the last equality is due to the fact that $|Du|$ is a bounded measure in Ω_{ϵ_0} and the Ω_ϵ are monotone ordered such that

$$\bigcap_{0<\epsilon\le\epsilon_0} \Omega_\epsilon = \emptyset. \tag{1.22}$$

□

Remark 1.6. By approximation we also obtain

$$\int_\Omega D_i u\eta = -\int_\Omega u D_i\eta + \int_{\partial\Omega} u\eta\nu_i \tag{1.23}$$

for all $\eta \in C^1_c(\Omega_\epsilon \cup \partial\Omega)$.

Remark 1.7. The definition of $BV(\Omega)$, when $\Omega \subset N$, where N is a Riemannian manifold is analogous to the case $N = \mathbb{R}^n$, namely,

$$\int_\Omega |Du| = \sup\{\int_\Omega u D_i\eta^i\colon g_{ij}\eta^i\eta^j \le 1,\ \eta^i \in C^\infty_c(\Omega)\,\}. \tag{1.24}$$

Even in the Euclidean case it is sometimes desirable to use the invariant definition of the total variation.

References

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