Regularity of Solutions of Nonlinear Variational Inequalities with a Gradient Bound as Constraint

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The aim of this paper is to obtain regularity theorems for solutions of *nonlinear variational inequalities* with a gradient bound as constraint, and in particular to generalize a result of BREZIS & STAMPACCHIA ([3; Théorème III.1]). Our results will be applicable to the elastic-plastic torsion of a cylindrical bar with a multiply connected cross section (cf. [6] for a description of this problem).

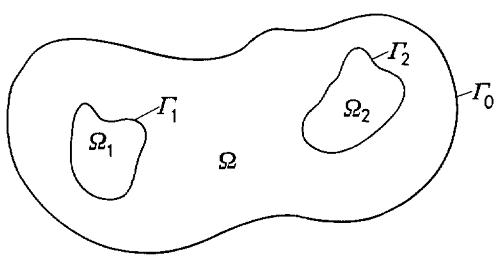

Fig. 1

Let Ω be a bounded multiply connected domain in $\mathbb{R}^N$, $N \geqq 2$, having finitely many holes Ω_k, $k=1, \ldots, n$, with respective boundaries $\Gamma_k = \partial\Omega_k$. The boundary of Ω is then the union of the disjoint family $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$. We assume moreover that $\partial\Omega$ is Lipschitz continuous and satisfies the following *outward sphere condition: for any boundary point x_0 there is a ball B of fixed radius R such that the intersection of $\bar{\Omega}$ and $\bar{B}$ consists of the point x_0 alone*[1].

We shall consider variational inequalities of the form

$$(*) \qquad \langle Au+f, v-u\rangle \geqq 0 \qquad \forall v \in \boldsymbol{K},$$

where

$$\boldsymbol{K} = \{v \in H^{1,\infty}(\Omega) \colon |Dv| \leqq 1,\ v_{|\Gamma_k} = c_k,\ k=0, \ldots, n\},$$

and where the c_k are given constants, f is a function belonging to $L^p(\Omega)$, $1<p<\infty$, and A is a quasilinear differential operator in divergence form

$$(1) \qquad A = -D^i\big(a_i(p)\big)$$

[1] Every open bounded set whose boundary is of class C^2 satisfies the *outward sphere condition* with some suitable R.

whose coefficients satisfy the conditions

(i) $$a_i \in C^1(\mathbb{R}^N)$$

and

(ii) $$\frac{\partial a_i}{\partial p^j} \xi^i \xi^j \geqq 0 \qquad \forall \xi \in \mathbb{R}^N.$$

If we assume that the convex set $\boldsymbol{K}$ is not empty, then the following result holds.

Theorem. *Under the assumptions stated above, the variational inequality* $(*)$ *has a solution* $u \in \boldsymbol{K}$ *such that*

(2) $$A u \in L^p(\Omega).$$

Furthermore, if $\partial\Omega \in C^{1,1}$ *and if* a_i *is a coercive vector field, then* $u \in H^{2,p}(\Omega)$ *provided* $p > N$.

Proof. The existence of a solution $u \in \boldsymbol{K}$ follows from well-known existence theorems for maximal monotone operators, since the convex set $\boldsymbol{K}$ is compact (cf. e.g. [3]). The crucial point is to show that the relation (2) is valid. To prove this assertion, it will be sufficient to demonstrate that the triple $\{u, \boldsymbol{K}, A\}$ is *J-compatible* in the sense of [3; Théorème I.1]; namely, if J_p is the duality mapping from $L^p(\Omega)$ to $L^q(\Omega)$ defined by

(3) $$J_p(v) = |v|^{p-2} v, \qquad 1/p + 1/q = 1,$$

then for every $\varepsilon > 0$ we shall prove the existence of an element w_ε in $L^p(\Omega)$, whose L^p norm is bounded independently of ε, such that the equation

(4) $$u_\varepsilon + \varepsilon J_p(A u_\varepsilon + w_\varepsilon) = u$$

has a solution $u_\varepsilon \in \boldsymbol{K}$ with $A u_\varepsilon \in L^p(\Omega)$. According to the results of BREZIS & STAMPACCHIA, Au then belongs to $L^p(\Omega)$ too.

To prove (4) let us consider the monotone, hemicontinuous operator A_0 from $H^{1,2}(\Omega)$ to $H^{-1,2}(\Omega)$ defined by

$$A_0 = -D^i(\tilde{a}_i(p)),$$

where $\tilde{a}_i(p) = a_i(p)$ on the compact set $|p| \leqq 2$. The existence of such an operator has been shown by BREZIS & STAMPACCHIA in [3]. We then define the *multivalued* operator $\tilde{A} = A_\sigma + B$ through the assignments:

(5) $$A_\sigma = A_0 - \sigma \Delta,$$

(6) $$B v = \mu \beta(v - \phi);$$

here σ is any positive number, ϕ is any given element of $\boldsymbol{K}$, μ is a positive constant to be determined later, and β is the following maximal monotone graph in $\mathbb{R} \times \mathbb{R}$

(7) $$\beta(t) = \begin{cases} -1, & t \leqq 0 \\ [-1, 1], & t = 0 \\ 1, & t \geqq 0. \end{cases}$$

$\tilde{A}$ maps elements of $H^{1,2}(\Omega)$ onto subsets of $H^{-1,2}(\Omega)$.

For later use we shall need the following result (compare [3; Lemme III.2]).

Lemma 1 (Comparison Lemma). *Let Θ be a nondecreasing real function with $\Theta(0)=0$. For $i=1,2$, let F_i, $\phi_i \in L^\infty(\Omega)$, and $u_i \in H^{1,2}(\Omega)$ be functions such that the relations*

(8) $$0 \in A_\sigma u_1 + \mu\,\beta(u_1-\phi_1)+\Theta(u_1-F_1)$$

and

(9) $$0 \in A_\sigma u_2 + \mu\,\beta(u_2-\phi_2)+\Theta(u_2-F_2)$$

hold in the sense of distributions. Then we have

(10) $$|u_2-u_1| \leqq \max\big(\sup_{\partial\Omega}|u_2-u_1|, \sup_{\Omega}|F_2-F_1|, \sup_{\Omega}|\phi_2-\phi_1|\big).$$

Proof. (i) First we shall show that

(11) $$u_1-u_2 \leqq T=\max\big(\sup_{\partial\Omega}(u_1-u_2), \sup_{\Omega}(F_1-F_2), \sup_{\Omega}(\phi_1-\phi_2)\big)$$

provided

(12) $$0 \leqq A_\sigma u_2 + \mu\,\beta(u_2-\phi_2)+\Theta(u_2-F_2)$$

(i.e. provided there is an element in $A_\sigma u_2+\mu\,\beta(u_2-\phi_2)+\Theta(u_2-F_2)$ such that (12) is satisfied in the distributional sense).

For any $\varepsilon>0$, we set

(13) $$\eta=\max(u_1-u_2, T+\varepsilon)-(T+\varepsilon) \in H_0^{1,2}(\Omega).$$

From (8) and (12) it is clear that

(14) $$\begin{aligned} 0 \geqq \langle A_\sigma u_1 - A_\sigma u_2, \eta\rangle &+ \mu \int_\Omega \{\beta(u_1-\phi_1)-\beta(u_2-\phi_2)\}\,\eta\,dx \\ &+ \int_\Omega \{\Theta(u_1-F_1)-\Theta(u_2-F_2)\}\,\eta\,dx. \end{aligned}$$

Now in the set $\{u_1-u_2-T-\varepsilon \geqq 0\}$ we have

(15) $$u_1-F_1 > u_2-F_2$$

and

(16) $$u_1-\phi_1 > u_2-\phi_2.$$

The last two integrals in (14) are consequently nonnegative by the definition of β and Θ; hence

(17) $$0 \geqq \int_{\{u_1-u_2 \geqq T+\varepsilon\}} |D(u_1-u_2)|^2\,dx$$

which implies the assertion (10).

(ii) The estimate (10) is obtained by permuting the indices in the first part of the proof.

Next we prove the crucial

Lemma 2. *Let Θ be a continuous, bounded, nondecreasing real function with $\Theta(0)=0$. Then for any $F\in\boldsymbol{K}$ there exists a solution $v\in\boldsymbol{K}$ of the relation*

$$0\in\tilde{A}\,v+\Theta(v-F), \tag{18}$$

provided μ is sufficiently large.

The proof will be given in three steps: First, (18) has a solution $v\in\phi+H_0^{1,2}(\Omega)$ (see the Appendix). Second, for any pair of points $x\in\Omega$, $x_0\in\partial\Omega$, we shall show that

$$|v(x)-v(x_0)|\leqq|x-x_0|. \tag{19}$$

Then, with the help of the *comparison lemma* we conclude that

$$|v(x)-v(y)|\leqq|x-y|\qquad\forall\,x,y\in\Omega. \tag{20}$$

To prove (19) we shall construct appropriate comparison functions δ^+ and δ^- following HARTMANN & STAMPACCHIA [5; Lemma 10.1]. Let $x_0\in\Gamma_k$. By assumption there is a ball with radius R which touches Γ_k in x_0. Without loss of generality we may assume that the center of the ball lies in the origin.

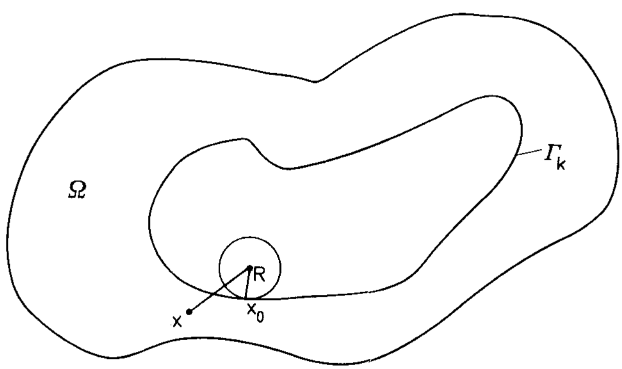

Fig. 2

Now define

$$\delta_0(x)=|x|-R. \tag{21}$$

One easily checks that

$$\delta_0(x)\geqq\delta_0(x_0)=0\qquad\text{for }|x|\geqq|x_0|=R,$$

$$D^i\delta_0=\frac{x^i}{|x|},\qquad D^iD^j\delta_0=\frac{\delta^{ij}}{|x|}-\frac{x^i x^j}{|x|^3}.$$

Moreover, for any function $u\in\boldsymbol{K}$ we have the estimate

$$|u(x)-c_k|\leqq\inf_{y\in\Gamma_k}|x-y|\leqq|x|-R=\delta_0(x)\qquad\forall x\in\Omega. \tag{22}$$

Now choose

$$\delta^+=\delta_0+c_k+\varepsilon, \tag{23}$$

where ε is any positive constant. The estimate (22) then implies

$$\Theta(\delta^+-F)\geqq\Theta(0)=0 \tag{24}$$

and

$$\beta(\delta^+ - \phi) = 1. \tag{25}$$

Hence,

$$\tilde{A}\,\delta^+ + \Theta(\delta^+ - F) \geqq -D^i\big(a_i(D\,\delta^+)\big) - \sigma\,\Delta\delta^+ + \mu \geqq -c + \mu, \tag{26}$$

where c depends only on the first derivatives of the a_i's on the compact set $|p|=1$, and on R, N, and diam Ω.

If we choose μ sufficiently large we therefore get

$$\tilde{A}\,\delta^+ + \Theta(\delta^+ - F) \geqq 0. \tag{27}$$

As we have shown in the first part of the proof of Lemma 1, it follows from (18) and (27) that

$$v \leqq \delta^+, \tag{28}$$

or in other words

$$v(x) - v(x_0) = v(x) - c_k \leqq |x| - |x_0| \leqq |x - x_0| \qquad \forall x \in \Omega,\ \forall x_0 \in \partial\Omega.$$

To prove (19), we set

$$\delta^- = -\delta_0 + c_k - \varepsilon. \tag{29}$$

From (22) and the relation

$$\tilde{A}\,\delta^- + \Theta(\delta^- - F) \leqq 0 \tag{30}$$

we get by similar considerations

$$\delta^- \leqq v. \tag{31}$$

Thus (19) is proved.

To complete the proof of Lemma 2, let $x_1, x_2 \in \Omega$, and make the definitions $h = x_2 - x_1$, $\Omega_h = \Omega - h$, $v_h(x) = v(x+h)$, $F_h(x) = F(x+h)$, and $\phi_h(x) = \phi(x+h)$. In the open set $\mathcal{O} = \Omega \cap \Omega_h$ we have

$$0 \in A_\sigma\, v + \mu\,\beta(v - \phi) + \Theta(v - F)$$

and

$$0 \in A_\sigma\, v_h + \mu\,\beta(v_h - \phi_h) + \Theta(v_h - F_h).$$

From the *comparison lemma* we now conclude that the inequality

$$|v_h - v| \leqq \max\big(\sup_{\partial\mathcal{O}} |v_h - v|,\ \sup_{\mathcal{O}} |F_h - F|,\ \sup_{\mathcal{O}} |\phi_h - \phi|\big) \leqq |h| \tag{32}$$

holds in $\mathcal{O}$ (here $x \in \partial\mathcal{O}$ is equivalent that x or $x+h$ belongs to $\partial\Omega$). Consequently we have

$$|v(x_1) - v(x_2)| \leqq |x_1 - x_2| \qquad \forall x_1, x_2 \in \Omega.$$

This completes the proof of Lemma 2.

The J-compatibility of $\{u, \boldsymbol{K}, A\}$ now follows easily. According to Lemma 2, for any $\sigma > 0$ there exists a solution $v_\sigma \in \boldsymbol{K}$ of the relation

$$0 \in A\,v_\sigma - \sigma\,\Delta v_\sigma + \Theta(v_\sigma - F) + \mu\,\beta(v_\sigma - \phi). \tag{33}$$

Now we choose a sequence of values σ tending to zero such that the corresponding functions v_σ converge uniformly to some function $v \in \boldsymbol{K}$ satisfying

$$0 \in A v + \Theta(v - F) + \mu \beta(v - \phi) \tag{34}$$

(here we use the fact that $A_0 + \Theta(\cdot - F)$ is a monotone, hemicontinuous operator and that $\beta(v_\sigma - \phi)$ converges weakly in $L^2(\Omega)$ to $\beta(v - \phi)$; see the Appendix for similar considerations).

For any $\varepsilon > 0$ let Θ be a function of the type indicated in Lemma 2, which agrees with the function

$$t \to |t/\varepsilon|^{q-2}\, t/\varepsilon$$

on the interval $[-C, C]$, $C > 2 \operatorname{diam} \Omega$. Also let $F = u$ in Lemma 2. Then there are elements $u_\varepsilon \in \boldsymbol{K}$ and $w_\varepsilon \in L^p(\Omega)$ such that

$$u_\varepsilon + \varepsilon J_p(A u_\varepsilon + w_\varepsilon) = 0,$$

where w_ε is some element in $\mu \beta(u_\varepsilon - \phi)$. Thus the relation (2) is proved.

The final assertion of the theorem is well-known (cf. e.g. [4; Appendix]).

Appendix

To prove that

$$0 \in A_\sigma v + \Theta(v - F) + \mu \beta(v - \phi) \tag{A.1}$$

has a solution $v_\sigma \in \phi + H_0^{1,2}(\Omega)$, consider the *regularized* graph

$$\beta_\varepsilon(t) = \begin{cases} -1, & t \leqq -\varepsilon \\ t/\varepsilon, & |t| \leqq \varepsilon \\ 1, & t \geqq \varepsilon. \end{cases} \tag{A.2}$$

Since β_ε is *continuous, bounded*, and *nondecreasing*, there exists a solution v_ε of

$$0 = A_\sigma v_\varepsilon + \Theta(v_\varepsilon - F) + \mu \beta_\varepsilon(v_\varepsilon - \phi). \tag{A.3}$$

Since the lower order terms in (A.3) are bounded, we have the estimates

$$\|v_\varepsilon\|_{1,2,\Omega} \leqq \operatorname{const}(\sigma) \tag{A.4}$$

and

$$\|v_\varepsilon\|_{2,p,\Omega'} \leqq \operatorname{const}(\sigma, p, \Omega') \tag{A.5}$$

for any $p > N$ and for all $\Omega' \subset\subset \Omega$.

Hence a subsequence of v_ε (which we again call v_ε) converges weakly in $\phi + H_0^{1,2}(\Omega)$ and uniformly on compact subsets of Ω to some function v.

The crucial step in the proof that v satisfies (A.1) is to show that $\beta_\varepsilon(v_\varepsilon - \phi)$ converges weakly in $L^2(\Omega)$ to some element of $\beta(v - \phi)$.

Since β_ε is bounded, a subsequence of $\beta_\varepsilon(v_\varepsilon - \phi)$ converges weakly to some function γ which has its range in the *convex* set $[-1, 1]$. If we could show that

$$\gamma(x) = \begin{cases} 1 \text{ almost everywhere in } \{v - \phi > 0\} \\ -1 \text{ almost everywhere in } \{v - \phi < 0\} \end{cases} \tag{A.6}$$

then γ would belong to $\beta(v - \phi)$.

We shall only prove the first assertion of (A.6): Since v and ϕ are continuous $G=\{x\in\Omega: v-\phi>0\}$ is open. Let K be a compact subset of G, then $(v-\phi)_{|K}\geqq\tau>0$ and

$$(v_\varepsilon-\phi)_{|K}\geqq\tau/2 \qquad \forall\, 0<\varepsilon\leqq\varepsilon_0. \tag{A.7}$$

On the other hand, for $\varepsilon<\min(\tau/2,\varepsilon_0)$ we have

$$\beta_\varepsilon(v_\varepsilon-\phi)=1 \qquad \text{in } K. \tag{A.8}$$

The remainder of the proof now follows by standard techniques and will be omitted.

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