

CLOSED WEINGARTEN HYPERSURFACES IN SEMI-RIEMANNIAN MANIFOLDS

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ABSTRACT. The existence of closed hypersurfaces of prescribed curvature in semi-riemannian manifolds is proved provided there are barriers.

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0. INTRODUCTION

We want to prove the existence of closed hypersurfaces of prescribed curvature in Riemannian or Lorentzian manifolds N of dimension $n + 1$, $n \geq 2$. Since we wish to treat both cases simultaneously, let us stipulate that terms which make only sense in Lorentzian manifolds should be ignored if the ambient space is Riemannian. With this in mind, let Ω be a connected, precompact, open subset of N , $f = f(x, \nu)$ a positive function defined for $x \in \bar{\Omega}$ and time-like

Date: November 6, 2001.

2000 Mathematics Subject Classification. 35J60, 53C21, 53C44, 53C50, 58J05.

Key words and phrases. Prescribed curvature, closed Weingarten hypersurfaces, semi-riemannian manifolds.

vectors $\nu \in T_x(N)$, and $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ a symmetric curvature function defined on the positive cone $\Gamma_+ \subset \mathbb{R}^n$. Then, we look for closed space-like hypersurfaces $M \subset \Omega$ such that

$$(0.1) \quad F|_M = f(x, \nu),$$

where f is evaluated at $x \in M$ and at the past directed normal $\nu \in T_x(N)$.

Various existence results have been proved for a wide range of curvature functions F if f only depends on x . In Euclidian space any monotone curvature function F can be considered with the property that $\log F$ is concave—at least in principle, cf. [2, 6]. The only possible obstruction could occur when one tries to prove C^1 - estimates. That particular difficulty arises in any Riemannian space, but in addition—in order to obtain C^2 - estimates—one has to assume that F satisfies a certain concavity estimate, i.e. a stronger property is needed than mere concavity of F or $\log F$. In [6, 8] we proved existence results for curvature functions of class (K) that satisfy such an estimate.

If the ambient space is Lorentzian, then, C^1 - estimates can be obtained for curvature functions for which a corresponding estimate in a Riemannian setting is known to be impossible, or only achievable with additional structural conditions on f . But still, the curvature functions have to satisfy the same stronger concavity property as in the Riemannian case in order to derive C^2 - estimates, and, in addition, one further estimate is needed, namely, there should exist $\epsilon_0 > 0$ such that

$$(0.2) \quad \epsilon_0 F H \leq F^{ij} h_{ik} h_j^k$$

for all admissible tensors (h_{ij}) , where as usual H stands for the mean curvature, cf. [12].

Recently, a concavity estimate has been proved for the scalar curvature operator H_2 , cf. [1]. We improved that estimate in [13], so that functions f depending on the normal can be considered, and had been able to prove the existence of closed space-like hypersurfaces satisfying (0.1), where N is Lorentzian, $F = H_2$, and $f = f(x, \nu)$ is general enough, so that solutions can be considered as having prescribed scalar curvature.

It is only natural to ask if similar generalizations with regard to f are also possible for other curvature functions or in Riemannian spaces instead of Lorentzian. Two difficulties arise from the presence of the normal vector in the right-hand side f that have to be dealt with separately. Let us first address the simple one, the C^2 - estimates. These estimates can be derived for all curvature functions F that obey a strong concavity condition mentioned above, and, in addition, an estimate of the form

$$(0.3) \quad n\epsilon_0 F \leq F_i H \quad \forall 1 \leq i \leq n,$$

where $\epsilon_0 = \epsilon_0(F)$ is a positive constant, $F_i = \frac{\partial F}{\partial \kappa_i}$, and the inequality should be valid in the convex cone associated with F . Furthermore, it is assumed that lower order estimates in the C^0 - and C^1 - norms are already established.

Obtaining the C^1 - estimates is the second difficulty that seems to be unsurmountable in general Riemannian spaces, e.g. for H_2 , and even in Euclidean space, it is only possible if special structural conditions on f are imposed. However, the only curvature functions F for which strong concavity estimates are known so far, are H_2 and those of class (K) . The latter are defined in Γ_+ , i.e. the admissible hypersurfaces have to be strictly convex, and, thus, the C^1 - estimates are easily obtained.

We have excluded the mean curvature or functions of mean curvature type, $F \in (H)$, from the solvability discussion, since they pose different problems. In their case the C^1 - estimates are the only challenge, and they can be derived in Lorentzian space under the assumptions

$$(0.4) \quad \|f_{\beta}(x, \nu)\| \leq c(1 + \|\nu\|),$$

and

$$(0.5) \quad \|f_{\nu^{\beta}}(x, \nu)\| \leq c,$$

cf. [13, Proposition 4.8].

To give a precise statement of the existence results we need a few definitions and assumptions. It seems advisable to treat the Lorentzian and Riemannian cases separately.

0.1. The Lorentzian case. We assume that N is a smooth, connected, *globally hyperbolic* manifold with a compact *Cauchy hypersurface* \mathcal{S}_0 , and suppose that Ω is bounded by two *achronal*, connected, space-like hypersurfaces M_1 and M_2 of class $C^{4,\alpha}$, where M_1 is supposed to lie in the past of M_2 .

Let F of class (K) satisfy (0.3), and $0 < \epsilon_1 \leq f$ be of class $C^{2,\alpha}$. Then, we assume that the boundary components act as barriers for (F, f) .

Definition 0.1. M_2 is an *upper barrier* for (F, f) , if M_2 is strictly convex and satisfies

$$(0.6) \quad F|_{M_2} \geq f(x, \nu),$$

where f is evaluated at $x \in M_2$, and at the past directed normal $\nu(x)$ of M_2 .

M_1 is a *lower barrier* for (F, f) , if at the points $\Sigma \subset M_1$, where M_1 is strictly convex, there holds

$$(0.7) \quad F|_{\Sigma} \leq f(x, \nu).$$

Σ may be empty.

Then, we can prove

Theorem 0.2. *Let M_1 be a lower and M_2 an upper barrier for (F, f) . Then, the problem*

$$(0.8) \quad F|_M = f(x, \nu)$$

has a strictly convex solution $M \subset \bar{\Omega}$ of class $C^{4,\alpha}$ that can be written as a graph over S_0 provided there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$.

0.2. The Riemannian case. Let N be a smooth connected Riemannian manifold with $K_N \leq 0$, and assume that the boundary components of Ω are both strictly convex hypersurfaces homeomorphic to S^n and of class $C^{4,\alpha}$, such that the mean curvature vector of M_1 points outside of Ω and the mean curvature vector of M_2 points inside of Ω .

Then, we can prove

Theorem 0.3. *Let F of class (K) satisfy (0.3), and let $0 < \epsilon_1 \leq f$ of class $C^{2,\alpha}$ be given. Assume that M_2 is an upper barrier for (F, f) and M_1 a lower barrier. Then, the problem*

$$(0.9) \quad F|_M = f(x, \nu)$$

has a strictly convex solution $M \subset \bar{\Omega}$ of class $C^{4,\alpha}$.

The paper is organized as follows: In Section 1 we take a closer look at curvature functions and show that the curvature functions in question coincide with a subclass of (K) , where the functions can be written as a product such that one factor is a power of the Gaussian curvature.

In Section 2 we introduce the notations and common definitions we rely on, and state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces in a semi-riemannian manifold.

In Section 3 we look at the curvature flow associated with our problem, and the corresponding evolution equations for the basic geometrical quantities of the flow hypersurfaces.

In Section 4 we prove lower order estimates for the evolution problem, while a priori estimates in the C^2 -norm are derived in Section 5 for the Lorentzian case, and in Section 6 for the Riemannian case.

The final existence result is contained in Section 7.

1. CURVATURE FUNCTIONS

Let $\Gamma_+ \subset \mathbb{R}^n$ be the open positive cone and $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ a symmetric function satisfying the condition

$$(1.1) \quad F_i = \frac{\partial F}{\partial \kappa^i} > 0;$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices \mathcal{S}_+ , for, let $(h_{ij}) \in \mathcal{S}_+$ with eigenvalues κ_i , $1 \leq i \leq n$, then define F on \mathcal{S}_+ by

$$(1.2) \quad F(h_{ij}) = F(\kappa_i).$$

If we define

$$(1.3) \quad F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

and

$$(1.4) \quad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$$

then,

$$(1.5) \quad F^{ij} \xi_i \xi_j = \frac{\partial F}{\partial \kappa_i} |\xi^i|^2 \quad \forall \xi \in \mathbb{R}^n,$$

$$(1.6) \quad F^{ij} \text{ is diagonal if } h_{ij} \text{ is diagonal,}$$

and

$$(1.7) \quad F^{ij,kl} \eta_{ij} \eta_{kl} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j} (\eta_{ij})^2,$$

for any $(\eta_{ij}) \in \mathcal{S}$, where \mathcal{S} is the space of all symmetric matrices. The second term on the right-hand side of (1.7) is non-positive if F is concave, and non-negative if F is convex, and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

In [12] we defined—or better redefined—the curvature functions of class (K) as

Definition 1.1. A symmetric curvature function $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ positively homogeneous of degree $d_0 > 0$ is said to be of class (K) if

$$(1.8) \quad F_i = \frac{\partial F}{\partial \kappa^i} > 0 \quad \text{in } \Gamma_+,$$

$$(1.9) \quad F|_{\partial \Gamma_+} = 0,$$

and

$$(1.10) \quad F^{ij,kl} \eta_{ij} \eta_{kl} \leq F^{-1} (F^{ij} \eta_{ij})^2 - F^{ik} \tilde{h}^{jl} \eta_{ij} \eta_{kl} \quad \forall \eta \in \mathcal{S},$$

or, equivalently, if we set $\hat{F} = \log F$,

$$(1.11) \quad \hat{F}^{ij,kl} \eta_{ij} \eta_{kl} \leq -\hat{F}^{ik} \tilde{h}^{jl} \eta_{ij} \eta_{kl} \quad \forall \eta \in \mathcal{S},$$

where F is evaluated at (h_{ij}) .

The preceding considerations are also applicable if the κ_i are the principal curvatures of a hypersurface M with metric (g_{ij}) . F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) with eigenvalues κ_i with respect to the metric.

$$(1.12) \quad F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is then a contravariant tensor of second order. Sometimes it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$(1.13) \quad h_j^i = g^{ik} h_{kj}.$$

Then,

$$(1.14) \quad F_i^j = \frac{\partial F}{\partial h_j^i}$$

is also a mixed tensor with contravariant index j and covariant index i .

Remark 1.2. Let $F \in (K)$, then $\log F$ is concave, and, if F is homogeneous of degree 1, then, F is already concave.

Proof. The concavity of $\log F$ follows immediately from (1.11), while, in case F is homogeneous of degree 1, the concavity of F can be derived from the inequality (1.10) by applying Schwartz inequality: Choose coordinates such that in a fixed point

$$(1.15) \quad g_{ij} = \delta_{ij} \quad \text{and} \quad h_{ij} = \kappa_i \delta_{ij}.$$

Let $\eta = (\eta_{ij})$ be an arbitrary symmetric tensor, then

$$\begin{aligned}
(1.16) \quad F^{-1}(F^{ij}\eta_{ij})^2 &= F^{-1}\left(\sum_i F_i^i \eta_{ii}\right)^2 \\
&= F^{-1}\left(\sum_i (F_i^i)^{\frac{1}{2}} \kappa_i^{\frac{1}{2}} (F_i^i)^{\frac{1}{2}} \kappa_i^{-\frac{1}{2}} \eta_{ii}\right)^2 \\
&\leq F^{-1}\left(\sum_i F_i^i \kappa_i\right) \left(\sum_i F_i^i \kappa_i^{-1} \eta_{ii}^2\right) \\
&\leq F^{ik} \tilde{h}^{jl} \eta_{ij} \eta_{kl},
\end{aligned}$$

hence, the right-hand side of (1.10) is non-positive. \square

The subclass (K^*) has been defined in [12, Definition 1.6] as

Definition 1.3. A function $F \in (K)$ is said to be of class (K^*) if there exists $0 < \epsilon_0 = \epsilon_0(F)$ such that

$$(1.17) \quad \epsilon_0 F H \leq F^{ij} h_{ik} h_j^k,$$

for any $(h_{ij}) \in \mathcal{S}_+$, where F is evaluated at (h_{ij}) . H represents the mean curvature, i.e. the trace of (h_{ij}) .

The condition (1.17) is crucial for solving curvature problems in Lorentzian manifolds; it is slightly weaker than the condition (0.3) which is also satisfied by $F = H_2$, see e.g. [13, (1.17)].

Lemma 1.4. *Let F be a symmetric curvature function defined in an open convex cone Γ satisfying the relations (1.1) and (0.3) in Γ , then it also satisfies the inequality (1.17) with the same constant ϵ_0 .*

Proof. We first observe that in view of the relation (0.3) H is positive in Γ . Next, choose coordinates as in (1.15), then,

$$\begin{aligned}
(1.18) \quad F^{ij} h_{ik} h_j^k &= H^{-1} \sum_i H F_i \kappa_i^2 \\
&\geq n \epsilon_0 H^{-1} F |A|^2 \geq \epsilon_0 F H,
\end{aligned}$$

where we used the usual abbreviation $|A|^2$ for $\sum_i \kappa_i^2$. \square

Remark 1.5. Special functions of class (K^*) are those that can be written as a product

$$(1.19) \quad F = GK^a, \quad a > 0,$$

where $G \in (K)$ and K is the Gaussian curvature. They are exactly those that satisfy the estimate

$$(1.20) \quad F_i \kappa_i \geq \epsilon_0 F \quad \forall 1 \leq i \leq n$$

with some positive constant $\epsilon_0 = \epsilon_0(F)$, cf. [12, Proposition 1.9].

Using the simple estimate $\kappa_i \leq H$, which is valid in Γ_+ , we conclude that these special functions also satisfy the condition (0.3).

The reverse is also true.

Lemma 1.6. *Let $F \in (K)$ be such that the relation (0.3) is valid, then, F also satisfies (1.20).*

Proof. Let κ_n be the largest component of the n -tuple $(\kappa_i) \in \Gamma_+$. Then, we conclude in view of (0.3)

$$(1.21) \quad F_n \kappa_n \geq \frac{1}{n} F_n H \geq \epsilon_0 F.$$

On the other hand, any curvature function of class (K) satisfies

$$(1.22) \quad F_i \kappa_i \geq F_n \kappa_n \quad \forall 1 \leq i \leq n,$$

cf. [8, Lemma 1.3]. □

2. NOTATIONS AND PRELIMINARY RESULTS

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for hypersurfaces. We shall formulate the governing equations of a hypersurface M in a semi-riemannian $(n+1)$ -dimensional space N , which is either Riemannian or Lorentzian. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc. Greek indices range from 0 to n and Latin from 1 to n ; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i) . Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function u in N , (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant

derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$(2.1) \quad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a *space-like* hypersurface, i.e. the induced metric is Riemannian, with a differentiable normal ν . We define the signature of ν , $\sigma = \sigma(\nu)$, by

$$(2.2) \quad \sigma = \bar{g}_{\alpha\beta} \nu^\alpha \nu^\beta = \langle \nu, \nu \rangle.$$

In case N is Lorentzian, $\sigma = -1$, and ν is time-like.

In local coordinates, (x^α) and (ξ^i) , the geometric quantities of the space-like hypersurface M are connected through the following equations

$$(2.3) \quad x_{ij}^\alpha = -\sigma h_{ij} \nu^\alpha$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$(2.4) \quad x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma.$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to $-\sigma\nu$.

The second equation is the *Weingarten equation*

$$(2.5) \quad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$(2.6) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauß equation*

$$(2.7) \quad R_{ijkl} = \sigma \{h_{ik} h_{jl} - h_{il} h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

For the rest of this section we treat the Riemannian and Lorentzian cases separately.

2.1. The Lorentzian case. Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface \mathcal{S}_0 . Then, N is topologically a product, $N = \mathbb{R} \times \mathcal{S}_0$, where \mathcal{S}_0 is a compact, n -dimensional Riemannian manifold, and there exists a Gaussian coordinate system $(x^\alpha)_{0 \leq \alpha \leq n}$ such that x^0 represents the time, the $(x^i)_{1 \leq i \leq n}$ are local coordinates for \mathcal{S}_0 , where we may assume that \mathcal{S}_0 is equal to the level hypersurface $\{x^0 = 0\}$ —we don't distinguish between \mathcal{S}_0 and $\{0\} \times \mathcal{S}_0$ —, and such that the Lorentzian metric takes the form

$$(2.8) \quad d\bar{s}_N^2 = e^{2\psi} \{-dx^{0^2} + \sigma_{ij}(x^0, x) dx^i dx^j\},$$

where σ_{ij} is a Riemannian metric, ψ a function on N , and x an abbreviation for the space-like components (x^i) , see [14], [16, p. 212], [15, p. 252], and [5, Section 6]. We also assume that the coordinate system is *future oriented*, i.e. the time coordinate x^0 increases on future directed curves. Hence, the *contravariant* time-like vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Furthermore, any achronal hypersurface can be written as a graph over \mathcal{S}_0 , cf. [12, Proposition 2.5].

Let $M = \text{graph } u|_{\mathcal{S}_0}$ be a space-like hypersurface

$$(2.9) \quad M = \{(x^0, x) : x^0 = u(x), x \in \mathcal{S}_0\},$$

then the induced metric has the form

$$(2.10) \quad g_{ij} = e^{2\psi} \{-u_i u_j + \sigma_{ij}\}$$

where σ_{ij} is evaluated at (u, x) , and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$(2.11) \quad g^{ij} = e^{-2\psi} \left\{ \sigma^{ij} + \frac{u^i u^j}{v} \right\},$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$(2.12) \quad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2. \end{aligned}$$

Hence, graph u is space-like if and only if $|Du| < 1$.

The covariant form of a normal vector of a graph looks like

$$(2.13) \quad (\nu_\alpha) = \pm v^{-1} e^\psi (1, -u_i).$$

and the contravariant version is

$$(2.14) \quad (\nu^\alpha) = \mp v^{-1} e^{-\psi} (1, u^i).$$

Thus, we have

Remark 2.1. Let M be space-like graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

$$(2.15) \quad (\nu^\alpha) = v^{-1} e^{-\psi} (1, u^i)$$

and the past directed

$$(2.16) \quad (\nu^\alpha) = -v^{-1} e^{-\psi} (1, u^i).$$

In the Gauß formula (2.3) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal for reasons that we have explained in [12, Section 2].

Look at the component $\alpha = 0$ in (2.3) and obtain in view of (2.16)

$$(2.17) \quad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{I}_{00}^0 u_i u_j - \bar{I}_{0i}^0 u_j - \bar{I}_{0j}^0 u_i - \bar{I}_{ij}^0.$$

Here, the covariant derivatives are taken with respect to the induced metric of M , and

$$(2.18) \quad -\bar{I}_{ij}^0 = e^{-\psi} \bar{h}_{ij},$$

where (\bar{h}_{ij}) is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

Sometimes, we need a Riemannian reference metric, e.g. if we want to estimate tensors. Since the Lorentzian metric can be expressed as

$$(2.19) \quad \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} \{-dx^{02} + \sigma_{ij} dx^i dx^j\},$$

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

$$(2.20) \quad \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} \{dx^{02} + \sigma_{ij} dx^i dx^j\}$$

and we abbreviate the corresponding norm of a vectorfield η by

$$(2.21) \quad \|\eta\| = (\tilde{g}_{\alpha\beta} \eta^\alpha \eta^\beta)^{1/2},$$

with similar notations for higher order tensors.

2.2. The Riemannian case. In view of our assumptions on N and Ω , we may assume that N is simply connected and that Ω is the difference of two convex bodies, cf. [6, Theorem 4.7], and, therefore, Ω can be covered by a geodesic polar coordinate system $(x^\alpha)_{0 \leq \alpha \leq n}$, where x^0 represents the radial distance to the center, and (x^i) are local coordinates for the geodesic sphere $S^n = \{x^0 = 1\}$.

The barriers M_i can be written as graphs over S^n , $M_i = \text{graph } u_i$, and the metric in N can be expressed as

$$(2.22) \quad d\bar{s}^2 = (dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j.$$

The 0-th component of the Gauß formula yields

$$(2.23) \quad v^{-1} h_{ij} = -u_{ij} + \bar{h}_{ij},$$

where

$$(2.24) \quad v^2 = 1 + \sigma^{ij} u_i u_j,$$

and (\bar{h}_{ij}) is the second fundamental form of the level hypersurfaces $\{x^0 = \text{const}\}$.

Remark 2.2. It is well known that these level hypersurfaces are strictly convex if $K_N \leq 0$, and, hence, that there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$, cf. [8, Remark 0.5].

3. THE EVOLUTION PROBLEM

Solving the problem (0.1) consists of two steps: first, one has to prove a priori estimates, and secondly, one has to find a procedure which, with the help of the a priori estimates, leads to a solution of the problem.

In Lorentzian manifolds the evolution method is the method of choice, but in Riemannian manifolds this approach requires the sectional curvatures of the ambient space to be non-positive. There is an alternative method—successive approximation—but it is only applicable when the a priori estimates also apply to the *elliptic regularizations* of the curvature functions in mind, cf. [8]. Though the class (K) is closed under elliptic regularization, see [8, Section 1], this is not valid for the subclass of functions satisfying the additional property (0.3). For that reason we require that a Riemannian space has non-positive sectional curvature.

We want to prove that the equation

$$(3.1) \quad F = f$$

has a solution. For technical reasons, it is convenient to solve instead the equivalent equation

$$(3.2) \quad \Phi(F) = \Phi(f),$$

where Φ is a real function defined on \mathbb{R}_+ such that

$$(3.3) \quad \dot{\Phi} > 0 \quad \text{and} \quad \ddot{\Phi} \leq 0.$$

For notational reasons, let us abbreviate

$$(3.4) \quad \tilde{f} = \Phi(f).$$

We also point out that we may—and shall—assume without loss of generality that F is homogeneous of degree 1 if F is of class (K) .

To solve (3.2) we look at the evolution problem

$$(3.5) \quad \begin{aligned} \dot{x} &= -\sigma(\Phi - \tilde{f})\nu, \\ x(0) &= x_0, \end{aligned}$$

where x_0 is an embedding of an initial strictly convex, compact, space-like hypersurface M_0 , $\Phi = \Phi(F)$, and F is evaluated at the principal curvatures of the flow hypersurfaces $M(t)$, or, equivalently, we may assume that F depends on the second fundamental form (h_{ij}) and the metric (g_{ij}) of $M(t)$; $x(t)$ is the embedding of $M(t)$ and σ the signature of the normal $\nu = \nu(t)$ —past directed, if N is Lorentzian, resp. the outward normal, if N is Riemannian.

This is a parabolic problem, so short-time existence is guaranteed—the proof in the Lorentzian case is identical to that in the Riemannian case, cf. [6, p. 622]—, and under suitable assumptions, we shall be able to prove that the solution exists for all time and converges to a stationary solution if t goes to infinity.

There is a slight ambiguity in the notation, since we also call the evolution parameter *time*, but this lapse shouldn't cause any misunderstandings.

Next, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives. The proofs are identical to those of the corresponding results in a Riemannian setting, cf. [6, Section 3], and will be omitted.

Lemma 3.1 (Evolution of the metric). *The metric g_{ij} of $M(t)$ satisfies the evolution equation*

$$(3.6) \quad \dot{g}_{ij} = -2\sigma(\Phi - \tilde{f})h_{ij}.$$

Lemma 3.2 (Evolution of the normal). *The normal vector evolves according to*

$$(3.7) \quad \dot{\nu} = \nabla_M(\Phi - \tilde{f}) = g^{ij}(\Phi - \tilde{f})_i x_j.$$

Lemma 3.3 (Evolution of the second fundamental form). *The second fundamental form evolves according to*

$$(3.8) \quad \dot{h}_i^j = (\Phi - \tilde{f})_i^j + \sigma(\Phi - \tilde{f})h_i^k h_k^j + \sigma(\Phi - \tilde{f})\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj}$$

and

$$(3.9) \quad \dot{h}_{ij} = (\Phi - \tilde{f})_{ij} - \sigma(\Phi - \tilde{f})h_i^k h_{kj} + \sigma(\Phi - \tilde{f})\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta.$$

Lemma 3.4 (Evolution of $(\Phi - \tilde{f})$). *The term $(\Phi - \tilde{f})$ evolves according to the equation*

$$(3.10) \quad \begin{aligned} (\Phi - \tilde{f})' - \dot{\Phi}F^{ij}(\Phi - \tilde{f})_{ij} &= \sigma\dot{\Phi}F^{ij}h_{ik}h_j^k(\Phi - \tilde{f}) \\ &+ \sigma\tilde{f}_\alpha\nu^\alpha(\Phi - \tilde{f}) - \tilde{f}_{\nu^\alpha}x_i^\alpha(\Phi - \tilde{f})_j g^{ij} \\ &+ \sigma\dot{\Phi}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta(\Phi - \tilde{f}), \end{aligned}$$

where

$$(3.11) \quad (\Phi - \tilde{f})' = \frac{d}{dt}(\Phi - \tilde{f})$$

and

$$(3.12) \quad \dot{\Phi} = \frac{d}{dr}\Phi(r).$$

From (3.8) we deduce with the help of the Ricci identities a parabolic equation for the second fundamental form

Lemma 3.5. *The mixed tensor h_i^j satisfies the parabolic equation*

$$(3.13) \quad \begin{aligned} &\dot{h}_i^j - \dot{\Phi}F^{kl}h_{i;kl}^j \\ &= \sigma\dot{\Phi}F^{kl}h_{rk}h_l^r h_i^j - \sigma\dot{\Phi}Fh_{ri}h^{rj} + \sigma(\Phi - \tilde{f})h_i^k h_k^j \\ &\quad - \tilde{f}_{\alpha\beta}x_i^\alpha x_k^\beta g^{kj} + \sigma\tilde{f}_\alpha\nu^\alpha h_i^j - \tilde{f}_{\alpha\nu^\beta}(x_i^\alpha x_k^\beta h^{kj} + x_l^\alpha x_k^\beta h_l^k g^{lj}) \\ &\quad - \tilde{f}_{\nu^\alpha\nu^\beta}x_l^\alpha x_k^\beta h_i^k h^{lj} - \tilde{f}_{\nu^\beta}x_k^\beta h_{i;l}^k g^{lj} + \sigma\tilde{f}_{\nu^\alpha}\nu^\alpha h_i^k h_k^j \\ &\quad + \dot{\Phi}F^{kl,rs}h_{kl;i}h_{rs}^j + 2\dot{\Phi}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_l^m g^{rj} \\ &\quad - \dot{\Phi}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_l^\gamma h_i^\delta g^{mj} - \dot{\Phi}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_l^\gamma x_i^\delta h^{mj} \\ &\quad + \sigma\dot{\Phi}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j - \sigma\dot{\Phi}F\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\ &\quad + \sigma(\Phi - \tilde{f})\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} + \ddot{\Phi}F_i F^j \\ &\quad + \dot{\Phi}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\}. \end{aligned}$$

The proof is identical to that of the corresponding result in [12, Lemma 3.5]; we only have to keep in mind that f now also depends on the normal.

If we had assumed F to be homogeneous of degree d_0 instead of 1, then, we would have to replace the explicit term F —occurring twice in the preceding lemma—by $d_0 F$.

Remark 3.6. In view of the maximum principle, we immediately deduce from (3.10) that the term $(\Phi - \tilde{f})$ has a sign during the evolution if it has one at the beginning, i.e., if the starting hypersurface M_0 is the upper barrier M_2 , then $(\Phi - \tilde{f})$ is non-negative, or equivalently,

$$(3.14) \quad F \geq f,$$

while in case $M_0 = M_1$, $(\Phi - \tilde{f})$ is non-positive, or equivalently,

$$(3.15) \quad F \leq f.$$

4. LOWER ORDER ESTIMATES

We consider the evolution problem (3.5) with $\Phi(r) = \log r$ and with initial hypersurface $M_0 = M_2$ if N is Lorentzian resp. $M_0 = M_1$ if N is Riemannian. Solutions exist in a maximal time interval $[0, T^*)$, $0 < T^* \leq \infty$, as long as the flow hypersurfaces stay in $\bar{\Omega}$ and are smooth and strictly convex.

Let us first consider the Lorentzian case in more detail.

4.1. The Lorentzian case. As we have already mentioned, the barriers M_i are then graphs over the compact Cauchy hypersurface \mathcal{S}_0 and this is also valid for the flow hypersurfaces $M(t)$, $M(t) = \text{graph } u(t)$.

The scalar version of (3.5) is

$$(4.1) \quad \frac{\partial u}{\partial t} = -e^{-\psi} v (\Phi - \tilde{f}),$$

where

$$(4.2) \quad v = \tilde{v}^{-1} = 1 - |Du|^2.$$

As we have shown in [12, Section 4], the flow hypersurfaces stay in $\bar{\Omega}$ and are uniformly space-like, i.e. the term \tilde{v} is uniformly bounded. Moreover, \tilde{v} satisfies a useful parabolic equation that we shall exploit to estimate the principal curvatures of the hypersurfaces $M(t)$ from above.

Lemma 4.1 (Evolution of \tilde{v}). *Consider the flow (3.5) in the distinguished coordinate system associated with \mathcal{S}_0 . Then, \tilde{v} satisfies the evolution equation*

$$(4.3) \quad \begin{aligned} \dot{\tilde{v}} - \dot{\Phi} F^{ij} \tilde{v}_{ij} &= -\dot{\Phi} F^{ij} h_{ik} h_j^k \tilde{v} + [(\Phi - \tilde{f}) - \dot{\Phi} F] \eta_{\alpha\beta} \nu^\alpha \nu^\beta \\ &\quad - 2\dot{\Phi} F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} - \dot{\Phi} F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha \\ &\quad - \dot{\Phi} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma \eta_\epsilon x_l^\delta g^{\epsilon kl} \\ &\quad - \tilde{f}_\beta x_i^\beta x_k^\alpha \eta_\alpha g^{ik} - \tilde{f}_{\nu\beta} x_k^\beta h^{ik} x_i^\alpha \eta_\alpha, \end{aligned}$$

where η is the covariant vector field $(\eta_\alpha) = e^\psi(-1, 0, \dots, 0)$.

For a proof see [12, Lemma 4.4]; we only have to keep in mind that, now, f also depends on the normal.

Corollary 4.2. *Let $\tilde{\varphi} = e^{\lambda\tilde{v}}$, then, $\tilde{\varphi}$ satisfies the evolution inequality*

$$(4.4) \quad \begin{aligned} \dot{\tilde{\varphi}} - \dot{\Phi} F^{ij} \tilde{\varphi}_{ij} &\leq -\frac{\lambda}{2} \dot{\Phi} F^{ij} h_{ik} h_j^k e^{\lambda\tilde{v}} - \frac{\lambda^2}{2} \dot{\Phi} F^{ij} \tilde{v}_i \tilde{v}_j e^{\lambda\tilde{v}} \\ &\quad + c\lambda \dot{\Phi} F^{ij} g_{ij} e^{\lambda\tilde{v}} + c[(\Phi - \tilde{f}) + 1] e^{\lambda\tilde{v}} \\ &\quad + c\lambda \dot{\Phi} F^{ij} g_{ij} e^{\lambda\tilde{v}} + c(\dot{\Phi})^{-1} \tilde{F}^{ij} g_{ij} e^{\lambda\tilde{v}}, \end{aligned}$$

where $(\tilde{F}^{ij}) = (F^{ij})^{-1}$, c is a known constant, and where we also used the estimate (3.14).

Proof. We have

$$(4.5) \quad \dot{\tilde{\varphi}} - \dot{\Phi} F^{ij} \tilde{\varphi}_{ij} = [\dot{\tilde{v}} - \dot{\Phi} F^{ij} \tilde{v}_{ij}] \lambda e^{\lambda\tilde{v}} - \lambda^2 \dot{\Phi} F^{ij} \tilde{v}_i \tilde{v}_j e^{\lambda\tilde{v}}.$$

The non-trivial terms in (4.3) are estimated as follows

$$(4.6) \quad -2\dot{\Phi} F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} \lambda e^{\lambda\tilde{v}} \leq \frac{\lambda}{2} \dot{\Phi} F^{ij} h_{ik} h_j^k e^{\lambda\tilde{v}} + c\lambda \dot{\Phi} F^{ij} g_{ij} e^{\lambda\tilde{v}},$$

and

$$(4.7) \quad \begin{aligned} |\tilde{f}_{\nu\beta} x_k^\beta h^{ik} x_i^\alpha \eta_\alpha| \lambda e^{\lambda\tilde{v}} &\leq |\tilde{f}_{\nu\beta} \tilde{v}_k x_l^\beta g^{kl}| \lambda e^{\lambda\tilde{v}} + c \|\tilde{f}_{\nu\beta}\| \lambda e^{\lambda\tilde{v}} \\ &\leq c \|\tilde{f}_{\nu\beta}\| \lambda e^{\lambda\tilde{v}} + \frac{\lambda^2}{2} \dot{\Phi} F^{ij} \tilde{v}_i \tilde{v}_j e^{\lambda\tilde{v}} \\ &\quad + c(\dot{\Phi})^{-1} \tilde{F}^{ij} g_{ij} e^{\lambda\tilde{v}}. \end{aligned}$$

With the help these estimates inequality (4.4) is easily derived; in the first inequality of (4.7) we used

$$(4.8) \quad \tilde{v}_i = \eta_{\alpha\beta} x_i^\beta \nu^\alpha + \eta_\alpha \nu_i^\alpha$$

together with the Weingarten equation. \square

4.2. The Riemannian case. As we have shown in [6, Sections 5 & 6], the flow hypersurfaces can be written as graphs over a geodesic unit sphere, $M(t) = \text{graph } u(t)$. The scalar version of (3.5) now looks like

$$(4.9) \quad \frac{\partial u}{\partial t} = -(\Phi - \tilde{f})v.$$

Moreover, all flow hypersurfaces stay in $\bar{\Omega}$ and v is uniformly bounded in view of the convexity of the $M(t)$.

Lemma 4.3 (Evolution of v). *The quantity v satisfies the parabolic equation*

$$(4.10) \quad \begin{aligned} \dot{v} - \dot{\Phi}F^{ij}v_{ij} &= -\dot{\Phi}F^{ij}h_{ik}h_j^k v - 2v^{-1}\dot{\Phi}F^{ij}v_i v_j \\ &+ [(\Phi - f) - \dot{\Phi}F]\eta_{\alpha\beta}\nu^\alpha\nu^\beta v^2 \\ &+ 2\dot{\Phi}F^{ij}h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} v^2 + \dot{\Phi}F^{ij}\eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha v^2 \\ &+ \dot{\Phi}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_k^\gamma \eta_\epsilon x_l^\epsilon g^{kl} v^2 \\ &+ \tilde{f}_\beta x_i^\beta x_k^\alpha \eta_\alpha g^{ik} v^2 + \tilde{f}_{\nu\beta} x_k^\beta h^{ik} x_i^\alpha \eta_\alpha v^2, \end{aligned}$$

where η is the covariant vector field $(\eta_\alpha) = (1, 0, \dots, 0)$.

For a proof see [6, Lemma 7.3].

Similar as in the Lorentzian case we obtain a parabolic inequality for $e^{\lambda v}$.

Corollary 4.4. *Let $\varphi = e^{\lambda v}$, then, φ satisfies the evolution inequality*

$$(4.11) \quad \begin{aligned} \dot{\varphi} - \dot{\Phi}F^{ij}\varphi_{ij} &\leq -\frac{\lambda}{2}\dot{\Phi}F^{ij}h_{ik}h_j^k e^{\lambda v} - \frac{\lambda^2}{2}\dot{\Phi}F^{ij}v_i v_j e^{\lambda v} \\ &+ c\lambda\dot{\Phi}F^{ij}g_{ij}e^{\lambda v} + c[-(\Phi - f) + 1]e^{\lambda v} \\ &+ c\lambda\dot{\Phi}F^{ij}g_{ij}e^{\lambda v} + c(\dot{\Phi})^{-1}\tilde{F}^{ij}g_{ij}e^{\lambda v}, \end{aligned}$$

where $(\tilde{F}^{ij}) = (F^{ij})^{-1}$, c is a known constant, and where we also used the estimate (3.15).

5. C^2 - ESTIMATES IN LORENTZIAN SPACE

Let $M(t)$ be a solution of the evolution problem (3.5) with initial hypersurface $M_0 = M_2$, defined on a maximal time interval $I = [0, T^*)$. We assume that F is of class (K) , homogeneous of degree 1, and satisfies the condition (0.3); we choose $\Phi(r) = \log r$.

Furthermore, we suppose that there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$, i.e. there holds

$$(5.1) \quad \chi_{\alpha\beta} \geq c_0 \bar{g}_{\alpha\beta}$$

with a positive constant c_0 .

We observe that

$$(5.2) \quad \begin{aligned} \dot{\chi} - \dot{\Phi} F^{ij} \chi_{ij} &= [(\Phi - \tilde{f}) - \dot{\Phi} F] \chi_{\alpha} \nu^{\alpha} - \dot{\Phi} F^{ij} \chi_{\alpha\beta} x_i^{\alpha} x_j^{\beta} \\ &\leq [(\Phi - \tilde{f}) - \dot{\Phi} F] \chi_{\alpha} \nu^{\alpha} - c_0 \dot{\Phi} F^{ij} g_{ij}, \end{aligned}$$

where we used the homogeneity of F .

From Remark 3.6 we infer

$$(5.3) \quad \Phi \geq \tilde{f} \quad \text{or} \quad F \geq f,$$

and from the results in Section 4 that the flow stays in the compact set $\bar{\Omega}$, and that \tilde{v} is uniformly bounded.

We are now able to prove

Lemma 5.1. *Let F be of class (K) satisfying (0.3). Then, the principal curvatures of the evolution hypersurfaces $M(t)$ are uniformly bounded.*

Proof. Let φ and w be defined respectively by

$$(5.4) \quad \varphi = \sup \{ h_{ij} \eta^i \eta^j : \|\eta\| = 1 \},$$

$$(5.5) \quad w = \log \varphi + e^{\lambda \tilde{v}} + \mu \chi,$$

where λ, μ are large positive parameters to be specified later. We claim that w is bounded for a suitable choice of λ, μ .

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(5.6) \quad \sup_{M_0} w < \sup \{ \sup_{M(t)} w : 0 < t \leq T \} = w(x_0).$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$(5.7) \quad g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h_n^n.$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$(5.8) \quad \tilde{\eta} = (0, \dots, 0, 1),$$

and set

$$(5.9) \quad \tilde{\varphi} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}.$$

$\tilde{\varphi}$ is well defined in neighbourhood of (t_0, ξ_0) .

Now, define \tilde{w} by replacing φ by $\tilde{\varphi}$ in (5.5); then, \tilde{w} assumes its maximum at (t_0, ξ_0) . Moreover, at (t_0, ξ_0) we have

$$(5.10) \quad \dot{\tilde{\varphi}} = \dot{h}_n^n,$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde{\varphi}$ satisfies the same differential equation (3.13) as h_n^n . For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that w is defined by

$$(5.11) \quad w = \log h_n^n + e^{\lambda\bar{v}} + \mu\chi.$$

At (t_0, ξ_0) we have $\dot{w} \geq 0$, and, in view of the maximum principle, we deduce from (0.3), (1.17), (3.13), (4.4), and (5.2)

$$(5.12) \quad \begin{aligned} 0 \leq & \dot{\Phi}Fh_n^n - (\Phi - \tilde{f})h_n^n + \lambda c_1 e^{\lambda\bar{v}} - \frac{\lambda}{2}\epsilon_0 \dot{\Phi}FHe^{\lambda\bar{v}} \\ & + \lambda c_1 \dot{\Phi}F^{ij}g_{ij}e^{\lambda\bar{v}} + c_1 He^{\lambda\bar{v}} \\ & + (\lambda e^{\lambda\bar{v}} + \mu)c_1[(\Phi - \tilde{f}) + \dot{\Phi}F] - \mu c_0 \dot{\Phi}F^{ij}g_{ij} \\ & + \dot{\Phi}F^{ij}(\log h_n^n)_i(\log h_n^n)_j \\ & + \{\ddot{\Phi}F_n F^n + \dot{\Phi}F^{kl,rs}h_{kl;n}h_{rs; \quad}^n\}(h_n^n)^{-1}, \end{aligned}$$

where we have estimated bounded terms by a constant c_1 , assumed that h_n^n , λ , and μ are larger than 1, and used (5.3).

Now, the last term in (5.12) is estimated from above by

$$(5.13) \quad \{\ddot{\Phi}F_n F^n + \dot{\Phi}F^{-1}F_n F^n\}(h_n^n)^{-1} - \dot{\Phi}F^{ij}h_{in;n}h_{jn; \quad}^n(h_n^n)^{-2},$$

cf. (1.10), where the sum in the braces vanishes, due to the choice of Φ . Moreover, because of the Codazzi equation, we have

$$(5.14) \quad h_{in;n} = h_{nn;i} + \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_n^\beta x_i^\gamma x_n^\delta,$$

and hence, using the abbreviation \bar{R}_i for the curvature term, we conclude that (5.13) is bounded from above by

$$(5.15) \quad -(h_n^n)^{-2} \dot{\Phi} F^{ij} (h_{n;i}^n + \bar{R}_i) (h_{n;j}^n + \bar{R}_j).$$

Thus, the terms in (5.12) containing the derivatives of h_n^n are estimated from above by

$$(5.16) \quad -2 \dot{\Phi} F^{ij} (\log h_n^n)_i \bar{R}_j (h_n^n)^{-1}.$$

Moreover, Dw vanishes at ξ_0 , i.e.

$$(5.17) \quad D \log h_n^n = -\lambda e^{\lambda \tilde{v}} D \tilde{v} - \mu D \chi,$$

where only $D \tilde{v}$ deserves further consideration.

Replacing then $D \tilde{v}$ by the right-hand side of (4.8), and using the Weingarten equation as well as the simple observation

$$(5.18) \quad |F^{ij} h_j^k \eta_k| \leq \|\eta\| F$$

for any vector field (η_k) , cf. [6, Lemma 7.4], we finally conclude from (5.12)

$$(5.19) \quad \begin{aligned} 0 &\leq \dot{\Phi} F h_n^n - (\Phi - \tilde{f}) h_n^n + \lambda c_1 e^{\lambda \tilde{v}} + c_1 H e^{\lambda \tilde{v}} - \frac{\lambda}{2} \epsilon_0 \dot{\Phi} F H e^{\lambda \tilde{v}} \\ &\quad + (\lambda e^{\lambda \tilde{v}} + \mu) c_1 [(\Phi - \tilde{f}) + \dot{\Phi} F] + \lambda c_1 e^{\lambda \tilde{v}} \dot{\Phi} F^{ij} g_{ij} \\ &\quad - \mu [c_0 - c_1 (h_n^n)^{-1}] \dot{\Phi} F^{ij} g_{ij} \end{aligned}$$

Then, if we suppose h_n^n to be so large that

$$(5.20) \quad c_1 \leq \frac{1}{2} c_0 h_n^n,$$

and if we choose λ, μ such that

$$(5.21) \quad 4 \leq \lambda \epsilon_0$$

and

$$(5.22) \quad 8 \lambda c_1 \leq \mu c_0$$

we derive

$$(5.23) \quad \begin{aligned} 0 &\leq -\frac{1}{4} \lambda \epsilon_0 \dot{\Phi} F H e^{\lambda \tilde{v}} - (\Phi - \tilde{f}) h_n^n + c_1 H e^{\lambda \tilde{v}} \\ &\quad + (\lambda e^{\lambda \tilde{v}} + \mu) c_1 [(\Phi - \tilde{f}) + \dot{\Phi} F]. \end{aligned}$$

We now observe that $\dot{\Phi}F = 1$, and deduce in view of (5.3) that h_n^n is a priori bounded at (t_0, ξ_0) . \square

The result of Lemma 5.1 can be restated as a uniform estimate for the functions $u(t) \in C^2(\mathcal{S}_0)$. Since, moreover, the principal curvatures of the flow hypersurfaces are not only bounded, but also uniformly bounded away from zero, in view of (5.3) and the assumption that F vanishes on $\partial\Gamma_+$, we conclude that F is uniformly elliptic on $M(t)$.

6. C^2 - ESTIMATES IN RIEMANNIAN SPACE

If N is Riemannian with $K_N \leq 0$, we use M_1 as the initial hypersurface for the evolution. This is the only change in the settings with regard to the Lorentzian case. Due to this choice we now have

$$(6.1) \quad \Phi - \tilde{f} \leq 0$$

during the evolution, cf. Remark 3.6..

For the C^2 - estimates we have to prove upper bounds for the principal curvatures as well as a strictly positive lower bound for F

$$(6.2) \quad 0 < \epsilon_2 \leq F,$$

since, in view of Remark 3.6, we presently only know that (3.15) is valid.

Furthermore, as we have seen in Remark 2.2, there exists a strictly convex function χ .

We now prove the corresponding result to Lemma 5.1

Lemma 6.1. *Let $F \in (K)$ such that (0.3) is satisfied, and let $M(t)$ be the solutions of (3.5) with initial hypersurface $M_0 = M_1$. Then, the principal curvatures of the $M(t)$ are uniformly bounded from above.*

Proof. The proof is identical to that of Lemma 5.1. We define φ and w as in (5.4) and (5.5), where of course \tilde{v} is replaced by v , and apply the maximum principle to w .

In a point where the maximum principle is applied we obtain in view of (1.17) an inequality that corresponds to the similar inequality (5.23)

$$(6.3) \quad \begin{aligned} 0 \leq & -\frac{\lambda}{4}\epsilon_0\dot{\Phi}FHe^{\lambda v} + (\Phi - \tilde{f})h_n^n + c_1He^{\lambda v} \\ & + (\lambda e^{\lambda v} + \mu)c_1[-(\Phi - \tilde{f}) + \dot{\Phi}F]. \end{aligned}$$

In deriving this inequality, we also used the simple estimate

$$(6.4) \quad \dot{\Phi} F^{ij} h_{ik} h_j^k \leq \dot{\Phi} F h_n^n.$$

From (6.3) we immediately get the required a priori estimate because of (6.1). \square

Lemma 6.2. *Assume that $K_N \leq 0$, then, there is a positive constant ϵ_2 such that the estimate (6.2) is valid.*

Proof. We proceed similar as in the proof of [6, Lemma 8.3]. Consider the function

$$(6.5) \quad w = -(\Phi - \tilde{f}) + \mu\chi,$$

where μ is large. Let $0 < T < T^*$ and suppose

$$(6.6) \quad \sup_{M_0} w < \sup\{ \sup_{M(t)} w : 0 \leq t \leq T \}.$$

Then, there exists $x_0 = x_0(t_0) \in \mathcal{S}_0$, $0 < t_0 \leq T$, such that

$$(6.7) \quad w(x_0) = \sup\{ \sup_{M(t)} w : 0 \leq t \leq T \}.$$

From (3.10), (5.2), and the maximum principle we then infer

$$(6.8) \quad 0 \leq -\dot{\Phi} F^{ij} h_{ik} h_j^k (\Phi - \tilde{f}) - \tilde{f}_\alpha \nu^\alpha (\Phi - \tilde{f}) + \tilde{f}_{\nu^\alpha} x_i^\alpha (\Phi - \tilde{f})_j g^{ij} \\ - \dot{\Phi} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta (\Phi - \tilde{f}) + \mu c_1 [1 - (\Phi - \tilde{f})] - \mu c_0 \dot{\Phi} F^{ij} g_{ij}.$$

Let κ be an upper bound for the principal curvatures, then, the first term on the right-hand side can be estimated from above by

$$(6.9) \quad -\dot{\Phi} F^{ij} h_{ik} h_j^k (\Phi - \tilde{f}) \leq -\dot{\Phi} F \kappa (\Phi - \tilde{f}).$$

The term involving the Riemann curvature tensor is non-positive since $K_N \leq 0$, and, hence, we deduce

$$(6.10) \quad 0 \leq \mu c_1 [1 - (\Phi - \tilde{f})] - \mu c_0 \dot{\Phi} F^{ij} g_{ij}$$

for $\mu \geq 1$, and we obtain an a priori estimate for $-(\Phi - \tilde{f})$, since

$$(6.11) \quad F^{ij} g_{ij} \geq F(1, \dots, 1)$$

and $\dot{\Phi} = F^{-1}$ is the dominating term in (6.10). \square

Remark 6.3. The assumption $K_N \leq 0$ was only necessary to obtain a uniform bound for the principal curvatures during the *evolution*. For stationary solutions

$$(6.12) \quad F|_M = f(x, \nu)$$

the proof of Lemma 6.1 would yield a priori estimates for the principal curvatures in arbitrary Riemannian manifolds as long as M is a graph in a Gaussian coordinate system, lower order estimates are valid, and there exists a strictly convex function in a neighbourhood of M .

This could be used to solve the Dirichlet problem for the equation (6.12), since in the existence proof for Dirichlet problems a deformation process is used instead of an evolutionary approximation, cf. [19].

7. CONVERGENCE TO A STATIONARY SOLUTION

We only consider the Lorentzian case since the essential arguments do not depend on the nature of the ambient space. Let $M(t)$ be the flow with initial hypersurface $M_0 = M_2$. Let us look at the scalar version of the flow (3.5)

$$(7.1) \quad \frac{\partial u}{\partial t} = -e^{-\psi} v(\Phi - \tilde{f}).$$

This is a scalar parabolic differential equation defined on the cylinder

$$(7.2) \quad Q_{T^*} = [0, T^*) \times \mathcal{S}_0$$

with initial value $u(0) = u_2 \in C^{4,\alpha}(\mathcal{S}_0)$. In view of the a priori estimates, which we have established in the preceding sections, we know that

$$(7.3) \quad |u|_{2,0,\mathcal{S}_0} \leq c$$

and

$$(7.4) \quad \Phi(F) \text{ is uniformly elliptic in } u$$

independent of t . Moreover, $\Phi(F)$ is concave, and thus, we can apply the regularity results of [17, Chapter 5.5] to conclude that uniform $C^{2,\alpha}$ -estimates are valid, leading further to uniform $C^{4,\alpha}$ -estimates due to the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e. $T^* = \infty$.

Now, integrating (6.1) with respect to t , and observing that the right-hand side is non-positive, yields

$$(7.5) \quad u(0, x) - u(t, x) = \int_0^t e^{-\psi} v(\Phi - \tilde{f}) \geq c \int_0^t (\Phi - \tilde{f}),$$

i.e.,

$$(7.6) \quad \int_0^\infty |\Phi - \tilde{f}| < \infty \quad \forall x \in \mathcal{S}_0$$

Hence, for any $x \in \mathcal{S}_0$ there is a sequence $t_k \rightarrow \infty$ such that $(\Phi - \tilde{f}) \rightarrow 0$.

On the other hand, $u(\cdot, x)$ is monotone decreasing and therefore

$$(7.7) \quad \lim_{t \rightarrow \infty} u(t, x) = \tilde{u}(x)$$

exists and is of class $C^{4,\alpha}(\mathcal{S}_0)$ in view of the a priori estimates. We, finally, conclude that \tilde{u} is a stationary solution of our problem, and that

$$(7.8) \quad \lim_{t \rightarrow \infty} (\Phi - \tilde{f}) = 0.$$

REFERENCES

- [1] P. Bayard, *Dirichlet problems for space-like hypersurfaces with prescribed scalar curvature in $\mathbb{R}^{n,1}$* , preprint 2
- [2] L. Caffarelli, L. Nirenberg & J. Spruck, *Nonlinear second order elliptic equations IV: Starshaped Weingarten hypersurfaces*, Current topics in partial differential equations (Y. Ohya, K. Kashara & N. Shimakura, eds.), Kinokunize, Tokyo, 1985, pp. 1–26. 1
- [3] K. Ecker & G. Huisken, *Immersed hypersurfaces with constant Weingarten curvature*, Math. Ann. **283** (1989) 329–332.
- [4] L. C. Evans, *Classical solutions of fully nonlinear, second-order elliptic equations*, Comm. Pure Appl. Math. XXV (1982) 333–363.
- [5] C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Commun. Math. Phys. **89** (1983) 523–553. 9
- [6] ———, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Differential Geom. **43** (1996) 512–641. 1, 11, 13, 16, 20, 21
- [7] ———, *Closed Weingarten hypersurfaces in space forms*, Geometric Analysis and the Calculus of Variations (J. Jost, ed.), International Press, Boston, 1996, pp. 71–98.
- [8] ———, *Hypersurfaces of prescribed Weingarten curvature*, Math. Z. **224** (1997) 167–194. 1, 8, 11, 12
- [9] ———, *Closed hypersurfaces of prescribed mean curvature in conformally flat Riemannian manifolds*, J. Differential Geom. **48** (1998) 587–613.
- [10] ———, *Erratum "Hypersurfaces of prescribed Weingarten curvature"*, Math. Z. **233** (2000) 829–830.

- [11] ———, *Hypersurfaces of prescribed mean curvature in Lorentzian manifolds*, Math. Z. **235** (2000) 83–97.
- [12] ———, *Hypersurfaces of prescribed curvature in Lorentzian manifolds*, Indiana Univ. Math. J. **49** (2000) 1125–1153. 2, 5, 7, 9, 10, 14, 15
- [13] ———, *Hypersurfaces of prescribed scalar curvature in Lorentzian manifolds*, J. reine angew. Math. **554** (2003), 157–199, math.DG/0207054. 2, 3, 7
- [14] R. P. Geroch, *The domain of dependence*, J. Math. Phys. **11** (1970) 437–449. 9
- [15] R. P. Geroch & G. T. Horowitz, *Global structure of space-time*, General Relativity. An Einstein centenary survey (S. W. Hawking & W. Israel, eds.), Cambridge University Press, Cambridge, 1979, pp. 212–293. 9
- [16] S. W. Hawking & G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973. 9
- [17] N. V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, Reidel, Dordrecht, 1987. 23
- [18] B. O’Neill, *Semi-riemannian geometry*, Academic Press, New-York, 1983.
- [19] O. C. Schnürer, *The Dirichlet problem for Weingarten hypersurfaces in Lorentzian manifolds*, Math. Z. **242** (2002) 159–181. 22

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