HYPERSURFACES OF PRESCRIBED SCALAR CURVATURE IN LORENTZIAN MANIFOLDS

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Dedicated to Robert Finn on the occasion of his eightieth birthday

Abstract. The existence of closed hypersurfaces of prescribed scalar curvature in globally hyperbolic Lorentzian manifolds is proved provided there are barriers.

Contents

0. Introduction

Consider the problem of finding a closed hypersurface of prescribed curvature F in a globally hyperbolic (n+1)-dimensional Lorentzian manifold N having a compact Cauchy hypersurface $\mathcal{S}_0$. To be more precise, let Ω be a connected open subset of $N, f \in C^{2,\alpha}(\bar{\Omega}), F$ a smooth, symmetric function defined in an open cone $\Gamma \subset \mathbb{R}^n$, then we look for a space-like hypersurface $M \subset \Omega$ such that

$$F_{|_M} = f(x) \qquad \forall x \in M, \tag{0.1}$$

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where $F_{|_M}$ means that F is evaluated at the vector $(\kappa_i(x))$ the components of which are the principal curvatures of M. The prescribed function f should satisfy natural structural conditions, e. g. if Γ is the positive cone and the hypersurface M is supposed to be convex, then f should be positive, but no further, merely technical, conditions should be imposed.

In [1, 2, 8, 14] the case $F = H$, the mean curvature, has been treated, and in [15] we solved the problem for curvature functions F of class K^* that includes the Gaussian curvature, see [15, Section 1] for the definition, but excludes the symmetric polynomials H_k for $1 < k < n$. Among these, H_2, that corresponds to the scalar curvature operator, is of special interest.

However, a solution of equation (0.1) with $F = H_2$ is in general not a hypersurface of prescribed scalar curvature—unless the ambient space has constant curvature—since the scalar curvature of a hypersurface also depends on $\bar R_{\alpha\beta}\nu^\alpha\nu^\beta$. Thus, we have to allow that the right-hand side f also depends on time-like vectors and look for hypersurfaces M satisfying

$$F_{|_M} = f(x,\nu) \qquad \forall x \in M, \tag{0.2}$$

where $\nu = \nu(x)$ is the past-directed normal of M in the point x.

To give a precise statement of the existence result we need a few definitions and assumptions. First, we assume that Ω is a precompact, connected, open subset of N, that is bounded by two *achronal*, connected, space-like hypersurfaces M_1 and M_2 of class $C^{4,\alpha}$, where M_1 is supposed to lie in the past of M_2.

Let $F = H_2$ be the scalar curvature operator defined on the open cone $\Gamma_2 \subset \mathbb{R}^n$, and $f = f(x,\nu)$ be of class $C^{2,\alpha}$ in its arguments such that

$$0 < c_1 \le f(x,\nu) \qquad \text{if} \quad \langle\nu,\nu\rangle = -1, \tag{0.3}$$

$$|||f_\beta(x,\nu)||| \le c_2(1 + |||\nu|||^2), \tag{0.4}$$

and

$$|||f_{\nu^\beta}(x,\nu)||| \le c_3(1 + |||\nu|||), \tag{0.5}$$

for all $x \in \bar\Omega$ and all past directed time-like vectors $\nu \in T_x(\Omega)$, where $|||\cdot|||$ is a Riemannian reference metric that will be detailed in Section 2.

We suppose that the boundary components M_i act as barriers for (F, f).

Definition 0.1. M_2 is an *upper barrier* for (F, f), if M_2 is *admissible*, i.e. its principal curvatures (κ_i) with respect to the past directed normal belong to Γ_2, and if

$$F_{|_{M_2}} \ge f(x,\nu) \qquad \forall\, x \in M_2. \tag{0.6}$$

M_1 is a lower barrier for (F, f), if at the points $\Sigma \subset M_1$, where M_1 is admissible, there holds

$$F_{|\Sigma} \le f(x,\nu) \qquad \forall\, x \in \Sigma. \tag{0.7}$$

Σ may be empty.

Remark 0.2. This definition of upper and lower barriers for a pair (F, f) also makes sense for other curvature functions F defined in an open convex cone Γ, with a corresponding meaning of the notion *admissable*.

Now, we can state the main theorem.

Theorem 0.3. *Let M_1 be a lower and M_2 an upper barrier for (F, f), where $F = H_2$. Then, the problem*

$$F_{|M} = f(x,\nu) \tag{0.8}$$

has an admissible solution $M \subset \bar\Omega$ of class $C^{4,\alpha}$ that can be written as a graph over $\mathcal{S}_0$ provided there exists a strictly convex function $\chi \in C^2(\bar\Omega)$.

Remark 0.4. As we have shown in [15, Lemma 2.7] the existence of a strictly convex function χ is guaranteed by the assumption that the level hypersurfaces $\{x^0 = \text{const}\}$ are strictly convex in $\bar\Omega$, where (x^α) is a Gaussian coordinate system associated with $\mathcal{S}_0$.

Looking at Robertson-Walker space-times it seems that the assumption of the existence of a strictly convex function in the neighbourhood of a given compact set is not too restrictive: in Minkowski space e.g. $\chi = -|x^0|^2 + |x|^2$ is a globally defined strictly convex function. The only obstruction we are aware of is the existence of a compact maximal slice. In the neighbourhood of such a slice a strictly convex function cannot exist.

The existence result of our main theorem would also be valid in Riemannian manifolds if one could prove C^1- estimates. For the C^2- estimates the nature of the ambient space is irrelevant though the proofs are slightly different.

For prescribed curvature problems it seems more natural to assume that the right-hand side f depends on (x, ν), and we shall prove in a subsequent paper existence results for curvature functions $F \in (K^*)$, where the ambient space can be Riemannian or Lorentzian, cf. [16].

The paper is organized as follows: In Section 1 we take a closer look at curvature functions and define the concept of *elliptic regularization* for these functions, and analyze some of its properties.

In Section 2 we introduce the notations and common definitions we rely on, and state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces.

In Section 3 we look at the curvature flow associated with our problem, and the corresponding evolution equations for the basic geometrical quantities of the flow hypersurfaces.

In Section 4 we prove lower order estimates for the evolution problem, while a priori estimates in the C^2-norm are derived in Section 5.

In Section 6, we demonstrate that the evolutionary solution converges to a stationary approximation of our problem, i.e. to a solution for a curvature problem, where F is replaced by its elliptic regularization F_ϵ.

The uniform C^1- estimates for the stationary approximations are derived in Sections 7 and 8, the C^2- estimates are given in Section 9, while the final existence result is contained in Section 10.

1. Curvature functions

Let $\Gamma \subset \mathbb{R}^n$ be an open cone containing the positive cone Γ_+, and $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma})$ a positive symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0\,; \tag{1.1}$$

then, F can also be viewed as a function defined on the space of symmetric matrices $\mathcal{C}$, the eigenvalues of which belong to Γ, namely, let $(h_{ij}) \in \mathcal{C}$ with eigenvalues κ_i, $1 \le i \le n$, then define F on $\mathcal{C}$ by

$$F(h_{ij}) = F(\kappa_i). \tag{1.2}$$

If we define

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{1.3}$$

and

$$F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\, \partial h_{kl}} \tag{1.4}$$

then,

$$F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}|\xi^i|^2 \quad \forall\, \xi \in \mathbb{R}^n, \tag{1.5}$$

in an appropriate coordinate system,

$$F^{ij} \text{ is diagonal if } h_{ij} \text{ is diagonal,} \tag{1.6}$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \frac{\partial^2 F}{\partial\kappa_i\,\partial\kappa_j}\eta_{ii}\eta_{jj} + \sum_{i\neq j}\frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2, \tag{1.7}$$

for any $(\eta_{ij}) \in \mathcal{S}$, where $\mathcal{S}$ is the space of all symmetric matrices. The second term on the right-hand side of (1.7) is non-positive if F is concave, and non-negative if F is convex, and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

The preceding considerations are also applicable if the κ_i are the principal curvatures of a space-like hypersurface M with metric (g_{ij}). F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) the eigenvalues of which belong to Γ. Such tensors will be called *admissible*; when the second fundamental form of M is admissible, then, we also call M admissible.

For an admissible tensor (h_{ij})

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{1.8}$$

is a contravariant tensor of second order. Sometimes it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$h^i_j = g^{ik}h_{kj}. \tag{1.9}$$

Then,

$$F^j_i = \frac{\partial F}{\partial h^i_j} \tag{1.10}$$

is also a mixed tensor with contravariant index j and covariant index i.

Such functions F are called curvature functions. Important examples are the symmetric polynomials of order k, H_k, $1 \le k \le n$,

$$H_k(\kappa_i) = \sum_{i_1<\cdots<i_k}\kappa_{i_1}\cdots\kappa_{i_k}. \tag{1.11}$$

They are defined on an open cone Γ_k that can be characterized as the connected component of $\{H_k > 0\}$ that contains Γ_+.

Since we have in mind that the κ_i are the principal curvatures of a hypersurface, we use the standard symbols H and $|A|$ for

$$H = \sum_i \kappa_i, \tag{1.12}$$

and

$$|A|^2 = \sum_i \kappa_i^2. \tag{1.13}$$

The scalar curvature function $F = H_2$ can then be expressed as

$$F = \frac{1}{2}(H^2 - |A|^2), \tag{1.14}$$

and we deduce that for $(\kappa_i) \in \Gamma_2$

$$|A|^2 \le H^2, \tag{1.15}$$

$$F_i = H - \kappa_i, \tag{1.16}$$

and hence,

$$HF_i \ge F, \tag{1.17}$$

for (1.17) is equivalent to

$$H\kappa_i \le \frac{1}{2}H^2 + \frac{1}{2}|A|^2, \tag{1.18}$$

which is obviously valid.

In important ingredient in our existence proof will be the method of *elliptic regularization*

Lemma 1.1. *For each $\epsilon > 0$, consider the linear isomorphism φ_ϵ in $\mathbb{R}^n$ given by*

$$(\tilde\kappa_i) = \varphi_\epsilon(\kappa_i) = (\kappa_i + \epsilon H). \tag{1.19}$$

Let $F \in C^2(\Gamma) \cap C^0(\bar\Gamma)$ be a curvature function such that

$$F_{|\partial\Gamma} = 0. \tag{1.20}$$

Then, $\Gamma_\epsilon = \varphi_\epsilon^{-1}(\Gamma)$ is an open cone and $F_\epsilon = F \circ \varphi_\epsilon \in C^2(\Gamma_\epsilon) \cap C^0(\bar\Gamma_\epsilon)$ a curvature function satisfying

$$F_{\epsilon|\partial\Gamma_\epsilon} = 0. \tag{1.21}$$

Assume furthermore, that

$$H > 0 \qquad \text{in} \quad \Gamma. \tag{1.22}$$

Then,

$$\varGamma \subset \varGamma_\epsilon, \tag{1.23}$$

and

$$H > 0 \quad \text{in} \quad \varGamma_\epsilon. \tag{1.24}$$

Proof. We only prove the assertions (1.23) and (1.24) since the other assertions are obvious. Let $(\kappa_i) \in \varGamma$ be fixed. Then,

$$0 < F(\kappa_i) \le F(\kappa_i + \epsilon H), \tag{1.25}$$

because F is monotone, and we deduce

$$(\kappa_i + \epsilon H) \in \varGamma \qquad \forall\, \epsilon > 0, \tag{1.26}$$

in view of (1.22) and the monotonicity of F, cf. (1.1).

To prove (1.24), we observe that

$$\sum_i \tilde\kappa_i = (1 + \epsilon n) \sum_i \kappa_i. \tag{1.27}$$

□

Remark 1.2. (i) Let F be as in Lemma 1.1 and assume moreover, that F is homogeneous of degree 1, and concave, then,

$$F(\kappa_i) \le \frac{1}{n} F(1, \dots, 1) H \qquad \forall\, (\kappa_i) \in \varGamma, \tag{1.28}$$

and we conclude that condition (1.22) is satisfied.

(ii) Let F be as in Lemma 1.1, but suppose that F is homogeneous of degree $d_0 > 0$ and $F^{\frac{1}{d_0}}$ concave, then, the relation (1.22) is also valid.

Proof. The inequality (1.28) follows easily from the concavity and homogeneity

$$\begin{aligned} F(\kappa_i) &\le F(1, \dots, 1) + \sum_i F_i(1, \dots, 1)(\kappa_i - 1) \\ &= \frac{1}{n} F(1, \dots, 1) H, \end{aligned} \tag{1.29}$$

since $F_i(1, \dots, 1) = \frac{1}{n} F(1, \dots, 1)$, while the other assertions are obvious. □

For better reference, we use a tensor setting in the next lemma, i.e. the $(\kappa_i) \in \Gamma$ are the eigenvalues of an admissible tensor (h_{ij}) with respect to a Riemannian metric (g_{ij}). In this setting the elliptic regularization of F is given by

$$\tilde{F}(h_{ij}) \equiv F(h_{ij} + \epsilon H g_{ij}). \tag{1.30}$$

Lemma 1.3. *Let $\tilde{F}$ be the elliptic regularization of a curvature function F of class C^2, then,*

$$\tilde{F}^{ij} = F^{ij} + \epsilon F^{rs} g_{rs} g^{ij}, \tag{1.31}$$

and

$$\begin{aligned} \tilde{F}^{ij,kl} &= F^{ij,kl} + \epsilon F^{ij,ab} g_{ab} g^{kl} \\ &\quad + \epsilon F^{rs,kl} g_{rs} g^{ij} + \epsilon^2 F^{rs,ab} g_{rs} g_{ab} g^{ij} g^{kl}. \end{aligned} \tag{1.32}$$

If F is concave, then, $\tilde{F}$ is also concave.

Proof. The relations (1.31) and (1.32) are straight-forward calculations.

To prove the concavity of $\tilde{F}$, let (η_{ij}) be a symmetric tensor, then,

$$\begin{aligned} \tilde{F}^{ij,kl} \eta_{ij} \eta_{kl} &= F^{ij,kl} \eta_{ij} \eta_{kl} + 2\epsilon F^{ij,rs} \eta_{ij} g_{rs} g^{kl} \eta_{kl} \\ &\quad + \epsilon^2 F^{rs,ab} g_{rs} g_{ab} (g^{ij} \eta_{ij})^2 \le 0. \end{aligned} \tag{1.33}$$

□

2. Notations and preliminary results

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces M in a (n+1)-dimensional Lorentzian space N. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function u in N, (u_α)

will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar R_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\bar R_{\alpha\beta\gamma\delta;i} = \bar R_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon \tag{2.1}$$

with obvious generalizations to other quantities.

Let M be a *space-like* hypersurface, i.e. the induced metric is Riemannian, with a differentiable normal ν that is time-like.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the space-like hypersurface M are connected through the following equations

$$x_{ij}^\alpha = h_{ij}\nu^\alpha \tag{2.2}$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar\Gamma_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma. \tag{2.3}$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to ν.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha, \tag{2.4}$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta \tag{2.5}$$

and the *Gauß equation*

$$R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar R_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta. \tag{2.6}$$

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. N is then a topological product $\mathbb{R}\times\mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that x^0 represents the time, the $(x^i)_{1\le i\le n}$ are local coordinates for $\mathcal{S}_0$, where we may assume that $\mathcal{S}_0$ is equal to the level hypersurface $\{x^0=0\}$—we don't distinguish between $\mathcal{S}_0$ and $\{0\}\times\mathcal{S}_0$—, and such that the Lorentzian metric takes the form

$$d\bar s_N^2 = e^{2\psi}\{-{dx^0}^2 + \sigma_{ij}(x^0,x)dx^i dx^j\}, \tag{2.7}$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the space-like components (x^i), see [17], [19, p. 212], [18, p. 252], and [8, Section 6]. We also assume that the coordinate system is *future oriented*, i.e. the time coordinate x^0 increases on future directed curves. Hence, the *contravariant* time-like vector$(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \operatorname{graph} u_{|\mathcal{S}_0}$ be a space-like hypersurface

$$M = \{\, (x^0, x) \colon x^0 = u(x),\ x \in \mathcal{S}_0 \,\}, \tag{2.8}$$

then the induced metric has the form

$$g_{ij} = e^{2\psi}\{-u_i u_j + \sigma_{ij}\} \tag{2.9}$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$g^{ij} = e^{-2\psi}\{\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\}, \tag{2.10}$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2. \end{aligned} \tag{2.11}$$

Hence, $\operatorname{graph} u$ is space-like if and only if $|Du| < 1$.

We also note that

$$v^{-2} = 1 + e^{2\psi} g^{ij} u_i u_j \equiv 1 + e^{2\psi} \|Du\|^2. \tag{2.12}$$

The covariant form of a normal vector of a graph looks like

$$(\nu_\alpha) = \pm v^{-1} e^{\psi}(1, -u_i). \tag{2.13}$$

and the contravariant version is

$$(\nu^\alpha) = \mp v^{-1} e^{-\psi}(1, u^i). \tag{2.14}$$

Thus, we have

Remark 2.1. Let M be space-like graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

$$(\nu^\alpha) = v^{-1} e^{-\psi}(1, u^i) \tag{2.15}$$

and the past directed

$$(\nu^\alpha) = -v^{-1} e^{-\psi}(1, u^i). \tag{2.16}$$

In the Gauß formula (2.2) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal for reasons that we have explained in [15].

Look at the component $\alpha = 0$ in (2.2) and obtain in view of (2.16)

$$e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{ij}. \tag{2.17}$$

Here, the covariant derivatives are taken with respect to the induced metric of M, and

$$-\bar{\Gamma}^0_{ij} = e^{-\psi} \bar{h}_{ij}, \tag{2.18}$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$\bar{h}_{ij} e^{-\psi} = -\tfrac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij}, \tag{2.19}$$

where the dot indicates differentiation with respect to x^0.

Next, let us analyze under which condition a space-like hypersurface M can be written as a graph over the Cauchy hypersurface $\mathcal{S}_0$.

We first need

Definition 2.2. Let M be a closed, space-like hypersurface in N. Then, M is said to be *achronal*, if no two points in M can be connected by a future directed time-like curve.

In [5] it is proved, see also [15, Proposition 2.5],

Proposition 2.3. *Let N be connected and globally hyperbolic, $\mathcal{S}_0 \subset N$ a compact Cauchy hypersurface, and $M \subset N$ a compact, connected space-like hypersurface of class $C^m, m \geq 1$. Then, $M = \operatorname{graph} u_{|\mathcal{S}_0}$ with $u \in C^m(\mathcal{S}_0)$ iff M is achronal.*

Remark 2.4. The M_i are barriers for the pair (F,f). Let us point out that without loss of generality we may assume

$$F_{|_{M_2}} > f(x,\nu) \qquad \forall\, x \in M_2, \tag{2.20}$$

and

$$F_{|_{\Sigma}} < f(x,\nu) \qquad \forall\, x \in \Sigma, \tag{2.21}$$

for let $\eta \in C^\infty(\bar{\Omega})$ be a function with support in a small neighbourhood of $M_1 \mathbin{\dot\cup} M_2$—the dot should indicate that the union is disjoint— such that

$$\eta_{|_{M_1}} > 0 \quad \text{and} \quad \eta_{|_{M_2}} < 0 \tag{2.22}$$

and define for $\delta > 0$

$$f_\delta = f + \delta\eta. \tag{2.23}$$

Then, if we assume f to be strictly positive with a positive lower bound, we have for small δ

$$f_\delta \geq \frac{1}{2} f, \tag{2.24}$$

and the M_i are barriers for (F, f_δ) satisfying the strict inequalities; since we shall derive $C^{4,\alpha}$ estimates independent of δ, we shall have proved the existence of a solution for f if we can prove it for f_δ.

Lemma 2.5. *Let M_i be barriers for (F,f) satisfying the strict inequalities* (2.20) *and* (2.21)*, where F is supposed to be monotone and concave. Then, they are also barriers for the elliptic regularizations F_ϵ for small ϵ.*

Proof. In view of Lemma 1.1, we know that $\Gamma \subset \Gamma_\epsilon$ and H is positive in Γ_ϵ. Hence, M_2 is certainly an upper barrier for (F_ϵ, f) because of the monotonicity of F.

Let Σ_ϵ resp. Σ be the points in M_1 where the principal curvatures belong to Γ_ϵ resp. Γ and assume that $\Sigma \neq \emptyset$. Suppose M_1 were not a lower barrier for (F_ϵ, f) for small ϵ, then, there exist a sequence $\epsilon \to 0$ and a corresponding convergent sequence $x_\epsilon \in \Sigma_\epsilon$, $x_\epsilon \to x_0 \in \Sigma$, such that

$$F_\epsilon \geq f(x_\epsilon, \nu), \tag{2.25}$$

and hence,

$$F \geq f(x_0, \nu) \tag{2.26}$$

contradicting (2.21). □

Remark 2.6. The condition (0.3) is reasonable as is evident from the Einstein equation

$$\bar{R}_{\alpha\beta} - \tfrac{1}{2}\bar{R}\bar{g}_{\alpha\beta} = T_{\alpha\beta}, \tag{2.27}$$

where the energy-momentum tensor $T_{\alpha\beta}$ is supposed to be positive semi-definite for time-like vectors (*weak energy condition*, cf. [19, p. 89]), and the relation

$$R = -[H^2 - h_{ij}h^{ij}] + \bar{R} + 2\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \tag{2.28}$$

for the scalar curvature of a space-like hypersurface; but it would be convenient for the approximations we have in mind, if the estimate in (0.3) would be valid for all time-like vectors.

In fact, we may assume this without loss of generality: Let ϑ be a smooth real function such that

$$\frac{c_1}{2} \le \vartheta \quad \text{and} \quad \vartheta(t) = t \quad \forall\, t \ge c_1, \tag{2.29}$$

then, we can replace f by $\vartheta \circ f$ and the new function satisfies our requirements for all time-like vectors.

We therefore assume in the following that the relation (0.3) holds for all time-like vectors $\nu \in T_x(N)$ and all $x \in \bar{\Omega}$.

Sometimes, we need a Riemannian reference metric, e.g. if we want to estimate tensors. Since the Lorentzian metric can be expressed as

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{2\psi}\{-{dx^0}^2 + \sigma_{ij}dx^i dx^j\}, \tag{2.30}$$

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

$$\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{2\psi}\{{dx^0}^2 + \sigma_{ij}dx^i dx^j\} \tag{2.31}$$

and we abbreviate the corresponding norm of a vectorfield η by

$$|||\eta||| = (\tilde{g}_{\alpha\beta}\eta^\alpha\eta^\beta)^{1/2}, \tag{2.32}$$

with similar notations for higher order tensors.

For a space-like hypersurface $M = \operatorname{graph} u$ the induced metrics with respect to $(\bar g_{\alpha\beta})$ resp. $(\tilde g_{\alpha\beta})$ are related as follows

$$\begin{aligned} \tilde g_{ij} &= \tilde g_{\alpha\beta} x_i^\alpha x_j^\beta = e^{2\psi}[u_i u_j + \sigma_{ij}] \\ &= g_{ij} + 2e^{2\psi} u_i u_j. \end{aligned} \tag{2.33}$$

Thus, if $(\xi^i) \in T_p(M)$ is a unit vector for (g_{ij}), then

$$\tilde g_{ij}\xi^i\xi^j = 1 + 2e^{2\psi}|u_i\xi^i|^2, \tag{2.34}$$

and we conclude for future reference

Lemma 2.7. *Let $M = \operatorname{graph} u$ be a space-like hypersurface in N, $p \in M$, and $\xi \in T_p(M)$ a unit vector, then*

$$|\!|\!| x_i^\beta \xi^i |\!|\!| \le c(1 + |u_i\xi^i|) \le c\tilde v, \tag{2.35}$$

where $\tilde v = v^{-1}$.

3. An auxiliary curvature problem

Solving the problem (0.2) involves two steps: first, proving a priori estimates, and secondly, applying a method to show the existence of a solution. In a general Lorentzian manifold the evolution method is the method of choice, but unfortunately, one cannot prove the necessary a priori estimates during the evolution when F is the scalar curvature operator. Both the C^1 and C^2- estimates fail for general $f = f(x,\nu)$.

Therefore, we use the *elliptic regularization* and consider the existence problem for the operators

$$F_\epsilon(\kappa_i) = F(\kappa_i + \epsilon H), \qquad \epsilon > 0, \tag{3.1}$$

i.e. we solve

$$F_{\epsilon|_M} = f(x,\nu). \tag{3.2}$$

Then, we prove uniform $C^{2,\alpha}$- estimates for the approximating solutions M_ϵ, and finally, let ϵ tend to zero.

The F_ϵ—or some positive power of it—belong to a class of curvature functions F that satisfy the following condition (H): $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar\Gamma)$, where $\Gamma \subset \mathbb{R}^n$ is an open cone containing Γ_+, F is symmetric, monotone, i.e. $F_i > 0$, homogeneous of degree 1, concave, vanishes on $\partial\Gamma$, and there exists $\epsilon_0 = \epsilon_0(F) > 0$ such that

$$F_i \ge \epsilon_0 \sum_k F_k \qquad \forall\, 1 \le i \le n. \tag{3.3}$$

Furthermore, the set

$$\Lambda_{\delta,\kappa} = \{\, (\kappa_i) \in \Gamma\colon 0 < \delta \le F(\kappa_i),\, \kappa_i \le \kappa \;\forall\, 1 \le i \le n \,\} \tag{3.4}$$

is compact.

Remark 3.1. If the original curvature function $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar\Gamma)$ is concave, homogeneous of degree 1, and vanishes on $\partial\Gamma$, then, the F_ϵ are of class (H) in the cone Γ_ϵ, and satisfy (3.3) with $\epsilon_0 = \epsilon$. The set

$$\tilde\Lambda_{\delta,\kappa} = \{\, (\kappa_i) \in \Gamma_\epsilon\colon 0 < \delta \le F_\epsilon(\kappa_i),\, \kappa_i \le \kappa \;\forall\, 1 \le i \le n \,\} \tag{3.5}$$

is compact for fixed ϵ.

If the parameters κ and δ are independent of ϵ, then the $\tilde\Lambda_{\delta,\kappa}$ are contained in a compact subset of Γ uniformly in ϵ, for small ϵ, $0 \le \epsilon \le \epsilon_1(\delta, \kappa, F)$.

Proof. In view of the results in Lemma 1.3 we only have to prove the compactness of $\tilde\Lambda_{\delta,\kappa}$. We shall also only consider the case when the estimates hold uniformly in ϵ.

Due to the concavity and homogeneity of F_ϵ we conclude from (1.28) that

$$F_\epsilon(\kappa_i) \le \frac{1}{n} F(1, \dots, 1)(1 + n\epsilon)H. \tag{3.6}$$

For $(\kappa_i) \in \tilde\Lambda_{\delta,\kappa}$ we, therefore, infer

$$\delta \le F_\epsilon(\kappa_i) \le \frac{1 + n\epsilon}{n} F(1, \dots, 1)H \le (1 + n\epsilon)F(1, \dots, 1)\kappa, \tag{3.7}$$

and thus,

$$\lim_{\epsilon \to 0} \epsilon H = 0, \tag{3.8}$$

uniformly in $\tilde\Lambda_{\delta,\kappa}$.

Suppose $\tilde\Lambda_{\delta,\kappa}$ would not stay in a compact subset of Γ for small $\epsilon, 0 < \epsilon \le \epsilon_1(\delta, \kappa, F)$. Then, there would exist a sequence $\epsilon \to 0$ and a corresponding sequence $(\kappa_i^\epsilon) \in \tilde\Lambda_{\delta,\kappa}$ converging to a point $(\kappa_i) \in \partial\Gamma$, which is impossible in view of (3.7), (3.8), and the continuity of F in $\bar\Gamma$. ☐

To prove the existence of hypersurfaces of prescribed curvature F for $F \in (H)$ we look at the evolution problem

$$\begin{aligned} \dot{x} &= (F - f)\nu, \\ x(0) &= x_0, \end{aligned} \tag{3.9}$$

where ν is the past-directed normal of the flow hypersurfaces $M(t)$, F the curvature evaluated at $M(t)$, $x = x(t)$ an embedding and x_0 an embedding of an initial hypersurface M_0, which we choose to be the upper barrier M_2.

Since F is an elliptic operator, short-time existence, and hence, existence in a maximal time interval $[0, T^*)$ is guaranteed. If we are able to prove uniform a priori estimates in $C^{2,\alpha}$, long-time existence and convergence to a stationary solution will follow immediately.

But before we prove the a priori estimates, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives. The proofs are identical to those of the corresponding results in a Riemannian setting, cf. [9, Section 3] and [15, Section 4], and will be omitted.

Lemma 3.2 (Evolution of the metric)**.** *The metric g_{ij} of $M(t)$ satisfies the evolution equation*

$$\dot{g}_{ij} = 2(F - f)h_{ij}. \tag{3.10}$$

Lemma 3.3 (Evolution of the normal)**.** *The normal vector evolves according to*

$$\dot{\nu} = \nabla_M(F - f) = g^{ij}(F - f)_i x_j. \tag{3.11}$$

Lemma 3.4 (Evolution of the second fundamental form)**.** *The second fundamental form evolves according to*

$$\dot{h}_i^j = (F - f)_i^j - (F - f)h_i^k h_k^j - (F - f)\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \tag{3.12}$$

and

$$\dot{h}_{ij} = (F - f)_{ij} + (F - f)h_i^k h_{kj} - (F - f)\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{3.13}$$

Lemma 3.5 (Evolution of $(F - f)$)**.** *The term $(F - f)$ evolves according to the equation*

$$\begin{aligned} (F - f)' - F^{ij}(F - f)_{ij} = &-F^{ij}h_{ik}h_j^k(F - f) - f_\alpha \nu^\alpha (F - f) \\ &- f_{\nu^\alpha} x_i^\alpha (F - f)_j g^{ij} - F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta (F - f), \end{aligned} \tag{3.14}$$

From (3.9) we deduce with the help of the Ricci identities a parabolic equation for the second fundamental form

Lemma 3.6. *The mixed tensor h_i^j satisfies the parabolic equation*

$$
\begin{aligned}
\dot h_i^j - &F^{kl} h_{i;kl}^j \\
&= -F^{kl} h_{rk} h_l^r h_i^j + f h_i^k h_k^j \\
&\quad - f_{\alpha\beta} x_i^\alpha x_k^\beta g^{kj} - f_\alpha \nu^\alpha h_i^j - f_{\alpha\nu^\beta}(x_i^\alpha x_k^\beta h^{kj} + x_l^\alpha x_k^\beta h_i^k\, g^{lj}) \\
&\quad - f_{\nu^\alpha\nu^\beta} x_l^\alpha x_k^\beta h_i^k h^{lj} - f_{\nu^\beta} x_k^\beta h_{i;l}^k\, g^{lj} - f_{\nu^\alpha} \nu^\alpha h_i^k h_k^j \\
&\quad + F^{kl,rs} h_{kl;i} h_{rs;}{}^j + 2F^{kl} \bar R_{\alpha\beta\gamma\delta} x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_l^m g^{rj} \\
&\quad - F^{kl} \bar R_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} - F^{kl} \bar R_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^{mj} \\
&\quad - F^{kl} \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j + f \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\
&\quad + F^{kl} \bar R_{\alpha\beta\gamma\delta;\epsilon} \{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\}.
\end{aligned}
\tag{3.15}
$$

The proof is identical to that of the corresponding result in the Riemannian case, cf. [9, Lemma 7.1 and Lemma 7.2]; the only difference is that f now also depends on ν.

Remark 3.7. In view of the maximum principle, we immediately deduce from (3.14) that the term $(F - f)$ has a sign during the evolution if it has one at the beginning, i.e., if the starting hypersurface M_0 is the upper barrier M_2, then $(F - f)$ is non-negative

$$F \geq f. \tag{3.16}$$

4. Lower order estimates for the auxiliary solutions

Since the two boundary components M_1, M_2 of $\partial\Omega$ are space-like, achronal hypersurfaces, they can be written as graphs over the Cauchy hypersurface $\mathcal{S}_0$, $M_i = \operatorname{graph} u_i$, $i = 1, 2$, and we have

$$u_1 \leq u_2, \tag{4.1}$$

for M_1 should lie in the past of M_2, and the enclosed domain is supposed to be connected.

Let us look at the evolution equation (3.9) with initial hypersurface M_0 equal to M_2 defined on a maximal time interval $I = [0, T^*), T^* \leq \infty$. Since the initial hypersurface is a graph over $\mathcal{S}_0$, we can write

$$M(t) = \operatorname{graph} u(t)_{|_{\mathcal{S}_0}} \quad \forall\, t \in I, \tag{4.2}$$

where u is defined in the cylinder $Q_{T^*} = I \times \mathcal{S}_0$. We then deduce from (3.9), looking at the component $\alpha = 0$, that u satisfies a parabolic equation of the form

$$\dot{u} = -e^{-\psi} v^{-1}(F - f), \tag{4.3}$$

where we use the notations in Section 2, and where we emphasize that the time derivative is a total derivative, i.e.

$$\dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i. \tag{4.4}$$

Since the past directed normal can be expressed as

$$(\nu^\alpha) = -e^{-\psi} v^{-1}(1, u^i), \tag{4.5}$$

we conclude from (3.9), (4.3), and (4.4)

$$\frac{\partial u}{\partial t} = -e^{-\psi} v(F - f). \tag{4.6}$$

Thus, $\frac{\partial u}{\partial t}$ is non-positive in view of Remark 3.7.

Next, let us state our first a priori estimate

Lemma 4.1. *Suppose that the boundary components act as barriers for (F, f), then the flow hypersurfaces stay in $\bar{\Omega}$ during the evolution.*

The proof is identical to that of the corresponding result in [15, Lemma 4.1].

For the C^1- estimate the term $\tilde{v} = v^{-1}$ is of great importance. It satisfies the following evolution equation

Lemma 4.2 (Evolution of $\tilde{v}$)**.** *Consider the flow* (3.9) *in the distinguished coordinate system associated with $\mathcal{S}_0$. Then, $\tilde{v}$ satisfies the evolution equation*

$$\begin{aligned}
\dot{\tilde{v}} - F^{ij}\tilde{v}_{ij} = &- F^{ij} h_{ik} h^k_j \tilde{v} - f\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\
&- 2F^{ij} h^k_j x^\alpha_i x^\beta_k \eta_{\alpha\beta} - F^{ij}\eta_{\alpha\beta\gamma} x^\beta_i x^\gamma_j \nu^\alpha \\
&- F^{ij} \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_k x^\delta_j \eta_\epsilon x^\epsilon_l g^{kl} \\
&- f_\beta x^\beta_i x^\alpha_k \eta_\alpha g^{ik} - f_{\nu^\beta} x^\beta_k h^{ik} x^\alpha_i \eta_\alpha,
\end{aligned} \tag{4.7}$$

where η is the covariant vector field $(\eta_\alpha) = e^\psi(-1, 0, \ldots, 0)$.

The proof uses the relation

$$\tilde{v} = \eta_\alpha \nu^\alpha \tag{4.8}$$

and is identical to that of [15, Lemma 4.4] having in mind that presently f also depends on ν.

Lemma 4.3. *Let $M(t) = \operatorname{graph} u(t)$ be the flow hypersurfaces, then, we have*

$$\begin{aligned} \dot{u} - F^{ij} u_{ij} = e^{-\psi} \tilde{v} f + \bar{\Gamma}^0_{00}\, F^{ij} u_i u_j \\ + 2F^{ij} \bar{\Gamma}^0_{0i}\, u_j + F^{ij} \bar{\Gamma}^0_{ij}, \end{aligned} \tag{4.9}$$

where all covariant derivatives a taken with respect to the induced metric of the flow hypersurfaces, and the time derivative $\dot{u}$ is the total time derivative, i.e. it is given by (4.4).

Proof. We use the relation (4.3) together with (2.17). □

As an immediate consequence we obtain

Lemma 4.4. *The composite function*

$$\varphi = e^{\mu e^{\lambda u}} \tag{4.10}$$

where μ, λ are constants, satisfies the equation

$$\begin{aligned} \dot{\varphi} - F^{ij} \varphi_{ij} = & f e^{-\psi} \tilde{v} \mu \lambda\, e^{\lambda u}\, \varphi + F^{ij} u_i u_j\, \bar{\Gamma}^0_{00}\, \mu\lambda\, e^{\lambda u} \varphi \\ & + 2F^{ij} u_i \bar{\Gamma}^0_{0j}\, \mu\lambda\, e^{\lambda u}\, \varphi + F^{ij} \bar{\Gamma}^0_{ij}\, \mu\lambda\, e^{\lambda u}\, \varphi \\ & - [1 + \mu\, e^{\lambda u}] F^{ij} u_i u_j\, \mu \lambda^2\, e^{\lambda u}\, \varphi. \end{aligned} \tag{4.11}$$

Before we can prove the C^1- estimates we need two more lemmata.

Lemma 4.5. *There is a constant $c = c(\Omega)$ such that for any positive function $0 < \epsilon = \epsilon(x)$ on $\mathcal{S}_0$ and any hypersurface $M(t)$ of the flow we have*

$$|||\nu||| \le c \tilde{v}, \tag{4.12}$$

$$g^{ij} \le c \tilde{v}^2 \sigma^{ij}, \tag{4.13}$$

$$F^{ij} \le F^{kl} g_{kl} g^{ij}, \tag{4.14}$$

$$|F^{ij}h^k_j x^\alpha_i x^\beta_k \eta_{\alpha\beta}| \le \frac{\epsilon}{2} F^{ij} h^k_i h_{kj} \tilde v + \frac{c}{2\epsilon} F^{ij} g_{ij} \tilde v^3, \tag{4.15}$$

$$|F^{ij}\eta_{\alpha\beta\gamma} x^\beta_i x^\gamma_j \nu^\alpha| \le c\tilde v^3 F^{ij} g_{ij}, \tag{4.16}$$

and

$$|F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_k x^\delta_j \eta_\epsilon x^\epsilon_l g^{kl}| \le c\tilde v^3 F^{ij} g_{ij}. \tag{4.17}$$

Proof. (i) The first three inequalities are obvious.

(ii) (4.15) follows from the generalized Schwarz inequality combined with (4.13) and (4.14).

(iii) (4.16) is a direct consequence of (4.13) and (4.14).

(iv) The proof of (4.17) is a bit more complicated and uses the symmetry properties of the Riemann curvature tensor.

Let

$$a_{ij} = \bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_k x^\delta_j \eta_\epsilon x^\epsilon_l g^{kl}. \tag{4.18}$$

We shall show that the symmetrization of a_{ij} satisfies

$$-c\tilde v^3 g_{ij} \le \frac{1}{2}(a_{ij} + a_{ji}) \le c\tilde v^3 g_{ij} \tag{4.19}$$

with a uniform constant $c = c(\Omega)$, which in turn yields (4.17).

Let $p \in M(t)$ be arbitrary, (x^α) be the special Gaussian coordinate of N, and (ξ^i) local coordinates around p such that

$$x^\alpha_i = \begin{cases} u_i, & \alpha = 0, \\ \delta^k_i, & \alpha = k. \end{cases} \tag{4.20}$$

We also note that all indices are raised with respect to g^{ij} with the exception of the contravariant vector

$$\check u^i = \sigma^{ij} u_j. \tag{4.21}$$

We point out that

$$\|Du\|^2 = g^{ij} u_i u_j = e^{-2\psi} \tilde v^2 \sigma^{ij} u_i u_j, \tag{4.22}$$

$$\tilde v^2 = 1 + e^{2\psi}\|Du\|^2, \tag{4.23}$$

$$(\nu^\alpha) = -\tilde v(1, \check u^i) e^{-\psi}, \tag{4.24}$$

and

$$\eta_\epsilon x_l^\epsilon g^{kl} = -e^\psi u^k. \tag{4.25}$$

First, let us observe that in view of (4.25) and the symmetry properties of the Riemann curvature tensor we have

$$a_{ij} u^j = 0. \tag{4.26}$$

Next, we shall expand the right-hand side of (4.18) explicitly.

$$\begin{aligned} a_{ij} &= \bar R_{0i0j}\tilde v \|Du\|^2 + \bar R_{0ik0}\tilde v u_j u^k + \bar R_{0ikj}\tilde v u^k \\ &\quad + \bar R_{l0k0}\tilde v u^k \check u^l u_i u_j + \bar R_{l00j}\tilde v \check u^l u_i \|Du\|^2 \\ &\quad + \bar R_{l0kj}\tilde v u^k \check u^l u_i + \bar R_{li0j}\tilde v \check u^l \|Du\|^2 \\ &\quad + \bar R_{lik0}\tilde v u^k \check u^l u_j + \bar R_{likj}\tilde v u^k \check u^l \end{aligned} \tag{4.27}$$

To prove the estimate (4.19), we may assume that $Du \neq 0$. Let e_i, $1 \le i \le n$ be an orthonormal base of $T_p(M(t))$ such that

$$e_1 = \frac{Du}{\|Du\|}, \tag{4.28}$$

then, for $2 \le k \le n$, the e_k are also orthonormal with respect to the metric $e^{2\psi}\sigma_{ij}$, and it is also valid that

$$\sigma_{ij}\check u^i e_k^j = 0 \qquad \forall\, 2 \le k \le n, \tag{4.29}$$

where $e_k = (e_k^i)$.

For $2 \le r, s \le n$ we deduce from (4.27)

$$\begin{aligned} a_{ij} e_r^i e_s^j &= \bar R_{0i0j}\tilde v \|Du\|^2 e_r^i e_s^j + \bar R_{0ikj}\tilde v u^k e_r^i e_s^j \\ &\quad + \bar R_{li0j}\tilde v \check u^l \|Du\|^2 e_r^i e_s^j + \bar R_{likj}\tilde v u^k \check u^l e_r^i e_s^j \end{aligned} \tag{4.30}$$

and hence,

$$|a_{ij} e_r^i e_s^j| \le c\tilde v^3 \qquad \forall\, 2 \le r, s \le n. \tag{4.31}$$

It remains to estimate $a_{ij} e_1^i e_r^j$ for $2 \le r \le n$, because of (4.26).

We deduce from (4.27)

$$a_{ij} e_1^i e_r^j = \bar R_{0i0j}\tilde v \|Du\|^2 \tilde v^{-2} e_1^i e_r^j + \bar R_{0ikj}\tilde v^{-1} u^k e_1^i e_r^j, \tag{4.32}$$

where we used the symmetry properties of the Riemann curvature tensor.

Hence, we conclude

$$|a_{ij}e_1^i e_r^j| \le c\tilde{v}^2 \qquad \forall\, 2 \le r \le n, \tag{4.33}$$

and the relation (4.19) is proved. □

Lemma 4.6. *Let $M \subset \bar{\Omega}$ be a graph over $\mathcal{S}_0$, $M = \operatorname{graph} u$, and $\epsilon = \epsilon(x)$ a function given in $\mathcal{S}_0$, $0 < \epsilon < \frac{1}{2}$. Let φ be defined through*

$$\varphi = e^{\mu e^{\lambda u}}, \tag{4.34}$$

where $0 < \mu$ and $\lambda < 0$. Then, there exists $c = c(\Omega)$ such that

$$\begin{aligned} 2|F^{ij}\tilde{v}_i\varphi_j| &\le cF^{ij}g_{ij}\tilde{v}^3|\lambda|\mu e^{\lambda u}\varphi + (1-2\epsilon)F^{ij}h_i^k h_{kj}\tilde{v}\varphi \\ &\quad + \frac{1}{1-2\epsilon}F^{ij}u_iu_j\mu^2\lambda^2 e^{2\lambda u}\tilde{v}\varphi. \end{aligned} \tag{4.35}$$

Proof. Since $\tilde{v} = \eta_\alpha \nu^\alpha$, we have

$$\begin{aligned} \tilde{v}_i &= \eta_{\alpha\beta}\nu^\alpha x_i^\beta + \eta_\alpha h_i^k x_k^\alpha \\ &= \eta_{\alpha\beta}\nu^\alpha x_i^\beta - e^{\psi}h_i^k u_k. \end{aligned} \tag{4.36}$$

Thus, we derive

$$\begin{aligned} 2|F^{ij}\tilde{v}_i\varphi_j| &= 2|F^{ij}\tilde{v}_iu_j||\lambda|\mu e^{\lambda u}\varphi \\ &\le cF^{ij}g_{ij}\tilde{v}^3|\lambda|\mu e^{\lambda u}\varphi + 2e^{\psi}|F^{ij}h_i^k u_k u_j||\lambda|\mu e^{\lambda u}\varphi. \end{aligned} \tag{4.37}$$

The last term of the preceding inequality can be estimated by

$$(1-2\epsilon)F^{ij}h_i^k h_{kj}\tilde{v}\varphi + \frac{1}{1-2\epsilon}\tilde{v}^{-1}e^{2\psi}\|Du\|^2 F^{ij}u_iu_j\mu^2\lambda^2 e^{2\lambda u}\varphi \tag{4.38}$$

and we obtain the desired estimate in view of (4.23). □

Applying Lemma 4.5 to the evolution equation for $\tilde{v}$ we conclude

Lemma 4.7. *There exists a constant $c = c(\Omega)$ such that for any function ϵ, $0 < \epsilon = \epsilon(x) < 1$, defined on $\mathcal{S}_0$ the term $\tilde{v}$ satisfies an evolution inequality of the form*

$$\begin{aligned}\dot{\tilde{v}} - F^{ij}\tilde{v}_{ij} &\le -(1-\epsilon)F^{ij}h_i^k h_{kj}\tilde{v} - f\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &\quad + \frac{c}{\epsilon}F^{ij}g_{ij}\tilde{v}^3 + c|||f_\beta|||\tilde{v}^2 + f_{\nu^\beta}x_l^\beta h^{kl}u_k e^\psi.\end{aligned} \tag{4.39}$$

We are now ready to prove the uniform boundedness of $\tilde{v}$.

Proposition 4.8. *Assume that there are positive constants c_i, $1 \le i \le 3$, such that for any $x \in \Omega$ and any past directed time-like vector ν there holds*

$$-c_1 \le f(x,\nu), \tag{4.40}$$

$$|||f_\beta(x,\nu)||| \le c_2(1 + |||\nu|||), \tag{4.41}$$

and

$$|||f_{\nu^\beta}(x,\nu)||| \le c_3. \tag{4.42}$$

Then, the term $\tilde{v}$ remains uniformly bounded during the evolution

$$\tilde{v} \le c = c(\Omega, c_1, c_2, c_3, \epsilon_0), \tag{4.43}$$

where ϵ_0 is the constant in (3.3). *Here, and in the following, the reference that a constant depends on Ω also means that it depends on the barriers and geometric quantities of the ambient space restricted to Ω.*

Proof. We proceed similar as in [14, Proposition 3.7] and show that the function

$$w = \tilde{v}\varphi, \tag{4.44}$$

φ as in (4.34), is uniformly bounded, if we choose

$$0 < \mu < 1 \quad \text{and} \quad \lambda << -1, \tag{4.45}$$

appropriately, and assume furthermore, without loss of generality, that $u \le -1$, for otherwise replace u by $(u - c)$, c large, in the definition of φ.

With the help of the lemmata 4.4, 4.6, and 4.7 we derive from the relation

$$\dot{w} - F^{ij}w_{ij} = [\dot{\tilde{v}} - F^{ij}\tilde{v}_{ij}]\varphi + [\dot{\varphi} - F^{ij}\varphi_{ij}]\tilde{v} - 2F^{ij}\tilde{v}_i\varphi_j \tag{4.46}$$

the parabolic inequality

$$
\begin{aligned}
\dot{w} - F^{ij} w_{ij} \le & -\epsilon F^{ij} h_i^k h_{kj} \tilde{v}\varphi + c[\epsilon^{-1} + |\lambda|\mu e^{\lambda u}] F^{ij} g_{ij} \tilde{v}^3 \varphi \\
& + [\frac{1}{1-2\epsilon} - 1] F^{ij} u_i u_j \mu^2 \lambda^2 e^{2\lambda u} \tilde{v}\varphi \\
& - F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \tilde{v}\varphi \\
& + f[-\eta_{\alpha\beta}\nu^\alpha\nu^\beta + e^{-\psi}\mu\lambda e^{\lambda u}\tilde{v}^2]\varphi \\
& + c\,|||f_\beta|||\tilde{v}^2\varphi + f_{\nu^\beta} x_l^\beta h^{kl} u_k e^\psi \varphi,
\end{aligned} \tag{4.47}
$$

where we have chosen the same function $\epsilon = \epsilon(x)$ in Lemma 4.6 resp. Lemma 4.7.

Setting $\epsilon = e^{-\lambda u}$ and using Lemma 2.7, the assumption (3.3), which can be rewritten as

$$
F^{ij} \ge \epsilon_0 F^{kl} g_{kl} g^{ij}, \tag{4.48}
$$

as well as the assumptions (4.40), (4.41), and (4.42), and observing, furthermore, that in view of the concavity and homogeneity of F

$$
F^{ij} g_{ij} \ge F(1, \dots, 1) > 0, \tag{4.49}
$$

we conclude

$$
\begin{aligned}
\dot{w} - F^{ij} w_{ij} \le & -\frac{1}{2} F^{ij} h_i^k h_{kj} e^{-\lambda u} \tilde{v}\varphi + c|\lambda|\mu e^{\lambda u} F^{ij} g_{ij} \tilde{v}^3 \varphi \\
& + \frac{2}{1-2\epsilon} F^{ij} u_i u_j \mu^2 \lambda^2 e^{\lambda u} \tilde{v}\varphi - F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \tilde{v}\varphi \\
& + c c_1 \mu |\lambda| e^{\lambda u} \tilde{v}^2 \varphi + c c_2 \tilde{v}^3 \varphi + c c_3^2 \epsilon_0^{-1} e^{\lambda u} \tilde{v}^3 \varphi,
\end{aligned} \tag{4.50}
$$

where $|\lambda|$ is chosen so large that

$$
e^{-\lambda u} \le \frac{1}{4}. \tag{4.51}
$$

Choosing, furthermore,

$$
\mu = \frac{1}{8}, \tag{4.52}
$$

we see that the terms involving $F^{ij}u_iu_j$ add up to a dominating negative quantity that can be estimated from above by

$$-\frac{1}{16}F^{ij}u_iu_j\lambda^2e^{\lambda u}\tilde{v}\varphi \le -\frac{\epsilon_0}{16}F^{kl}g_{kl}\|Du\|^2\lambda^2e^{\lambda u}\tilde{v}\varphi, \tag{4.53}$$

in view of (4.48).

$\|Du\|^2$ is of the order $\tilde{v}^2$ for large $\tilde{v}$, hence, the parabolic maximum principle yields a uniform estimate for w if $|\lambda|$ is chosen large enough. □

5. C^2- ESTIMATES FOR THE AUXILIARY SOLUTIONS

We want to prove that the principal curvatures of the flow hypersurfaces are uniformly bounded.

Proposition 5.1. *Let $M(t)$, $0 \le t < T^*$, be solutions of the evolution problem* (3.9) *with $M(0) = M_2$, $F \in (H)$, and $f \in C^{2,\alpha}$ strictly positive,*

$$0 < c_0 \le f. \tag{5.1}$$

Then, the principal curvatures of the flow hypersurfaces are uniformly bounded provided the $M(t)$ are uniformly space-like, i.e. uniform C^1- estimates are valid.

Proof. As we have already mentioned in Remark 3.7, we know that

$$0 < c_0 \le f \le F \tag{5.2}$$

during the evolution, thus, it is sufficient to estimate the principal curvatures from above.

Let φ be defined by

$$\varphi = \sup\{\, h_{ij}\eta^i\eta^j : \|\eta\| = 1\,\}. \tag{5.3}$$

We claim that φ is uniformly bounded.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0}\varphi < \sup\{\, \sup_{M(t)}\varphi : 0 < t \le T\,\} = \varphi(x_0). \tag{5.4}$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h^n_n. \tag{5.5}$$

Let $\tilde\eta = (\tilde\eta^i)$ be the contravariant vector field defined by

$$\tilde\eta = (0, \dots, 0, 1), \tag{5.6}$$

and set

$$\tilde\varphi = \frac{h_{ij}\tilde\eta^i\tilde\eta^j}{g_{ij}\tilde\eta^i\tilde\eta^j}. \tag{5.7}$$

$\tilde\varphi$ is well defined in neighbourhood of (t_0, ξ_0), and $\tilde\varphi$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$\dot{\tilde\varphi} = \dot h^n_n, \tag{5.8}$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde\varphi$ satisfies the same differential equation (3.15) as h^n_n. For the sake of greater clarity, let us therefore treat h^n_n like a scalar and pretend that $\varphi = h^n_n$.

At (t_0, ξ_0) we have $\dot\varphi \ge 0$, and, in view of the maximum principle, we deduce from Lemma 3.6

$$\begin{aligned} 0 \le &-\epsilon_0 F^{ij} g_{ij} |A|^2 h^n_n + f|h^n_n|^2 + cF^{ij} g_{ij}(h^n_n + 1) \\ &+ c(1 + |A|^2)(1 + f + |||Df||| + |||D^2 f|||), \end{aligned} \tag{5.9}$$

where we used the concavity of F, the Codazzi equations, (4.48), and where

$$|A|^2 = g^{ij} h^k_i h_{kj}. \tag{5.10}$$

Thus, φ is uniformly bounded in view of (4.49). $\square$

6. Convergence to a stationary solution

We shall show that the solution of the evolution problem (3.9) exists for all time, and that it converges to a stationary solution.

Proposition 6.1. *The solutions $M(t) = \operatorname{graph} u(t)$ of the evolution problem* (3.9) *with $F \in (H)$, and $M(0) = M_2$ exist for all time and converge to a stationary solution provided $f \in C^{2,\alpha}$ satisfies the conditions* (4.41), (4.42), *and* (5.1).

Proof. Let us look at the scalar version of the flow as in (4.6)

$$\frac{\partial u}{\partial t} = -e^{-\psi} v(F - f). \tag{6.1}$$

This is a scalar parabolic differential equation defined on the cylinder

$$Q_{T^*} = [0, T^*) \times \mathcal{S}_0 \tag{6.2}$$

with initial value $u(0) = u_2 \in C^{4,\alpha}(\mathcal{S}_0)$. In view of the a priori estimates, which we have established in the preceding sections, we know that

$$|u|_{2,0,\mathcal{S}_0} \le c \tag{6.3}$$

and

$$F \text{ is uniformly elliptic in } u \tag{6.4}$$

independent of t due to the definition of the class (H). Thus, we can apply the known regularity results, see e.g. [20, Chapter 5.5], where even more general operators are considered, to conclude that uniform $C^{2,\alpha}$-estimates are valid, leading further to uniform $C^{4,\alpha}$-estimates due to the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e. $T^* = \infty$.

Now, integrating (6.1) with respect to t, and observing that the right-hand side is non-positive, yields

$$u(0, x) - u(t, x) = \int_0^t e^{-\psi} v(F - f) \ge c \int_0^t (F - f), \tag{6.5}$$

i.e.,

$$\int_0^\infty |F - f| < \infty \qquad \forall x \in \mathcal{S}_0. \tag{6.6}$$

Hence, for any $x \in \mathcal{S}_0$ there is a sequence $t_k \to \infty$ such that $(F - f) \to 0$.
On the other hand, $u(\cdot, x)$ is monotone decreasing and therefore

$$\lim_{t \to \infty} u(t, x) = \tilde{u}(x) \tag{6.7}$$

exists and is of class $C^{4,\alpha}(\mathcal{S}_0)$ in view of the a priori estimates. We, finally, conclude that $\tilde{u}$ is a stationary solution, and that

$$\lim_{t \to \infty} (F - f) = 0. \tag{6.8}$$

□

An immediate consequence of the results we have proved so far—cf. especially Lemma 2.5 and Remark 3.1— is the following theorem which is of independent interest.

Theorem 6.2. *Let $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma})$ be a concave curvature function vanishing on $\partial\Gamma$ and homogeneous of degree 1. Let $f = f(x,\nu)$ of class $C^{2,\alpha}$ satisfy the conditions* (4.41), (4.42), *and* (5.1), *and suppose that the boundary components M_i act as barriers for (F, f), then, there exists an admissible hypersurface $M = \operatorname{graph} u$, $u \in C^{4,\alpha}(\bar{\mathcal{S}}_0)$, solving*

$$F_{\epsilon|_M} = f(x,\nu) \tag{6.9}$$

for small $\epsilon > 0$.

7. Stationary approximations

We want to solve the equation

$$H_{2|_M} = f(x,\nu), \tag{7.1}$$

where f satisfies the conditions of Theorem 0.3. The curvature function $F = H_2^{\frac{1}{2}}$ is concave and the elliptic regularization F_ϵ of class (H), cf. (3.1) and Remark 3.1.

Thus, we would like to apply the preceding existence result to find hypersurfaces $M_\epsilon \subset \bar{\Omega}$ such that

$$F_{\epsilon|_{M_\epsilon}} = f^{\frac{1}{2}}. \tag{7.2}$$

But, unfortunately, the derivatives f_β grow quadratically in $|||\nu|||$ contrary to the assumption (4.41) in Proposition 4.8.

Therefore, we define a smooth cut-off function $\theta \in C^\infty(\mathbb{R}_+)$, $0 < \theta \le 2k$, where $k \ge k_0 > 1$ is to be determined later, by

$$\theta(t) = \begin{cases} t, & 0 \le t \le k, \\ 2k, & 2k \le t, \end{cases} \tag{7.3}$$

such that

$$0 \le \dot{\theta} \le 4, \tag{7.4}$$

and consider the problem

$$F_{\epsilon|_{M_\epsilon}} = \tilde f(x,\tilde\nu), \tag{7.5}$$

where for a space-like hypersurface $M = \operatorname{graph} u$ with past directed normal vector ν as in (2.16), we set

$$\tilde\nu = \theta(\tilde v)\tilde v^{-1}\nu \tag{7.6}$$

and

$$\tilde f(x,\tilde\nu) = f^{\frac12}(x,\tilde\nu). \tag{7.7}$$

Then,

$$|||\tilde\nu||| \le ck, \tag{7.8}$$

so that the assumptions in Proposition 4.8 are certainly satisfied.

The constant k_0 should be so large that $\tilde\nu = \nu$ in case of the barriers M_i, $i = 1, 2$.

If we now start with the evolution equation

$$\dot x = (F_\epsilon - \tilde f)\nu, \tag{7.9}$$

then, the M_i are barriers for $(F_\epsilon, \tilde f)$ for small ϵ, cf. Lemma 2.5 and we conclude

Lemma 7.1. *The flow hypersurfaces $M_\epsilon(t) = \operatorname{graph} u_\epsilon$ stay in $\bar\Omega$ during the evolution if ϵ is small $0 < \epsilon \le \bar\epsilon(\Omega)$.*

Remark 7.2. When we consider the elliptic regularizations F_ϵ, we would like to generalize the meaning of *admissible* hypersurface by calling a hypersurface admissible if the tensor $h_{ij} + \epsilon H g_{ij}$ is admissible, i.e. if its eigenvalues belong to Γ_2.

Next, let us consider the evolution equations for $\tilde v$ and h_i^j which look slightly different: In (4.7) the term involving f_{ν_β} has to be replaced by

$$-\tilde f_{\tilde\nu_\beta}[\theta(\tilde v)\tilde v^{-1}\nu_i^\beta + \dot\theta\tilde v_i\tilde v^{-1}\nu^\beta - \theta\tilde v^{-2}\tilde v_i\nu^\beta]x_k^\alpha g^{ik}\eta_\alpha. \tag{7.10}$$

But in view of (4.36), the additional terms do not cause any new problems in the proof of Proposition 4.8, and hence, the uniform C^1- estimates are still valid for the modified evolution problem, where the estimates depend on k.

The C^2- estimates in Section 5 remain valid, too, since the second derivatives of $\tilde f$, $\tilde f^j_i$, that occur on the right-hand side of (3.15), can be expressed as—we only consider the covariant form $\tilde f_{ii}$, no summation over i—

$$\begin{aligned} -\tilde f_{ii} = &-\tilde f_{\alpha\beta}x^\alpha_i x^\beta_i - 2\tilde f_{\alpha\tilde\nu^\beta}x^\alpha_i\tilde\nu^\beta_i \\ &- \tilde f_\alpha\nu^\alpha h_{ii} - \tilde f_{\tilde\nu^\alpha\tilde\nu^\beta}\tilde\nu^\alpha_i\tilde\nu^\beta_i - \tilde f_{\tilde\nu^\alpha}\tilde\nu^\alpha_{i;i}, \end{aligned} \tag{7.11}$$

where

$$\tilde\nu^\alpha_i = \theta\tilde v^{-1}\nu^\alpha_i + \dot\theta\tilde v_i\tilde v^{-1}\nu^\alpha - \theta\tilde v^{-2}\tilde v_i\nu^\alpha, \tag{7.12}$$

$$\begin{aligned} \tilde\nu^\alpha_{i;i} = &\,2\dot\theta\tilde v_i\tilde v^{-1}\nu^\alpha_i - 2\theta\tilde v^{-2}\tilde v_i\nu^\alpha_i + \theta\tilde v^{-1}\nu^\alpha_{i;i} \\ &+ \ddot\theta\tilde v_i\tilde v_i\tilde v^{-1}\nu^\alpha - 2\dot\theta\tilde v_i\tilde v_i\tilde v^{-2}\nu^\alpha + \dot\theta\tilde v_{ii}\tilde v^{-1}\nu^\alpha \\ &+ 2\theta\tilde v^{-3}\tilde v_i\tilde v_i\nu^\alpha - \theta\tilde v^{-2}\tilde v_{ii}\nu^\alpha, \end{aligned} \tag{7.13}$$

and

$$\tilde v_{ii} = \eta_{\alpha\beta\gamma}x^\beta_i x^\gamma_i\nu^\alpha + \eta_{\alpha\beta}\nu^\alpha\nu^\beta h_{ii} + 2\eta_{\alpha\beta}x^\beta_i\nu^\alpha_i + \eta_\alpha\nu^\alpha_{i;i}. \tag{7.14}$$

Hence, the result of Proposition 5.1 is still valid since no additional bad terms occur in inequality (5.9) as one easily checks, and since, furthermore, we also have

$$\tilde f \le F_\epsilon \tag{7.15}$$

during the evolution, for the modified version of (3.14) now has the form

$$\begin{aligned} (F_\epsilon - \tilde f)' &- F^{ij}_\epsilon(F_\epsilon - \tilde f)_{ij} \\ &= -F^{ij}_\epsilon h_{ik}h^k_j(F_\epsilon - \tilde f) - \tilde f_\alpha\nu^\alpha(F_\epsilon - \tilde f) \\ &\quad - \tilde f_{\tilde\nu^\gamma}\nu^\gamma[\dot\theta\tilde v^{-1} - \theta\tilde v^{-2}]\eta_{\alpha\beta}\nu^\alpha\nu^\beta(F_\epsilon - \tilde f) \\ &\quad - [\dot\theta\tilde v^{-1} - \theta\tilde v^{-2}]\tilde f_{\tilde\nu^\beta}\nu^\beta\eta_\alpha x^\alpha_i(F_\epsilon - \tilde f)_j g^{ij} \\ &\quad - \theta\tilde v^{-1}\tilde f_{\tilde\nu^\alpha}x^\alpha_i(F_\epsilon - \tilde f)_j g^{ij} \\ &\quad - F^{ij}_\epsilon\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i\nu^\gamma x^\delta_j(F_\epsilon - \tilde f). \end{aligned} \tag{7.16}$$

Here, we used the relation

$$\dot{\tilde v} = \eta_{\alpha\beta}\dot x^\beta\nu^\alpha + \eta_\alpha\dot\nu^\alpha, \tag{7.17}$$

which follows immediately from (4.8), together with (3.9) and (3.11).

The conclusions of Section 6 are therefore applicable leading to a solution of equation (7.5).

8. C^1- ESTIMATES FOR THE STATIONARY APPROXIMATIONS

Consider the solutions $M_\epsilon = \operatorname{graph} u_\epsilon$ of equation (7.5), which at the moment not only depend on ϵ but also on k, the parameter of the cut-off function θ, cf. (7.3). We shall prove that the hypersurfaces M_ϵ are uniformly space-like independent of ϵ and k, or, equivalently, that there exists a constant m_1 such that

$$\tilde{v} = (1 - |Du_\epsilon|^2)^{-\frac{1}{2}} \leq m_1 \qquad \forall\, \epsilon, k, \tag{8.1}$$

where the parameter ϵ is supposed to be small and k to be large, so that the barrier condition is satisfied.

Lemma 8.1. *Let u_ϵ be a solution of* (7.5)*, then, the estimate* (8.1) *is valid uniformly in ϵ and k. Hence, $M_\epsilon = \operatorname{graph} u_\epsilon$ is a solution of equation* (7.2)*, if we choose $k \geq 2m_1$.*

Proof. For arbitrary but fixed values of ϵ and k, let us introduce the notation $\tilde{F}$ for F_ϵ, where from now on through the rest of the article

$$F = H_2, \tag{8.2}$$

and where $f = f(x, \nu)$ satisfies

$$0 < c_1 \leq f(x, \nu), \tag{8.3}$$

$$|||f_\beta(x, \nu)||| \leq c_2(1 + |||\nu|||^2), \tag{8.4}$$

and

$$|||f_{\nu^\beta}(x, \nu)||| \leq c_3(1 + |||\nu|||), \tag{8.5}$$

for all $x \in \bar{\Omega}$ and all past directed time-like vectors $\nu \in T_x(\Omega)$.

Thus, F is homogeneous of degree 2, and we recall that

$$F^{ij} = Hg^{ij} - h^{ij}, \tag{8.6}$$

and

$$\tilde{F}^{ij} = F^{ij} + \epsilon(n-1)(1 + \epsilon n)Hg^{ij}, \tag{8.7}$$

where $\tilde F^{ij}$ is evaluated at h_{ij} and F^{ij} at $(h_{ij}+\epsilon H g_{ij})$.

We also drop the index ϵ, writing u for u_ϵ and M for M_ϵ, i.e. M solves the equation

$$\tilde F_{|_M} = f(x,\tilde\nu). \tag{8.8}$$

The C^1- estimate will follow the arguments in the proof of Proposition 4.8, where at one point we shall introduce an additional observation especially suitable for the curvature function $F=H_2$.

Remark 8.2. The former parabolic equations and inequalities, (4.11), (4.39), and (4.47) can now be read as elliptic equations resp. inequalities by simply assuming that the terms involved are time independent. Though, to be absolutely precise, one has to observe that the present curvature function is homogeneous of degree 2, which means that, whenever the term F—not derivatives of F—occurs explicitly in the equations or inequalities just mentioned, it has to be replaced by $2F$ because it was obtained as a result of Euler's formula for homogeneous functions of degree d_0

$$d_0 F = F^{ij}h_{ij}. \tag{8.9}$$

We mention it as a matter of fact only, since it doesn't affect the estimates at all.

However, we have to be aware that f now depends on $\tilde\nu$ instead of ν, i.e. the elliptic version of inequality (4.47) now takes the form

$$\begin{aligned}
-\tilde F^{ij}w_{ij} \le{}& -\delta\,\tilde F^{ij}h^k_i h_{kj}\tilde v\varphi + c[\delta^{-1}+|\lambda|\mu e^{\lambda u}]\tilde F^{ij}g_{ij}\tilde v^3\varphi\\
&+[\frac{1}{1-2\delta}-1]\tilde F^{ij}u_iu_j\mu^2\lambda^2e^{2\lambda u}\tilde v\varphi\\
&-\tilde F^{ij}u_iu_j\mu\lambda^2e^{\lambda u}\tilde v\varphi\\
&+2f[-\eta_{\alpha\beta}\nu^\alpha\nu^\beta+e^{-\psi}\mu\lambda e^{\lambda u}\tilde v^2]\varphi + c\,|||f_\beta|||\tilde v^2\varphi\\
&+f_{\tilde\nu^\beta}[\theta\tilde v^{-1}\nu^\beta_i+\dot\theta\tilde v_i\tilde v^{-1}\nu^\beta-\theta\tilde v^{-2}\tilde v_i\nu^\beta]u^ie^\psi\varphi.
\end{aligned} \tag{8.10}$$

Here, we used the notation $\delta=\delta(x)$ for the small parameter in the Schwarz inequality instead of ϵ, which has a different meaning in the present context, w is defined as in (4.44), where the parameters μ,λ should satisfy the conditions in (4.45), and u is supposed to be less than -1.

We claim that w is uniformly bounded provided μ and λ are chosen appropriately. Following the arguments in Section 4, we shall use the maximum principle and consider a point $x_0 \in M$, where

$$w(x_0) = \sup_M w. \tag{8.11}$$

As before, we choose $\delta = e^{-\lambda u}$. But the further conclusions are no longer valid, since we have a really bad term on the right-hand side of (8.10) that is of the order $\tilde{v}^4$ due to the assumption (8.4).

The only possible good term which can balance it, is

$$-\delta \tilde{F}^{ij} h_i^k h_{kj} \tilde{v} \varphi. \tag{8.12}$$

To exploit this term we use the fact that $Dw(x_0) = 0$, or, equivalently

$$\begin{aligned} -\tilde{v}_i &= \mu\lambda e^{\lambda u} \tilde{v} u_i \\ &= e^{\psi} h_i^k u_k - \eta_{\alpha\beta} \nu^\alpha x_i^\beta, \end{aligned} \tag{8.13}$$

where the second equation follows from (4.8) and the definition of the covariant vectorfield $\eta = e^{\psi}(-1, 0, \dots, 0)$.

Next, we choose a coordinate system (ξ^i) such that in the critical point

$$g_{ij} = \delta_{ij} \qquad \text{and} \qquad h_i^k = \kappa_i \delta_i^k, \tag{8.14}$$

and the labelling of the principal curvatures corresponds to

$$\kappa_1 \le \kappa_2 \le \cdots \le \kappa_n. \tag{8.15}$$

Then, we deduce from (8.13)

$$e^{\psi} \kappa_i u_i = \mu\lambda e^{\lambda u} \tilde{v} u_i + \eta_{\alpha\beta} \nu^\alpha x_i^\beta. \tag{8.16}$$

Assume that $\tilde{v}(x_0) \ge 2$, and let $i = i_0$ be an index such that

$$|u_{i_0}|^2 \ge \frac{1}{n} \|Du\|^2. \tag{8.17}$$

Setting $(e^i) = \frac{\partial}{\partial \xi^{i_0}}$ and assuming without loss of generality that $0 \le u_i e^i$ in x_0 we infer from Lemma 2.7

$$\begin{aligned} e^{\psi} \kappa_{i_0} u_i e^i &= \mu\lambda e^{\lambda u} \tilde{v} u_i e^i + \eta_{\alpha\beta} \nu^\alpha x_i^\beta e^i \\ &\le \mu\lambda e^{\lambda u} \tilde{v} u_i e^i + c\tilde{v}^2, \end{aligned} \tag{8.18}$$

and we deduce further in view of (2.12) and (8.17) that

$$\kappa_{i_0} \le [\mu\lambda e^{\lambda u} + c]\tilde{v}e^{-\psi} \le \frac{1}{2}\mu\lambda e^{\lambda u}\tilde{v}e^{-\psi}, \tag{8.19}$$

if $|\lambda|$ is sufficiently large, i.e. κ_{i_0} is *negative* and of the same order as $\tilde{v}$.

The Weingarten equation and Lemma 2.7 yield

$$\|\nu_i^\beta u^i\| = \|h_i^k u^i x_k^\beta\| \le c\tilde{v}[h_i^k u^i h_{kl} u^l]^{\frac{1}{2}}, \tag{8.20}$$

and therefore, we infer from (8.13)

$$|\tilde{v}_i u^i| + \|\nu_i^\beta u^i\| \le c\mu|\lambda| e^{\lambda u}\tilde{v}^3 \tag{8.21}$$

in critical points of w, and hence, that in those points, the term involving $f_{\tilde{\nu}^\beta}$ on the right-hand side of inequality (8.10) can be estimated from above by

$$|f_{\tilde{\nu}^\beta}[\theta\tilde{v}^{-1}\nu_i^\beta + \dot{\theta}\tilde{v}_i\tilde{v}^{-1}\nu^\beta - \theta\tilde{v}^{-2}\tilde{v}_i\nu^\beta]u^i e^\psi\varphi| \le c\mu|\lambda|e^{\lambda u}\tilde{v}^4\varphi. \tag{8.22}$$

Next, let us estimate the crucial term in (8.12). Using (8.7), the particular coordinate system (8.14), as well as the inequalities (8.15), together with the fact that κ_{i_0} is negative, we conclude

$$\begin{aligned} -\tilde{F}^{ij}h_i^k h_{kj} \le -F^{ij}h_i^k h_{kj} &\le -\sum_{i=1}^{i_0} F_i^i \kappa_i^2 \\ &\le -\sum_{i=1}^{i_0} F_i^i \kappa_{i_0}^2, \end{aligned} \tag{8.23}$$

where we recall that the argument of F_i^i is the n- tupel with components

$$\tilde{\kappa}_j = \kappa_j + \epsilon H \tag{8.24}$$

and observe that in the present coordinate system

$$F_i^i = \frac{\partial F}{\partial \tilde{\kappa}_i}. \tag{8.25}$$

Let $\hat{F} = \log F$, then, $\hat{F}$ is concave, and therefore, we have in view of (8.15)

$$\hat{F}_1^1 \ge \hat{F}_2^2 \ge \cdots \ge \hat{F}_n^n, \tag{8.26}$$

cf. [6, Lemma 2], or equivalently,

$$F_1^1 \geq F_2^2 \geq \cdots \geq F_n^n. \tag{8.27}$$

Hence, we conclude

$$\begin{aligned} -\sum_{i=1}^{i_0} F_i^i \leq -F_1^1 &\leq -\frac{1}{n}\sum_{i=1}^{n} F_i^i \\ &= -\frac{1}{n}(n-1)[H+\epsilon nH] \leq -\frac{n-1}{n}H, \end{aligned} \tag{8.28}$$

where we also used (8.6) and (8.24).

Combining (8.19), (8.23), (8.28), and the estimate (1.15), we deduce further

$$\begin{aligned} -\tilde F^{ij}h_i^k h_{kj} &\leq -\frac{n-1}{n}H\kappa_{i_0}^2 \\ &\leq -\frac{n-1}{2n}c_n|\kappa_{i_0}|^3 - \frac{n-1}{2n}H\kappa_{i_0}^2 \\ &\leq -a_0\mu^3|\lambda|^3 e^{3\lambda u}\tilde v^3 - a_1 H\mu^2\lambda^2 e^{2\lambda u}\tilde v^2 \end{aligned} \tag{8.29}$$

with some positive constants $a_0 = a_0(n,\Omega)$ and $a_1 = a_1(n,\Omega)$.

Inserting this estimate, and the estimate in (8.22) in the elliptic inequality (8.10), with $\delta = e^{-\lambda u}$, we finally obtain

$$\begin{aligned} -\tilde F^{ij}w_{ij} \leq{}& -a_0\mu^3|\lambda|^3 e^{2\lambda u}\tilde v^4\varphi - a_1 H\mu^2\lambda^2 e^{\lambda u}\tilde v^3\varphi \\ &+ \frac{2}{1-2\delta}\tilde F^{ij}u_iu_j\mu^2\lambda^2 e^{\lambda u}\tilde v\varphi + c[1+|\lambda|\mu]He^{\lambda u}\tilde v^3\varphi \\ &- \tilde F^{ij}u_iu_j\mu\lambda^2 e^{\lambda u}\tilde v\varphi + c[c_2 + c_3\mu|\lambda|e^{\lambda u}]\tilde v^4\varphi \\ &+ 2f[c + e^{-\psi}\mu\lambda e^{\lambda u}]\tilde v^2\varphi. \end{aligned} \tag{8.30}$$

Choosing, now, $\mu = \frac{1}{4}$ and $|\lambda|$ large, the right-hand side of the preceding inequality is negative, contradicting the maximum principle, i.e. the maximum of w cannot occur at point where $\tilde v \geq 2$. Thus, the desired uniform estimate for w and, hence, $\tilde v$ is proved. □

Let us close this section with an interesting observation that is an immediate consequence of the preceding proof, we have especially (8.23) and the first line of inequality (8.28) in mind,

Lemma 8.3. *Let $F \in C^2(\Gamma)$ be a positive symmetric curvature function such that the partial derivatives F_i are positive and $\hat{F} = \log F$ is concave. Suppose F is evaluated at a point (κ_i), and assume that κ_{i_0} is a component that is either negative or the smallest component of that particular n- tupel, then*

$$\sum_{i=1}^n F_i \kappa_i^2 \ge \frac{1}{n} \sum_{i=1}^n F_i \kappa_{i_0}^2. \tag{8.31}$$

9. C^2- ESTIMATES FOR THE STATIONARY APPROXIMATIONS

We want to prove uniform C^2- estimates for admissible solutions M of

$$\tilde{F}_{|_M} = f(x,\nu), \tag{9.1}$$

where we use the notations and conventions of the preceding section.

The starting point is an elliptic equation for the second fundamental form.

Lemma 9.1. *The tensor h_i^j satisfies the elliptic equation*

$$\begin{aligned}
-\tilde{F}^{kl} & h^j_{i;kl} \\
&= -\tilde{F}^{kl} h_{rk} h_l^r h_i^j + 2 f h_i^k h_k^j \\
&\quad - f_{\alpha\beta} x_i^\alpha x_k^\beta g^{kj} - f_\alpha \nu^\alpha h_i^j - f_{\alpha\nu^\beta}(x_i^\alpha x_k^\beta h^{kj} + x_l^\alpha x_k^\beta h_i^k\, g^{lj}) \\
&\quad - f_{\nu^\alpha\nu^\beta} x_l^\alpha x_k^\beta h_i^k h^{lj} - f_{\nu^\beta} x_k^\beta h^k_{i;l}\, g^{lj} - f_{\nu^\alpha}\nu^\alpha h_i^k h_k^j \\
&\quad + \tilde{F}^{kl,rs} h_{kl;i} h_{rs;}{}^j + 2\tilde{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_l^m g^{rj} \\
&\quad - \tilde{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} - \tilde{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^{mj} \\
&\quad - \tilde{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j + 2 f \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\
&\quad + \tilde{F}^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\}.
\end{aligned} \tag{9.2}$$

Proof. The elliptic equation can be immediately derived from the corresponding parabolic equation (3.15) by dropping the time-derivative, replacing F by $\tilde{F}$, and observing that the present curvature function is homogeneous of degree 2, cf. Remark 8.2. $\square$

Contracting over the indices (i,j) in (9.2) we obtain a differential equation for H

$$
\begin{aligned}
&-\tilde F^{kl}H_{kl}\\
&\quad= -\tilde F^{kl}h_{rk}h^r_l H + 2f|A|^2\\
&\qquad - f_{\alpha\beta}x^\alpha_i x^\beta_k g^{ki} - f_\alpha \nu^\alpha H - 2 f_{\alpha\nu^\beta}x^\alpha_i x^\beta_k h^{ki}\\
&\qquad - f_{\nu^\alpha\nu^\beta}x^\alpha_l x^\beta_k h^k_i h^{li} - f_{\nu^\beta}(H^k x^\beta_k + |A|^2\nu^\beta)\\
&\qquad - \bar R_{\alpha\beta}\nu^\alpha x^\beta_k g^{kl}x^\gamma_l f_{\nu^\gamma} + \tilde F^{kl,rs}h_{kl;i}h_{rs;}{}^i\\
&\qquad + 2\tilde F^{kl}\bar R_{\alpha\beta\gamma\delta}x^\alpha_m x^\beta_i x^\gamma_k x^\delta_r h^m_l g^{ri} - 2\tilde F^{kl}\bar R_{\alpha\beta\gamma\delta}x^\alpha_m x^\beta_k x^\gamma_i x^\delta_l h^{mi}\\
&\qquad - \tilde F^{kl}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_k \nu^\gamma x^\delta_l H + 2f\bar R_{\alpha\beta}\nu^\alpha\nu^\beta\\
&\qquad + \tilde F^{kl}\bar R_{\alpha\beta\gamma\delta;\epsilon}\{\nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\epsilon_m g^{mi} + \nu^\alpha x^\beta_i x^\gamma_k x^\delta_m x^\epsilon_l g^{mi}\},
\end{aligned}
\tag{9.3}
$$

where we also used the symmetry properties of the Riemann curvature tensor and the Codazzi equations at one point.

Next, let us improve the estimate in Proposition 5.1.

Lemma 9.2. *Let $M = \operatorname{graph} u$ be an admissible solution of* (9.1)*, then the principal curvatures of M satisfy the estimate*

$$\epsilon|A|^2 \le \text{const}, \tag{9.4}$$

where the constant depends on $|||Df|||, |||D^2 f|||$, the constant c_1 in (8.3)*, and on known estimates of the C^0 and C^1- norm of u.*

Proof. We argue as in the proof of Proposition 5.1 and define

$$\varphi = \sup\{\, h_{ij}\eta^i\eta^j : \|\eta\| = 1\,\}. \tag{9.5}$$

Let $x_0 \in M$ be a point, where φ achieves its maximum, and assume without loss of generality that, after having introduced normal Riemannian coordinates around x_0, we may write $\varphi = h^n_n$, cf. the corresponding arguments in the proof of Proposition 5.1.

Applying the maximum principle in x_0, we deduce from (9.2) the following inequality

$$
\begin{aligned}
0 &\le -\epsilon(n-1)H|A|^2 h^n_n + 2f|h^n_n|^2 + \tilde F^{kl,rs}h_{kl;n}h_{rs;}{}^n\\
&\quad + c(1 + |||Df||| + |||D^2 f|||)(1+|A|^2) + c(\tilde F^{ij}g_{ij} + f),
\end{aligned}
\tag{9.6}
$$

where we also used (8.7), the Weingarten and Codazzi equations, and the fact that the pair (x,ν) stays in a compact subset of $\bar\Omega\times C_-(\bar\Omega)$, where $C_-(\bar\Omega)$ stands for the set of past directed time-like vectorfields in $\bar\Omega$.

Furthermore, we know that

$$\tilde F^{ij}g_{ij}\le cH, \tag{9.7}$$

and

$$\begin{aligned}\tilde F^{kl,rs}h_{kl;n}h_{rs;}{}^n &\le \tilde F^{-1}\tilde F_{;n}\tilde F_;{}^n = f^{-1}f_nf^n\\ &\le c\,c_1^{-1}(1+|A|^2),\end{aligned} \tag{9.8}$$

since $\log\tilde F$ is concave, cf. Lemma 1.3.

Thus, we conclude that in x_0 the following inequality is valid

$$0\le -\epsilon|A|^4+c(1+|A|^2) \tag{9.9}$$

with a known constant c, and the lemma is proved. □

The estimate (9.4) will play an important role in the final a priori estimate.

Lemma 9.3. *Let $F=H_2$, $M=M^n$ a Riemannian manifold with metric g_{ij}, h_{ij} a symmetric tensor field on M the eigenvalues of which belong to Γ_2, and $p\in M$ an arbitrary point. Choose local coordinates around p such that the relations* (8.14) *and* (8.15) *are satisfied. Then, we have for $1\le j\le n$*

$$\sum_{i\ne j}\kappa_i^2+2F=|F^j_j|^2+2F^j_j\kappa_j, \tag{9.10}$$

$$\sum_{i\ne n}\kappa_i^2+2F\le cF^n_n\kappa_n, \tag{9.11}$$

and

$$\sum_{i\ne j}\Big|\frac{F_{;j}}{F^j_j}+\frac{F_{;j}}{H}(1-\frac{F^i_i}{F^j_j})\Big|^2\le c|F_{;j}|^2F^{-1}, \tag{9.12}$$

with $c=c(n)$, F is evaluated at h_{ij}, and where we point out that the summation convention is not used.

Proof. Throughout the proof we shall use the ambivalent meaning of F as a function depending on κ_i or on h_{ij} switching freely from one viewpoint to the other.

(i) From the definition of F

$$F = \frac{1}{2}(H^2 - |A|^2), \tag{9.13}$$

and (8.6) we conclude

$$\begin{aligned} 2F &= (F^j_j + \kappa_j)^2 - \sum_i \kappa_i^2 \\ &= |F^j_j|^2 + 2F^j_j\kappa_j - \sum_{i\neq j}\kappa_i^2, \end{aligned} \tag{9.14}$$

which proves (9.10).

(ii) If $j = n$, and thus κ_n the largest eigenvalue, then, we derive from (8.6)

$$F^n_n \le H \le n\kappa_n, \tag{9.15}$$

and (9.11) follows at once from (9.10).

(iii) A simple algebraic transformation yields

$$\begin{aligned} \frac{F_{;j}}{F^j_j} + \frac{F_{;j}}{H}(1 - \frac{F^i_i}{F^j_j}) &= \frac{F_{;j}}{HF^j_j}(H - \kappa_j + \kappa_i) \\ &= \frac{F_{;j}}{HF^j_j}(\sum_{k\neq j}\kappa_k + \kappa_i), \end{aligned} \tag{9.16}$$

and hence,

$$\sum_{i\neq j}\Big|\frac{F_{;j}}{F^j_j} + \frac{F_{;j}}{H}(1 - \frac{F^i_i}{F^j_j})\Big|^2 \le c\,\frac{|F_{;j}|^2}{H^2|F^j_j|^2}\sum_{i\neq j}\kappa_i^2. \tag{9.17}$$

We, now, treat the cases $j = n$ and $j \neq n$ separately.

If $j = n$, we apply (9.11) and (1.17) and conclude

$$\sum_{i\neq j}\Big|\frac{F_{;j}}{F^j_j} + \frac{F_{;j}}{H}(1 - \frac{F^i_i}{F^j_j})\Big|^2 \le c\,\frac{|F_{;j}|^2}{HF^j_j} \le c\,\frac{|F_{;j}|^2}{F}. \tag{9.18}$$

If $j \neq n$, we deduce from (9.10)

$$\kappa_n^2 \le 6|F_j^j|^2, \tag{9.19}$$

and deduce further

$$\sum_{i\neq j} \kappa_i^2 \le 8|F_j^j|^2, \tag{9.20}$$

where apparently we only had to worry about the case $0 \le \kappa_j$.

Thus, the right-hand side of (9.17) is estimated from above by

$$c\,\frac{|F_{;j}|^2}{H^2}, \tag{9.21}$$

which in turn is less than

$$c\,\frac{|F_{;j}|^2}{F}. \tag{9.22}$$

□

Corollary 9.4. *Let M be an admissible solution of* (9.1) *and $p \in M$ arbitrary. Choose local coordinates around p such that the relations* (8.14) *and* (8.15) *are valid. Then, for any $1 \le j \le n$, the following inequality is valid in p*

$$\sum_{i\neq j} \Big|\frac{f_j}{\tilde F_j^j} + \frac{f_j}{\widetilde H}\big(1 - \frac{\tilde F_i^i}{\tilde F_j^j}\big)\Big|^2 \le c|f_j|^2 f^{-1}, \tag{9.23}$$

where we use the notation $\widetilde H = (1+\epsilon n)H$, and do not apply the summation convention.

Proof. Let us recall the relation (8.7), which we can also express in the form

$$\tilde F^{ij} = \widetilde H g^{ij} - \tilde h^{ij} + \epsilon F^{rs} g_{rs} g^{ij}, \tag{9.24}$$

where

$$\tilde h_{ij} = h_{ij} + \epsilon H g_{ij}. \tag{9.25}$$

Consider each summand in (9.23) separately. We have

$$
\begin{aligned}
\Big|\frac{f_j}{\tilde F^j_j}+\frac{f_j}{\widetilde H}\big(1-\frac{\tilde F^i_i}{\tilde F^j_j}\big)\Big|^2 &= \frac{f_j^2}{\widetilde H^2|\tilde F^j_j|^2}|\widetilde H+\tilde F^j_j-\tilde F^i_i|^2\\
&\le \frac{f_j^2}{\widetilde H^2|F^j_j|^2}\Big|\sum_{k\ne j}\tilde\kappa_k+\tilde\kappa_i\Big|^2,
\end{aligned}\tag{9.26}
$$

in view of (9.24), i.e. we are exactly in the same situation as in the proof of Lemma 9.3 after the equation (9.16) with the following modifications: we replace h_{ij},κ_i and H by $\tilde h_{ij},\tilde\kappa_i$ resp. $\widetilde H$ and observe that $F(\tilde h_{ij})=f$. □

Lemma 9.5. *Let M be an admissible solution of equation* (9.1)*, then, the estimate*

$$
\tilde F^{ij,kl}h_{ij;r}h_{kl;}{}^r \;+\; H^{-1}\tilde F^{ij}H_iH_j \;\le\; cf^{-1}\|Df\|^2 \;+\; c\epsilon\|DH\|^2 \;+\; c \tag{9.27}
$$

is valid in every point, where the smallest principal curvature κ_1 satisfies

$$
\max(-\kappa_1,0)\le\frac{1}{2(n-1)}H\equiv\epsilon_1H. \tag{9.28}
$$

Proof. The proof is a modification of a similar result in [3, Sections 6.1.4 and 6.1.5] or [4, Section 5.1.1].

It follows immediately from the definition of F that

$$
F^{ij,kl}=g^{ij}g^{kl}-\frac12(g^{ik}g^{jl}+g^{il}g^{jk}) \tag{9.29}
$$

and

$$
\begin{aligned}
\tilde F^{ij,kl}h_{ij;r}h_{kl;s}g^{rs} &= F^{ij,kl}h_{ij;r}h_{kl;s}g^{rs}\\
&\quad+\epsilon[2(n-1)+\epsilon(n-1)n]\|DH\|^2.
\end{aligned}\tag{9.30}
$$

In a fixed point $p\in M$ introduce normal Riemannian coordinates such that the relations (8.14) and (8.15) are valid, and define the matrix (a^{kl}) through

$$
a^{kl}=\begin{cases}1, & k\ne l,\\ 0, & k=l.\end{cases} \tag{9.31}
$$

We also set

$$
h_{ijk}=h_{ij;k}. \tag{9.32}
$$

Then, we conclude from (9.29)

$$
\begin{aligned}
F^{ij,kl}h_{ijr}h_{kls}g^{rs} &= \|DH\|^2 - h_{ijk}h^{ijk} \\
&= \sum_{i,k,l}(h_{kki}h_{lli} - h_{kli}h_{kli}) \\
&= \sum_i a^{kl}(h_{kki}h_{lli} - h^2_{kli}) \\
&= \sum_i a^{kl}h_{kki}h_{lli} - \sum_l \sum_{k,i} a^{kl}h^2_{kli}.
\end{aligned} \tag{9.33}
$$

In the last summand let us interchange the roles of i and l to obtain

$$
-\sum_l \sum_{k,i} a^{kl}h^2_{kli} = -\sum_i \sum_{k,l} a^{ki}h^2_{kil}. \tag{9.34}
$$

The Codazzi equations yield

$$
h_{kil} = h_{kli} + c_{kil}, \tag{9.35}
$$

where (c_{kil}) is a uniformly bounded tensor in $\bar{\Omega}$

$$
\|\!|c_{kil}|\!\| \le \text{const}, \tag{9.36}
$$

and hence,

$$
h^2_{kil} = h^2_{kli} + 2h_{kli}c_{kil} + c^2_{kil} \tag{9.37}
$$

and

$$
\begin{aligned}
-\sum_i \sum_{k,l} a^{ki}h^2_{kil} &\le -\sum_i \sum_{k,l} a^{ki}h^2_{kli} - 2\sum_{i,k,l} a^{ki}h_{kli}c_{kil} \\
&\le -2\sum_i \sum_k a^{ki}h^2_{kki} - \sum_{[i,j,k]} h^2_{ijk} \\
&\quad - 2\sum_{i,k,l} a^{ki}h_{kli}c_{kil} - 2\sum_{i,k} a^{ki}h_{iik}c_{iki},
\end{aligned} \tag{9.38}
$$

where $\sum_{[i,j,k]}$ means that the summation is carried out over those triples (i,j,k) where all three indices are different from each other.

The first linear term in the last inequality can be estimated from above by

$$
\begin{aligned}
-2\sum_{i,k,l} a^{ki}h_{kli}c_{kil} &= -2\sum_i\sum_k a^{ki}h_{kki}c_{kik} - 2\sum_i\sum_{k\neq l} a^{ki}h_{kli}c_{kil} \\
&\le \frac{\delta}{2}\sum_i\sum_k a^{ki}h^2_{kki} + \frac{\delta}{2}\sum_{[i,j,k]} h^2_{ijk} + c\delta^{-1}
\end{aligned} \tag{9.39}
$$

for any $\delta > 0$, and the second term similarly. Thus, we deduce from (9.33) and (9.38)

$$
\begin{aligned}
F^{ij,kl}h_{ijr}h_{kls}g^{rs} \le -2(1-\frac{\delta}{2})\sum_i\sum_k a^{ki}h^2_{kki} \\
+\sum_i a^{kl}h_{kki}h_{lli} + c\delta^{-1}
\end{aligned} \tag{9.40}
$$

for any $0 < \delta \le 1$.

Next, let us consider the second term on the left-hand side of (9.27); we have

$$
\begin{aligned}
H^{-1}\tilde{F}^{ij}H_iH_j &= \widetilde{H}^{-1}\tilde{F}^{ij}H_iH_j + \frac{\epsilon n}{1+\epsilon n}H^{-1}\tilde{F}^{ij}H_iH_j \\
&\le \widetilde{H}^{-1}\sum_i \tilde{F}^{ii}(\sum_k h_{kki})^2 + \epsilon c\|DH\|^2,
\end{aligned} \tag{9.41}
$$

where $c = c(n)$, in view of (8.7), (9.25), and (1.15).

Combining (9.40) and the preceding estimate, we conclude

$$
\begin{aligned}
&F^{ij,kl}h_{ij;r}h_{kl;s}g^{rs} + H^{-1}\tilde{F}^{ij}H_iH_j \\
&\quad\le -2(1-\frac{\delta}{2})\sum_i\sum_k a^{ki}h^2_{kki} + \sum_i a^{kl}h_{kki}h_{lli} + c\delta^{-1} \\
&\quad\quad + \widetilde{H}^{-1}\sum_i \tilde{F}^{ii}(\sum_k h_{kki})^2 + \epsilon c\|DH\|^2.
\end{aligned} \tag{9.42}
$$

For each index i, let us estimate the corresponding summand separately, i.e. let us look at—no summation over i—

$$
-(1-\frac{\delta}{2})\sum_k a^{ki}h^2_{kki} + \frac{1}{2}a^{kl}h_{kki}h_{lli} + \frac{1}{2\widetilde{H}}\tilde{F}^{ii}(\sum_k h_{kki})^2, \tag{9.43}
$$

where we have divided the terms by 2.

Denote by $\sum'$ a sum where the index i is omitted during the summation, then, (9.43) can be expressed as

$$
\begin{aligned}
&-(1-\frac{\delta}{2})\sum_k a^{ki}h_{kki}^2 + h_{iii}\sum_k a^{ki}h_{kki} + {\sum_{k<l}}' h_{kki}h_{lli} \\
&\qquad + \frac{1}{2\widetilde{H}}\tilde{F}^{ii}(\sum_k h_{kki})^2.
\end{aligned} \tag{9.44}
$$

To replace h_{iii} in the preceding expression we use the chain rule

$$
f_i \equiv \tilde{F}_i = \tilde{F}^{kk}h_{kki} \tag{9.45}
$$

to derive

$$
h_{iii} = \frac{1}{\tilde{F}^{ii}}(f_i - {\sum_k}' \tilde{F}^{kk}h_{kki}). \tag{9.46}
$$

Inserting (9.46) in (9.44) we obtain, after some simple algebraic manipulations, cf. [3, equ. (36) on p. 78] or [4, equ. (17)],

$$
\begin{aligned}
&-(1-\frac{\delta}{2}){\sum_k}' h_{kki}^2 - {\sum_k}' \Big[\frac{\tilde{F}_k^k}{\tilde{F}_i^i} - \frac{\tilde{F}_i^i}{2\widetilde{H}}\Big(1-\frac{\tilde{F}_k^k}{\tilde{F}_i^i}\Big)^2\Big]h_{kki}^2 \\
&-{\sum_{k<l}}'\Big[\frac{\tilde{F}_k^k+\tilde{F}_l^l}{\tilde{F}_i^i} - 1 - \frac{\tilde{F}_i^i}{\widetilde{H}}\Big(1-\frac{\tilde{F}_k^k}{\tilde{F}_i^i}\Big)\Big(1-\frac{\tilde{F}_l^l}{\tilde{F}_i^i}\Big)\Big]h_{kki}h_{lli} \\
&+{\sum_k}'\Big[\frac{f_i}{\tilde{F}_i^i} + \frac{f_i}{\widetilde{H}}\Big(1-\frac{\tilde{F}_k^k}{\tilde{F}_i^i}\Big)\Big]h_{kki} + \frac{\tilde{F}_i^i}{2\widetilde{H}}\Big(\frac{f_i}{\tilde{F}_i^i}\Big)^2.
\end{aligned} \tag{9.47}
$$

Let us write (9.47) as the sum of three expressions $I_1 + I_2 + I_3$, where

$$
I_1 = -\frac{\delta}{2}{\sum_k}' h_{kki}^2 + {\sum_k}'\Big[\frac{f_i}{\tilde{F}_i^i} + \frac{f_i}{\widetilde{H}}\Big(1-\frac{\tilde{F}_k^k}{\tilde{F}_i^i}\Big)\Big]h_{kki}, \tag{9.48}
$$

$$
I_2 = \frac{\tilde{F}_i^i}{2\widetilde{H}}\Big(\frac{f_i}{\tilde{F}_i^i}\Big)^2, \tag{9.49}
$$

and

$$
\begin{aligned}
I_3 = &-(1-\delta){\sum_k}' h_{kki}^2 - {\sum_k}'\Big[\frac{\tilde{F}_k^k}{\tilde{F}_i^i} - \frac{\tilde{F}_i^i}{2\widetilde{H}}\Big(1-\frac{\tilde{F}_k^k}{\tilde{F}_i^i}\Big)^2\Big]h_{kki}^2 \\
&-{\sum_{k<l}}'\Big[\frac{\tilde{F}_k^k+\tilde{F}_l^l}{\tilde{F}_i^i} - 1 - \frac{\tilde{F}_i^i}{\widetilde{H}}\Big(1-\frac{\tilde{F}_k^k}{\tilde{F}_i^i}\Big)\Big(1-\frac{\tilde{F}_l^l}{\tilde{F}_i^i}\Big)\Big]h_{kki}h_{lli}.
\end{aligned} \tag{9.50}
$$

In view of (9.23) we can estimate I_1 from above by

$$I_1 \le c\delta^{-1} f^{-1}|f_i|^2. \tag{9.51}$$

I_2 is estimated by

$$I_2 = \frac{1}{2\widetilde{H}\tilde{F}_i^i}|f_i|^2 \le cf^{-1}|f_i|^2, \tag{9.52}$$

because of (1.17).

Finally, we claim that $I_3 \le 0$ if we choose $\delta = \frac{1}{4}$. To verify this assertion, let us multiply I_3 by $2\widetilde{H}\tilde{F}_i^i$ to obtain

$$\begin{aligned} &-2(1-\delta)\widetilde{H}\tilde{F}_i^i {\sum_k}' h_{kki}^2 - {\sum_k}' [2\widetilde{H}\tilde{F}_k^k - (\tilde{F}_i^i - \tilde{F}_k^k)^2] h_{kki}^2 \\ &- {\sum_{k<l}}' [2\widetilde{H}(\tilde{F}_k^k + \tilde{F}_l^l) - 2\widetilde{H}\tilde{F}_i^i - 2(\tilde{F}_i^i - \tilde{F}_k^k)(\tilde{F}_i^i - \tilde{F}_l^l)] h_{kki}h_{lli}. \end{aligned} \tag{9.53}$$

Now, we use (8.7) and replace *any* $\tilde{F}_j^j$, $1 \le j \le n$, by

$$F_j^j + \epsilon(n-1)(1+\epsilon n)Hg_j^j \equiv F_j^j + \epsilon\gamma_\epsilon H. \tag{9.54}$$

The expression in (9.53) is then equal to the sum of two terms $I_4 + I_5$, where

$$\begin{aligned} I_4 = &-2(1-\delta)\widetilde{H}F_i^i {\sum_k}' h_{kki}^2 - {\sum_k}' [2\widetilde{H}F_k^k - (F_i^i - F_k^k)^2] h_{kki}^2 \\ &- {\sum_{k<l}}' [2\widetilde{H}(F_k^k + F_l^l) - 2\widetilde{H}F_i^i - 2(F_i^i - F_k^k)(F_i^i - F_l^l)] h_{kki}h_{lli}, \end{aligned} \tag{9.55}$$

and

$$\begin{aligned} I_5 = -2(1-\delta)\widetilde{H}\epsilon\gamma_\epsilon H {\sum_k}' h_{kki}^2 - 2\widetilde{H}\epsilon\gamma_\epsilon H {\sum_k}' h_{kki}^2 \\ - 2\widetilde{H}\epsilon\gamma_\epsilon H {\sum_{k<l}}' h_{kki}h_{lli}. \end{aligned} \tag{9.56}$$

From the the binomial formula

$$\Big({\sum_k}' h_{kki}\Big)^2 = {\sum_k}' h_{kki}^2 + 2{\sum_{k<l}}' h_{kki}h_{lli} \tag{9.57}$$

we infer that $I_5 \le 0$, while I_4 is non-positive provided we choose $\delta = \frac{1}{4}$ and assume

$$\max(-\tilde\kappa_1, 0) \le \frac{1}{2(n-1)}\widetilde{H}, \tag{9.58}$$

cf. [3, pp. 81–85] or [4, pp. 23–29].

But the condition (9.58) is certainly satisfied in view of (9.28).

Combining (9.30), (9.42), (9.51), and (9.52) gives (9.27), and thus, the Lemma is proved. □

From Lemma 9.2 and Lemma 9.5 we conclude

Corollary 9.6. *Let $M = \operatorname{graph} u$ be an admissible solution of equation (9.1) in Ω. Then, the estimate*

$$\begin{aligned} \tilde F^{ij,kl} h_{ij;r} h_{kl;s} g^{rs} H^{-1} + \tilde F^{ij} (\log H)_i (\log H)_j \\ \le c H^{-1} f^{-1} \|Df\|^2 + c H^{-1} \|D \log H\|^2 + c H^{-1} \end{aligned} \tag{9.59}$$

is valid in every point $p \in M$, where (9.28) is satisfied. The constant c depends on Ω, $|||Df|||$, $|||D^2 f|||$, the constant c_1 in (8.3), and on known estimates of the C^0 and C^1- norm of u.

As we already mentioned we have to assume the existence of a strictly convex function $\chi \in C^2(\bar\Omega)$, i.e. χ satisfies

$$\chi_{\alpha\beta} \ge c_0 \bar g_{\alpha\beta} \tag{9.60}$$

with a positive constant c_0.

We observe that then the restriction $\chi = \chi_{|_M}$ of χ to an admissible solution $M \subset \bar\Omega$ of (9.1) satisfies the elliptic inequality

$$\begin{aligned} -\tilde F^{ij} \chi_{ij} &= -2\tilde F \chi_\alpha \nu^\alpha - \tilde F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta \\ &\le -2\tilde F \chi_\alpha \nu^\alpha - c_0 \tilde F^{ij} g_{ij}, \end{aligned} \tag{9.61}$$

where we used the homogeneity of $\tilde F$.

We can now prove uniform C^2- estimates.

Theorem 9.7. *Let $M = \operatorname{graph} u$ be an admissible solution of equation (9.1) in Ω, where f satisfies the estimates (8.3), (8.4) and (8.5). Then, the principal curvatures of M are uniformly bounded.*

Proof. Let χ be the strictly convex function and μ a large positive constant. We shall prove that $w = \log H + \mu\chi$ is uniformly bounded from above.

Let $x_0 \in M$ be such that

$$w(x_0) = \sup_M w, \tag{9.62}$$

and choose in x_0 a local coordinate system satisfying (8.14) and (8.15). Applying the maximum principle, we conclude from (9.3) and (9.61)

$$\begin{aligned} 0 \le & -\tilde{F}^{kl} h_{kr} h_l^r + c\tilde{F}^{ij} g_{ij} + c\mu f - \mu c_0 \tilde{F}^{ij} g_{ij} \\ & + c(1 + f + |||Df||| + |||D^2 f|||)(1 + H + \|D \log H\|) \\ & + \tilde{F}^{ij,kl} h_{ij;r} h_{kl;s} g^{rs} H^{-1} + \tilde{F}^{ij} (\log H)_i (\log H)_j, \end{aligned} \tag{9.63}$$

where we also assumed H to be larger than 1.

We now consider two cases.

Case 1: Suppose that

$$|\kappa_1| \ge \epsilon_1 H \equiv \frac{1}{2(n-1)} H. \tag{9.64}$$

Then, we infer from Lemma 8.3 and (9.24)

$$-\tilde{F}^{kl} h_{kr} h_l^r \le -\frac{n-1}{n} H \kappa_1^2 \le -\frac{n-1}{n} \epsilon_1^2 H^3 \equiv -\epsilon_2 H^3. \tag{9.65}$$

Moreover, the concavity of $\log \tilde{F}$ implies

$$\begin{aligned} \tilde{F}^{ij,kl} h_{ij;r} h_{kl;s} g^{rs} H^{-1} & \le \tilde{F}^{-1} g^{ij} \tilde{F}^{kl} h_{kl;i} \tilde{F}^{rs} h_{rs;j} H^{-1} \\ & = f^{-1} \|Df\|^2 H^{-1} \\ & \le c f^{-1} |||Df|||^2 |A|^2 H^{-1} \\ & \le c f^{-1} |||Df|||^2 H. \end{aligned} \tag{9.66}$$

Furthermore, $Dw(x_0) = 0$, or,

$$(\log H)_i = -\mu \chi_i. \tag{9.67}$$

Inserting the last three relations in (9.63) we obtain

$$0 \le -\epsilon_2 H^3 + c(1 + H + \mu) + c\mu^2 H, \tag{9.68}$$

where, now, c depends on f and its derivatives in the ambient space.
Hence, H, and therefore w, are a priori bounded in x_0.

Case 2: Suppose that

$$|\kappa_1| < \epsilon_1 H. \tag{9.69}$$

Then, Corollary 9.6 is applicable, and we infer from (9.63) and (9.67)

$$0 \le c(1 + H + \mu + \mu^2 H^{-1}) + (c - \mu c_0)\tilde{F}^{ij} g_{ij}. \tag{9.70}$$

Choosing now μ sufficiently large we obtain an a priori bound for $H(x_0)$, since

$$\tilde{F}^{ij} g_{ij} \ge (n-1)H. \tag{9.71}$$

Thus, w, or equivalently H, are uniformly bounded. □

10. Existence of a solution

We can now demonstrate the final step in the proof of Theorem 0.3. Let $M_\epsilon = \operatorname{graph} u_\epsilon$ be the stationary approximations. In the preceding sections we have proved uniform estimates for u_ϵ up to the order two. Since, by assumption, f is strictly positive, the principal curvatures of M_ϵ stay in a compact subset of the cone $\mathit{\Gamma}_2$ for small ϵ, cf. Remark 3.1, and therefore, the operator $\tilde{F}$ is uniformly elliptic for those ϵ. Taking the square root on both sides of equation (9.1) without changing the notation, we also know that $\tilde{F}$ is concave.
Hence the $C^{2,\alpha}$- estimates of Evans and Krylov are applicable, cf. [7] and [20], and we deduce

$$|u_\epsilon|_{2,\alpha,\mathcal{S}_0} \le \text{const} \tag{10.1}$$

uniformly in ϵ. If ϵ tends to zero, a subsequence converges to a solution $u \in C^{2,\alpha}(\mathcal{S}_0)$ of our problem. From the Schauder estimates we further conclude $u \in C^{4,\alpha}(\mathcal{S}_0)$.

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