Partial differential equations 2

Based on lectures by Claus Gerhardt

Contents

Chapter 1

Distributions and Sobolev spaces

1.1 Distributions

1.1.1 Definition. Let $\Omega \subset \mathbb{R}^n$ be open, $K \subset \Omega$ compact. We set

$$\mathcal{D}_K(\Omega) := \{\phi \in C_c^\infty(\Omega) \colon \operatorname{supp} \phi \subset K\}.$$

On $\mathcal{D}_K(\Omega)$ we define the following norms:

$$\forall m \in \mathbb{N} \colon p_m(\phi) = |\phi|_{m,K}.$$

1.1.2 Remark. Those norms define a topology on $\mathcal{D}_K(\Omega)$, using the base

$$U_{m,\epsilon} := \{\phi : p_m(\phi) < \epsilon\}, \ \epsilon > 0, \ m \in \mathbb{N},$$

such that $\mathcal{D}_K(\Omega)$ becomes a topological vector space, i.e., all the other neighborhood bases are formed by translation. This topology is then generated by the metric

$$d(\phi, \eta) := \sum_{m \in \mathbb{N}} 2^{-m} \frac{|\phi - \eta|_m}{1 + |\phi - \eta|_m}.$$

1.1.3 Proposition. *$T \in \mathcal{D}_K(\Omega)^*$ is continuous, if and only if*

$$\exists m \in \mathbb{N} \ \exists c > 0 \ \forall \phi \in \mathcal{D}_K(\Omega) \colon |\langle T, \phi \rangle| \leq c p_m(\phi).$$

Proof. Exercise. □

1.1.4 Remark. Let $K_i \nearrow \Omega$ be an exhaustion, such that $K_i \subset \mathring{K}_{i+1}$. Then

$$C_c^\infty(\Omega) = \bigcup_{i \in \mathbb{N}} \mathcal{D}_{K_i}(\Omega) =: \mathcal{D}(\Omega).$$

Let the topology $\mathcal{T}$ of $\mathcal{D}(\Omega)$ be defined by the requirement

$$\forall i \in \mathbb{N}\colon \mathcal{T}_{|\mathcal{D}_{K_i}(\Omega)} \subset \mathcal{T}_{\mathcal{D}_{K_i}(\Omega)}.$$

The topology $\mathcal{T}$ does not depend on the exhaustion.

Proof. Exercise. □

1.1.5 Definition. (i) A linear form T on $\mathcal{D}(\Omega)$ is called *distribution*, if it is continuous. For the set of all continuous linear forms on $\mathcal{D}(\Omega)$ we write $\mathcal{D}'(\Omega)$.
(ii) $\mathcal{D}'(\Omega)$ obtains the $*$-weak topology, i.e.

$$T_i \stackrel{*}{\rightharpoonup} T \Leftrightarrow \ \forall \phi \in \mathcal{D}(\Omega)\colon \langle T_i, \phi\rangle \to \langle T, \phi\rangle.$$

1.1.6 Remark. From the previous constructions we deduce

$$T \in \mathcal{D}'(\Omega) \Leftrightarrow \ \forall K \Subset \Omega \ \exists m \in \mathbb{N} \ \exists c > 0 \ \forall \phi \in \mathcal{D}_K(\Omega)\colon |\langle T, \phi\rangle| \le c p_m(\phi).$$

If m can be chosen independently of K, the minimal such m is called *order* of T, $\operatorname{ord}(T)$.

1.1.7 Definition. A distribution of order 0 is called *measure*.

1.1.8 Remark. Let $f \in L^1_{loc}(\Omega)$, then

$$\langle f, \phi\rangle = \int_\Omega f\phi$$

defines a measure.

Proof. Exercise. □

1.1.9 Definition. Let $T \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{N}^n$. We define the *α-th weak derivative* or *distributional derivative* of T, $D^\alpha T$ by

$$\langle D^\alpha T, \phi\rangle := (-1)^{|\alpha|}\langle T, D^\alpha \phi\rangle.$$

1.1.10 Remark. We have $D^\alpha T \in \mathcal{D}'(\Omega)$ and $\operatorname{ord}(D^\alpha T) \le \operatorname{ord}(T) + |\alpha|$, if both sides are defined.

1.1.11 Example. Let

$$\theta(t) := \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

Then, as one easily verifies, $\theta' = 2\delta_0$.

1.1.12 Remark. According to the fundamental lemma of the calculus of variations,

$$\Psi\colon L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$$

is an embedding.
The derivative $D^\alpha u$ of a function $u \in L^1_{loc}(\Omega)$ is always to be understood as distributional derivative.

1.1.13 Remark. For $\Psi(L^p_{loc}(\Omega))$ we simply write $L^p_{loc}(\Omega)$ and consider this to be a subspace of $\mathcal{D}'(\Omega)$.

1.2 Sobolev-Spaces

1.2.1 Definition. Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}$, $1 \leq p \leq \infty$. By

$$H^{m,p}(\Omega) := \{u \in L^p(\Omega)\colon D^\alpha u \in L^p(\Omega)\ \forall |\alpha| \leq m\}$$

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}},\ 1 \leq p < \infty,$$

$$\|u\|_{m,\infty} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty,$$

we denote the space of *Sobolev functions* of class (m,p). On $H^{m,2}(\Omega)$ we define the scalar product

$$\langle u, v \rangle := \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v.$$

1.2.2 Remark. $H^{m,p}(\Omega)$ is complete for $1 \leq p \leq \infty$.

Proof. Exercise. $\square$

1.2.3 Lemma. *(i) Let $u \in H^{m,p}(\mathbb{R}^n)$, $1 \leq p < \infty$ and (η_ϵ) be a Dirac sequence, then we have for*

$$u_\epsilon(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y) u(y) dy$$

$$\begin{aligned} &(a)\ \forall |\alpha| \leq m\colon D^\alpha u_\epsilon = (D^\alpha u)_\epsilon \\ &(b)\ u_\epsilon \to u \text{ in } H^{m,p}(\mathbb{R}^n) \end{aligned}$$

(ii) Let $\Omega' \Subset \Omega \subset \mathbb{R}^n$ be open and $u \in H^{m,p}(\Omega)$, $1 \leq p < \infty$. Extend u to $\mathbb{R}^n$ by 0. Then

$$u_\epsilon \to u \text{ in } H^{m,p}(\Omega'),\ \epsilon < \operatorname{dist}(\Omega', \partial\Omega).$$

Proof. Exercise. $\square$

1.2.4 Lemma. *(Product rule)*
Let $f \in H^{1,p}(\Omega)$ and $g \in H^{1,p'}(\Omega)$, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$f \cdot g \in H^{1,1}(\Omega)$$

and

$$D(fg) = Df \cdot g + f \cdot Dg.$$

Proof. By symmetry we may assume $p < \infty$. Extend f, g to $\mathbb{R}^n$ by 0 and let f_ϵ be the mollified sequence as in 1.2.3. Let $\zeta \in C_c^\infty(\Omega)$. Then there holds

$$\int_\Omega (\zeta f_\epsilon)\partial_i g = -\int_\Omega (\zeta \partial_i f_\epsilon g + f_\epsilon \partial_i \zeta g).$$

Taking the limit $\epsilon \to 0$ via Hoelder's theorem we obtain

$$\forall \zeta \in C_c^\infty(\Omega) \colon \int_\Omega \zeta (f \partial_i g + \partial_i f g) = -\int_\Omega f g \partial_i \zeta.$$

Again by Hoelder's inequality we obtain

$$D(fg) \in L^1(\Omega).$$

□

1.2.5 Lemma. *(Chain rule)*
Let $\Omega \Subset \mathbb{R}^n$, $g \in C^m(\mathbb{R})$ and $|g|_m \le c$. Then for $u \in H^{m,p}(\Omega)$ we have $g \circ u \in H^{m,p}(\Omega)$ and

$$D(g \circ u) = g'(u) Du.$$

Proof. Let $m = 1$ and $1 \le p < \infty$. Let $\phi \in C_c^\infty(\Omega)$ and $\Omega' \Subset \Omega$, such that $\phi \in C_c^\infty(\Omega')$. Let $u_\epsilon \in C^\infty(\bar{\Omega}')$ such that

$$\|u - u_\epsilon\|_{m,p,\Omega'} \to 0$$

and

$$(u_\epsilon, Du_\epsilon) \to (u, Du) \text{ a.e.}$$

$$\Rightarrow \int_{\Omega'} (g \circ u) D_i \phi = \lim_{\epsilon \to 0} \int_{\Omega'} (g \circ u_\epsilon) D_i \phi = \lim_{\epsilon \to 0} \left(-\int_{\Omega'} g'(u_\epsilon) D_i u_\epsilon \phi \right) \qquad (1.1)$$

There holds $g'(u_\epsilon) \to g'(u)$ a.e. and $|g'| \le L$.

$$\Rightarrow |\phi g'(u_\epsilon) Du| \le L |Du| |\phi|.$$

Dominated convergence implies

$$\begin{aligned} \int_{\Omega'} |g'(u_\epsilon) D_i u_\epsilon \phi - g'(u) D_i u \phi| &\le \int_{\Omega'} |g'(u_\epsilon)(D_i u_\epsilon - D_i u)\phi| \\ &\quad + \int_{\Omega'} |g'(u_\epsilon) - g'(u)| |D_i u| |\phi| \to 0. \end{aligned}$$

(1.1) implies the chain rule. Furthermore we have

$$\|g \circ u\|_{1,p,\Omega'} \le c \|u\|_{1,p,\Omega} + c |\Omega|^{\frac{1}{p}}$$

$$\Rightarrow g \circ u \in H^{1,p}(\Omega).$$

From this estimate we deduce, using $p \to \infty$, the claim for $p = \infty$. For $m > 1$ use induction and the product rule. □

1.2.6 Theorem. *Let $\tilde{x} \in \mathrm{Diff}^m(\Omega, \tilde{\Omega})$ such that $\tilde{x}$ and $\tilde{x}^{-1}$ have a bounded C^m-norm and $1 \leq p \leq \infty$.*
Then the map

$$\Phi : H^{m,p}(\Omega) \to H^{m,p}(\tilde{\Omega})$$
$$u \mapsto \tilde{u} = u \circ \tilde{x}^{-1}$$

is a topological isomorphism.

Proof. We show this for $m = 1$, the rest follows by induction.
Let $\Omega' \Subset \Omega$, $u \in H^{1,p}(\Omega)$, $u_\epsilon \to u$ in $H^{1,p}(\Omega')$.

$$\tilde{u}_\epsilon = u_\epsilon \circ \tilde{x}^{-1}$$
$$\Rightarrow \tilde{D}_i \tilde{u}_\epsilon = D_k u_\epsilon \frac{\partial x^k}{\partial \tilde{x}^i}.$$

Let the sequence also satisfy

$$\tilde{u}_\epsilon \to \tilde{u} \text{ a.e.}$$

and

$$\tilde{D}_i \tilde{u}_\epsilon \to D_k u \frac{\partial x^k}{\partial \tilde{x}^i} \text{ a.e.}$$

By the transformation theorem and the boundedness of the Jacobians we have

$$\tilde{u} \in H^{1,p}(\tilde{\Omega}')$$

and

$$\forall \Omega' \Subset \Omega \colon \|\tilde{u}\|_{1,p,\tilde{\Omega}'} \leq c\|u\|_{1,p,\Omega'}$$
$$\Rightarrow \|\tilde{u}\|_{1,p,\tilde{\Omega}} \leq c\|u\|_{1,p,\Omega}.$$

By symmetry this also holds for the inverse. For $p = \infty$ the claim holds by taking the limit. $\square$

1.2.7 Lemma. *Let $u \in H^{1,p}(\Omega)$, then*

$$u^+ = \max(u, 0), \; u^- = \min(u, 0) \text{ and } |u|$$

are in $H^{1,p}(\Omega)$ and a.e. there holds

$$Du^+ = \begin{cases} Du, & u > 0 \\ 0, & u \leq 0 \end{cases}$$

$$Du^- = \begin{cases} Du, & u < 0 \\ 0, & u \geq 0 \end{cases}$$

and

$$D|u| = \begin{cases} Du, & u > 0 \\ 0, & u = 0 \\ -Du, & u < 0 \end{cases}.$$

Proof. Let $\epsilon > 0$.

$$g_\epsilon(t) := \begin{cases} \sqrt{t^2+\epsilon^2} - \epsilon, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Then $g_\epsilon \in C^1$ and $|g_\epsilon'| \leq 1$.

$$g_\epsilon \to \max(\cdot, 0) \text{ locally uniformly.}$$

The chain rule implies

$$u_\epsilon := g_\epsilon \circ u \in H^{1,p}(\Omega)$$

and

$$Du_\epsilon = g_\epsilon'(u) Du = \begin{cases} \frac{uDu}{\sqrt{u^2+\epsilon^2}}, & u > 0 \\ 0, & u \leq 0. \end{cases}$$

Let $\eta \in C_c^\infty(\Omega)$.

$$\begin{aligned} \int_\Omega u_\epsilon D_i \eta &= -\int_\Omega D_i u_\epsilon \eta \\ &= -\int_{\{u>0\}} \frac{uD_i u}{\sqrt{u^2+\epsilon^2}} \eta \\ &= -\int_\Omega \frac{uD_i u}{\sqrt{u^2+\epsilon^2}} \chi_{\{u>0\}} \eta \to -\int_\Omega \chi_{\{u>0\}} D_i u \eta. \end{aligned}$$

Since the left hand side converges to

$$\int_\Omega u^+ D_i \eta,$$

we obtain the claim. Using

$$u^- = -(-u)^+$$

and

$$|u| = u^+ - u^-$$

the other cases also follow. □

1.2.8 Corollary. *Let $u \in H^{1,p}(\Omega)$, $c \in \mathbb{R}$, $E := \{u = c\}$.*

$$\Rightarrow Du_{|E} = 0 \text{ a.e.}$$

Proof. Wlog $c = 0$. There holds $u = u^+ + u^-$. Apply the previous lemma. □

1.2.9 Theorem. *Let $\Omega \Subset \mathbb{R}^n$, $u \in H^{1,p}(\Omega)$ and let $g \in C^{0,1}(\mathbb{R})$ such that $Lip(g) \le L$ and suppose g' has only at most countably many points of discontinuity. Let M be the set of those points. Then*

$$v := g \circ u \in H^{1,p}(\Omega)$$

and we have

$$Dv = \begin{cases} g'(u)Du, & u(x) \notin M \\ 0, & u(x) \in M. \end{cases}$$

Proof. Let g_ϵ be a mollification of g

$$\Rightarrow g_\epsilon \to g \text{ locally uniformly}$$

and

$$g'_\epsilon \to g' \text{ locally uniformly in } M^c,$$

as well as

$$|g'_\epsilon| \le L.$$

Then

$$v_\epsilon := g_\epsilon \circ u \in H^{1,p}(\Omega)$$

and

$$Dv_\epsilon = g'_\epsilon(u)Du.$$

Let $M = \{t_k : k \in H \subset \mathbb{N}\}$ and

$$E_k := \{u = t_k\},\ E := \bigcup_{k \in H} E_k.$$

$$\Rightarrow Du_{|E} = 0 \text{ a.e.}$$

There holds $g'(u)Du \in L^p(\Omega)$ and for a.e. $x \in \Omega$ we have

$$\lim_{\epsilon \to 0} g'_\epsilon(u(x))Du(x) = \begin{cases} g'(u(x))Du(x), & x \notin E \\ 0, & x \in E. \end{cases}$$

□

1.2.10 Remark. This theorem also holds for arbitrary $g \in C^{0,1}(\mathbb{R})$, $|g'| \le L$, c.f. Ziemer: Weakly differentiable functions.

1.2.11 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open, $\partial\Omega \in C^{0,1}$. Then there holds*

$$\forall u \in C^1(\bar{\Omega}) \colon \int_{\partial\Omega} |u| \le \sqrt{1+L^2} \int_\Omega |Du| + c \int_\Omega |u|,$$

where L is an upper bound for the Lipschitz constants of the boundary representations.

Proof. (i) Let $x_0 \in \partial\Omega$ and ϕ be a local graph representation around $0 \in \mathbb{R}^{n-1}$,

$$\Gamma = \{(\hat{x}, \phi(\hat{x}))\colon |\hat{x}| < \rho\}.$$

Furthermore let $0 < a$, such that

$$U = \{(\hat{x}, x^n) : \phi(\hat{x}) < x^n < a,\ \hat{x} \in \hat{B}_\rho(0)\} \subset \Omega.$$

For a function u having support in this chart we then have

$$\int_\Gamma |u| = \int_{\hat{B}_\rho(0)} |u(\hat{x}, \phi(\hat{x}))| \sqrt{1 + |D\phi|^2} \le \sqrt{1 + L^2} \int_{\hat{B}_\rho(0)} |u|.$$

Also suppose, that $u(\cdot, a) = 0$. Then

$$\begin{aligned} u(\hat{x}, \phi(\hat{x})) &= \int_a^{\phi(\hat{x})} D_n u(\hat{x}, t) dt \\ \Rightarrow |u(\hat{x}, \phi(\hat{x}))| &\le \int_{\phi(\hat{x})}^a |D_n u| \le \int_{\phi(\hat{x})}^a |Du|. \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_\Gamma |u| &\le \sqrt{1 + L^2} \int_{\hat{B}_\rho(0)} |u(\hat{x}, \phi(\hat{x}))| \\ &\le \int_{\hat{B}_\rho(0)} \int_{\phi(\hat{x})}^a |Du| \sqrt{1 + L^2} \\ &= \sqrt{1 + L^2} \int_U |Du|. \end{aligned}$$

(ii) Now consider an open covering $(B_{\rho_i}),\ 1 \le i \le N$, of $\partial\Omega$, such that $\partial\Omega \cap B_{\rho_i}$ can be represented as a graph locally and also such that the conditions of (i) are satisfied.
Let (η_i) be a subordinate finite partition of unity for $\partial\Omega$. Then

$$u = \sum_{i=1}^N u\eta_i \text{ on } \partial\Omega.$$

$$\begin{aligned} \Rightarrow \int_{\partial\Omega} |u| &\le \sum_{i=1}^N \int_{\partial\Omega} |u\eta_i| \le \sum_{i=1}^N \sqrt{1 + L^2} \int_\Omega |D(u\eta_i)| \\ &\le \sqrt{1 + L^2} \int_\Omega |Du| \sum_{i=1}^N \eta_i + \sqrt{1 + L^2} \int_\Omega |u| \sum_{i=1}^N |D\eta_i| \\ &\le \sqrt{1 + L^2} \int_\Omega |Du| + c \int_\Omega |u|. \end{aligned}$$

□

1.2.12 Remark.
(i) $\partial\Omega \in C^1 \Rightarrow \forall u \in C^1(\bar{\Omega})\colon \int_{\partial\Omega} |u| \leq (1+\epsilon)\int_\Omega |Du| + c_\epsilon \int_\Omega |u|$
(ii) $\partial\Omega \in C^2 \Rightarrow \forall u \in C^1(\bar{\Omega})\colon \int_{\partial\Omega} |u| \leq \int_\Omega |Du| + c\int_\Omega |u|$.

Proof. Exercise □

1.2.13 Definition. We say Ω satisfies the $H^{m,p}$- extension property, if there exists $\Omega \subset \Omega_0 \Subset \mathbb{R}^n$ and a continuous linear map

$$F : H^{m,p}(\Omega) \to H_0^{m,p}(\Omega_0),$$

such that

$$\forall u \in H^{m,p}(\Omega)\colon Fu_{|\Omega} = u.$$

F is then called *extension operator.*

1.2.14 Definition. Let $E \subset \mathbb{R}^n$ be measurable. Then the Sobolev spaces $H^{m,p}(E)$ and $H_0^{m,p}(E)$ respectively are defined as the closure of

$$\{u \in C^m(E)\colon \|u\|_{m,p,E} < \infty\}$$

and $C_c^m(E)$ respectively with respect to the norm $\|\cdot\|_{m,p}$.

1.2.15 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^m$, then there holds for $1 \leq p < \infty$*

$$H^{m,p}(\bar{\Omega}) = H^{m,p}(\Omega).$$

Proof. First choose a local boundary neighborhood U, such that 1.2.6 implies

$$H^{m,p}(U) = H^{m,p}(B_1^+(0)).$$

Let $u \in H_c^{m,p}(B_1^+(0) \cup \{x^n = 0\})$. Define

$$u_h(\hat{x}, x^n) := u(\hat{x}, x^n + h),\ \ h > 0.$$

Then u_h is defined in $B_1^+(0) - he_n$. For small $\epsilon = \epsilon(h)$ we then find

$$u_{h,\epsilon} = u_h * \eta_\epsilon \in C^\infty(\bar{B_1^+}(0)).$$

Later we will show, that

$$\|u_h - u\|_{m,p} \to 0,\ \ h \to 0.$$

Thus we find

$$u_{h_k,\epsilon_k} \to u \text{ in } H^{m,p}(B_1^+(0)).$$
$$\Rightarrow u \in H^{m,p}(\overline{B_1^+(0)}).$$

Using a partition of unity we obtain the claim. The other inclusions follow immediately from the definitions. □

1.2.16 Lemma. *(Lions-Magenes)*
Let $c_1, ..., c_{m+1}$ be solutions of the system

$$\sum_{k=1}^{m+1} (-1)^j k^j c_k = 1,\ 0 \le j \le m.$$

Then

$$\tilde{u}(\hat{x}, x^n) = \sum_{k=1}^{m+1} c_k u(\hat{x}, -kx^n),\ x^n < 0$$

defines an extension for $u \in C^m(\bar{\mathbb{R}^n_+}) \cap H^{m,p}(\mathbb{R}^n_+)$ into all of $\mathbb{R}^n$, such that

$$\tilde{u} \in C^m(\mathbb{R}^n)$$

and

$$\|\tilde{u}\|_{m,p,\mathbb{R}^n} \le c\|u\|_{m,p,\mathbb{R}^n_+},\ c = c(m,n,p),\ 1 \le p \le \infty.$$

Proof. Exercise □

1.2.17 Corollary. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^m$. Then Ω satisfies the $H^{m,p}$- extension property for all $1 \le p < \infty$.*

Proof. Clear by the previous theorem and lemma. □

1.2.18 Remark. (i) $\Omega \Subset \mathbb{R}^n \Rightarrow H_0^{m,p}(\Omega) \hookrightarrow H_c^{m,p}(\mathbb{R}^n)$.
(ii) $\partial\Omega \in C^{0,1} \Rightarrow \Omega$ satisfies the $H^{m,p}$ extension property (Calderon-Zygmund, without proof).
(iii) For $1 \le p < \infty$, $\partial\Omega \in C^{0,1} \Rightarrow H^{m,p}(\Omega) = H^{m,p}(\bar{\Omega})$.

Proof. (i) is clear and (iii) follows from (ii) immediately. □

1.2.19 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open, $\partial\Omega \in C^{0,1}$. Then there exists a continuous trace operator*

$$t : H^{1,p}(\Omega) \to L^p(\partial\Omega),\ 1 \le p < \infty,$$

such that

$$t_{|H^{1,p}(\Omega)\cap C^0(\bar{\Omega})} = \cdot_{|\partial\Omega}.$$

Proof. Since we have $H^{1,p}(\Omega) = H^{1,p}(\bar{\Omega})$, it suffices to prove the claim for $u \in C^\infty(\bar{\Omega})$.
(i) For $u \in C^1(\bar{\Omega})$ define $t(u) = u_{|\partial\Omega}$. We have

$$\int_{\partial\Omega} |u| \le \sqrt{1+L^2}\int_\Omega |Du| + c\int_\Omega |u|,$$

which also holds for Lipschitz functions by approximation. We apply this estimate to $|u|^p$ yielding

$$\begin{aligned}\int_{\partial\Omega} |u|^p &\le p\sqrt{1+L^2}\int_\Omega |Du||u|^{p-1} + c\int_\Omega |u|^p \\ &\le c_0\left(\int_\Omega |Du|^p\right)^{\frac{1}{p}}\left(\int_\Omega |u|^p\right)^{\frac{p-1}{p}} + c\int_\Omega |u|^p \\ &\Rightarrow \|t(u)\|_{p,\partial\Omega} \le c\|u\|_{1,p,\Omega}.\end{aligned}$$

(ii) Let $u \in H^{1,p}(\Omega)$ and

$$u_\epsilon = u * \eta_\epsilon \in C_c^\infty(\mathbb{R}^n)$$

$$\Rightarrow u_\epsilon \to u \text{ in } H^{1,p}(\bar{\Omega}).$$

$$\Rightarrow \|t(u_\epsilon)\|_{p,\partial\Omega} \le c\|u_\epsilon\|_{1,p,\Omega}.$$

Thus we can define

$$t(u) := \lim_{\epsilon\to 0} t(u_\epsilon).$$

(iii) Let $u \in H^{1,p}(\Omega) \cap C^0(\bar{\Omega})$. We may suppose $u \in H^{1,p}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$.

$$t(u_\epsilon) \to t(u) \text{ in } L^p(\partial\Omega)$$

and

$$u_\epsilon \to u \text{ in } C^0(\bar{\Omega})$$

imply the claim.

□

1.2.20 Proposition. $u \in H_0^{1,p}(\Omega) \Rightarrow t(u) = 0$.

Proof. Follows immediately from the preceding proof. □

1.2.21 Proposition. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$. Let $m \ge 1$, $1 \le p < \infty$. Then for $u \in H^{m,p}(\Omega)$ all the $D^\beta u$, $|\beta| \le m-1$, are defined on $\partial\Omega$ in the sense of traces.*

Proof. All those functions are in $H^{1,p}(\Omega)$. □

1.2.22 Proposition. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$. For $u, v \in H^{1,p}(\Omega)$ there holds*

$$\begin{aligned}t(\max(u,v)) &= \max(t(u), t(v)) \\ t(\min(u,v)) &= \min(t(u), t(v)).\end{aligned}$$

Proof. By approximation. □

1.2.23 Lemma. *Let $\Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^{0,1}$, $u \in H^{1,p}(\Omega)$, $1 \le p < \infty$. Then we have for large k*

(i) $k^p \int_{\Omega_{\frac{1}{k}}} |u|^p \le c_p k^{p-1}\sqrt{1+L^2}\int_{\partial\Omega}|u|^p + c\sqrt{1+L^2}^p \int_{\Omega_{\frac{\sqrt{1+L^2}}{k}}} (|Du|^p + |u|^p)$.

(ii) $\int_{\partial\Omega}|u| \le k\sqrt{1+L^2}\int_{\Omega_{\frac{1}{k}}}|u| + c\int_{\Omega_{\frac{1}{k}}}(|Du| + |u|)$

(iii) $\limsup_{k\to\infty} k\int_{\Omega_{\frac{1}{k}}}|u| \le \sqrt{1+L^2}\int_{\partial\Omega}|u| \le (1+L^2)\liminf_{k\to\infty} k\int_{\Omega_{\frac{1}{k}}}|u|$

(iv) $t(u) = 0 \Rightarrow \limsup_{k\to\infty} k^p \int_{\Omega_{\frac{1}{k}}}|u|^p = 0$,

where

$$c_p = \begin{cases} 1, & \text{if } p = 1 \\ c(p,\partial\Omega), & \text{if } p > 1, \end{cases}$$

$\Omega_k = \{x \in \Omega \colon d(x,\partial\Omega) < k\}$ and $d = \operatorname{dist}(\cdot,\partial\Omega)$.

Proof. (i) Let $u \in C^1(\bar\Omega)$, wlog $\operatorname{supp}(u) \cap \bar\Omega \subset \hat B_R(0) \times (0,a) =: G$. Let $\frac{1}{k} < \min(a,R)$, then

$$\Omega_{\frac{1}{k}} \cap G = \left\{(\hat x, x^n) \in \Omega \colon |\hat x| < R \ \wedge \ d(\hat x, x^n) < \frac{1}{k}\right\}.$$

$$\forall (\hat x, x^n) \in \Omega_{\frac{1}{k}} \cap G \ \exists \hat y \in \hat B_{2R}(0) \colon d(\hat x, x^n) = \sqrt{|\hat x - \hat y|^2 + |\phi(\hat y) - x^n|^2},$$

where $\partial\Omega \cap \hat B_{2R}(0) \times (0,a) = \operatorname{graph}\phi$. Thus for all $(\hat x, x^n) \in \Omega_{\frac{1}{k}} \cap G$ we have

$$\begin{aligned}\Rightarrow |x^n - \phi(\hat x)| &\le |x^n - \phi(\hat y)| + |\phi(\hat y) - \phi(\hat x)| \\ &\le |x^n - \phi(\hat y)| + L|\hat x - \hat y| \\ &\le \sqrt{1+L^2}\sqrt{|x^n - \phi(\hat y)|^2 + |\hat x - \hat y|^2} \\ &\le k^{-1}\sqrt{1+L^2}\end{aligned}$$

$$\Rightarrow \Omega_{\frac{1}{k}} \cap G \subset \left\{(\hat x, x^n) : |\hat x| < R,\ \phi(\hat x) < x^n < \phi(\hat x) + \frac{1}{k}\sqrt{1+L^2}\right\}.$$

$$|u(\hat x, x^n) - u(\hat x, \phi(\hat x))| \le \int_{\phi(\hat x)}^{x^n} |D_n u(\hat x, t)|dt.$$

$$|u(\hat x, x^n)| \le |u(\hat x, x^n) - u(\hat x, \phi(\hat x))| + |u(\hat x, \phi(\hat x))|$$

$$|u(\hat x, x^n)|^p \le 2^p(|u(\hat x, x^n) - u(\hat x, \phi(\hat x))|^p + |u(\hat x, \phi(\hat x))|^p).$$

Set

$$c_p = \begin{cases} 1, & \text{if } p = 1 \\ p^{-1}2^p, & \text{if } p > 1. \end{cases}$$

Then we find

$$\begin{aligned}\int_{\hat{B}_R}\int_{\phi(\hat{x})}^{\phi(\hat{x})+\frac{\sqrt{1+L^2}}{k}} |u(\hat{x},x^n)|^p &\le \sqrt{1+L^2}^p c_p k^{-p} \int_{\hat{B}_R}\int_{\phi(\hat{x})}^{\phi(\hat{x})+\frac{\sqrt{1+L^2}}{k}} |D_n u|^p \\ &+ c_p k^{-1}\sqrt{1+L^2}\int_{\hat{B}_R} |u(\hat{x},\phi(\hat{x}))|^p \\ &\le c_p k^{-p}\sqrt{1+L^2}^p \int_{\hat{B}_R}\int_{\phi(\hat{x})}^{\phi(\hat{x})+\frac{\sqrt{1+L^2}}{k}} |Du|^p \\ &+ c_p k^{-1}\sqrt{1+L^2}\int_{\partial\Omega} |u|^p.\end{aligned}$$

Furthermore we have

$$\{(\hat{x},x^n)\in\Omega\colon |\hat{x}|<R,\ \phi(\hat{x})<x^n<\phi(\hat{x})+\frac{\sqrt{1+L^2}}{k}\}\subset\Omega_{\frac{\sqrt{1+L^2}}{k}}.$$

$$\begin{aligned}\Rightarrow \int_{\Omega_{\frac{1}{k}}\cap G} |u|^p &\le \int_{\hat{B}_R}\int_{\phi(\hat{x})}^{\phi(\hat{x})+\frac{\sqrt{1+L^2}}{k}} |u(\hat{x},x^n)|^p \\ &\le c_p k^{-p}\sqrt{1+L^2}^p \int_{\Omega_{\frac{\sqrt{1+L^2}}{k}}} |Du|^p \qquad (1.2) \\ &+ c_p k^{-1}\sqrt{1+L^2}\int_{\partial\Omega} |u|^p.\end{aligned}$$

(ii) From

$$\begin{aligned}|u(\hat{x},\phi(\hat{x}))| &\le |u(\hat{x},x^n)-u(\hat{x},\phi(\hat{x}))| + |u(\hat{x},x^n)| \\ &\le |u(\hat{x},x^n)| + \int_{\phi(\hat{x})}^{x^n} |D_n u(\hat{x},t)|dt\end{aligned}$$

we deduce

$$k^{-1}\int_{\hat{B}_R} |u(\hat{x},\phi(\hat{x}))| \le \int_{\hat{B}_R}\int_{\phi(\hat{x})}^{\phi(\hat{x})+\frac{1}{k}} |u(\hat{x},x^n)| + k^{-1}\int_{\hat{B}_R}\int_{\phi(\hat{x})}^{\phi(\hat{x})+\frac{1}{k}} |Du|.$$

$$\begin{aligned}k^{-1}\int_{\partial\Omega}|u| &\le k^{-1}\int_{\hat{B}_R} |u(\hat{x},\phi(\hat{x}))|\sqrt{1+L^2} \\ &\le \sqrt{1+L^2}\int_{\Omega_{\frac{1}{k}}} |u| + k^{-1}\sqrt{1+L^2}\int_{\Omega_{\frac{1}{k}}} |Du|. \qquad (1.3)\end{aligned}$$

This also holds for all $u \in H^{1,p}(\Omega)$ sucht that $\operatorname{supp}(u) \cap \bar{\Omega} \subset G$.
Let $u \in H^{1,p}(\Omega)$ and consider a covering of $\bar{\Omega}_{\frac{1}{k_0}}$ by u_i, $1 \le i \le N$, together with a subordinate partition of unity (η_i), such that (1.2) and (1.3) are applicable to $u\eta_i$. Thus

$$\int_{\Omega_{\frac{1}{k}}} |u|^p \le ck^{-p}\sqrt{1+L^2}^p \int_{\Omega_{\frac{1}{k}}} (|Du|^p + |u|^p) + c_p k^{-1}\sqrt{1+L^2}N^p \int_{\partial\Omega} |u|^p$$

and

$$\int_{\partial\Omega} |u| \le \sqrt{1+L^2}k \int_{\Omega_{\frac{1}{k}}} |u| + c\int_{\Omega_{\frac{1}{k}}} (|Du| + |u|).$$

(iii) and (iv) follow from (i) and (ii) easily.

□

1.2.24 Lemma. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$, $1 \le p < \infty$. Let $u \in H^{1,p}(\Omega)$, $t(u) = 0$. Then there holds*

$$u \in H_0^{1,p}(\Omega).$$

Proof. $d = \operatorname{dist}(\cdot, \partial\Omega) \in C^{0,1}(\mathbb{R}^n)$ and $|Dd| = 1$ a.e. Set

$$\eta_k := \min(1, kd), \;\; k \ge 1.$$

Let Ω_k be the corresponding boundary strip. Then we find

$$\eta_k = 1 \text{ in } \Omega \backslash \Omega_{\frac{1}{k}}.$$

(i) **Claim:** $u \in H^{1,p}(\Omega) \Rightarrow u\eta_k \in H_0^{1,p}(\Omega)$.
Proof: Let $u \in C^{0,1}(\bar{\Omega})$

$$\Rightarrow v := u\eta_k \in C^{0,1}(\bar{\Omega}) \wedge u\eta_{k|\partial\Omega} = 0.$$

Let $\epsilon > 0$ and using a decomposition into v^+ and v^- we may as well suppose $v \ge 0$.

$$v_\epsilon := \max(v - \epsilon, 0) \in C_c^{0,1}(\Omega) \subset H_0^{1,p}(\Omega),$$

which follows from approximation. We have

$$Dv_\epsilon = \begin{cases} Dv, & \text{if } v > \epsilon \\ 0, & \text{if } v \le \epsilon. \end{cases}$$

$$\int_\Omega |Dv - Dv_\epsilon|^p = \int_{\{v \le \epsilon\}} |Dv|^p \to 0, \text{ since } |\Omega| < \infty.$$

$$\int_\Omega |v - v_\epsilon|^p = \epsilon^p \int_{\{v > \epsilon\}} 1 + \int_{\{v \le \epsilon\}} |v|^p \to 0.$$

Let $u \in H^{1,p}(\Omega)$, $t(u) = 0$. Then for a mollification u_ϵ we have

$$u_\epsilon \to u \text{ in } H^{1,p}(\mathbb{R}^n)$$
$$\Rightarrow u_\epsilon \eta_k \to u\eta_k \text{ in } H^{1,p}(\Omega).$$
$$\Rightarrow u\eta_k \in H_0^{1,p}(\Omega).$$

(ii) Furthermore we have

$$\int_\Omega |Du - D(u\eta_k)| \leq k \int_{\Omega_{\frac{1}{k}}} |u| + \int_{\Omega_{\frac{1}{k}}} |Du| \to 0,$$

by the preceding lemma.
$p > 1$:

$$\int_\Omega |Du - D(u\eta_k)|^p \leq 2^p \int_{\Omega_{\frac{1}{k}}} |Du|^p + 2^p k^p \int_{\Omega_{\frac{1}{k}}} |u|^p.$$

Analogously

$$\int_\Omega |u - u\eta_k|^p \leq \int_{\Omega_{\frac{1}{k}}} |u|^p.$$

Thus $u \in H_0^{1,p}(\Omega)$.

□

1.2.25 Proposition. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$. Let $u \in H^{1,p}(\Omega)$, $t(u) \leq k$ a.e. on $\partial\Omega$. Then*

$$\max(u - k, 0) \in H_0^{1,p}(\Omega).$$

Proof. $t(\max(u-k, 0)) = \max(t(u)-k, 0) = 0$ and use the preceding lemma.

□

1.2.26 Corollary. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^1$, $u \in H^{1,1}(\Omega)$. Then*

$$k \int_{\Omega_{\frac{1}{k}}} |u| \to \int_{\partial\Omega} |u|.$$

Proof. For C^1 boundary it is possible to obtain $L \leq \epsilon$ for all $\epsilon > 0$. □

1.2.27 Lemma. *For $h \in \mathbb{R}^n$, $v \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ define*

$$v_h(x) = v(x + h).$$

(i) This defines an isometry of $L^p(\mathbb{R}^n)$, $\|v\|_p = \|v_h\|_p$,
(ii) $\lim_{h\to 0} \|v - v_h\|_p = 0$ and
(iii) For $\Omega \subset \mathbb{R}^n$ and $L^p(\Omega) \to L^p(\mathbb{R}^n)$ extending by zero we have

$$\|v_h\|_{p,\Omega} \leq \|v\|_{p,\Omega}$$

and

$$\|v_h - v\|_{p,\Omega} \to 0.$$

Proof. Exercise. □

1.3 The difference quotient

In this chapter we consider for a given function u the so-called difference quotient

$$\Delta_h u(x) = \frac{u(x+he_n)-u(x)}{h}, \quad 0 \neq h \in \mathbb{R}.$$

Abusing notation, let

$$h = he_n.$$

1.3.1 Lemma. *Let $\Omega \subset \mathbb{R}^n$ be open. For $\Omega' \Subset \Omega$ and $h < \operatorname{dist}(\Omega', \partial\Omega)$ we have that*

$$\Delta_h : L^p(\Omega) \to L^p(\Omega')$$

is continuous and

$$\|\Delta_h u\|_{p,\Omega'} \leq 2|h|^{-1}\|u\|_{p,\Omega}.$$

Furthermore there holds

$$\langle \Delta_h u, v\rangle_{L^2} = -\langle u, \Delta_{-h} v\rangle_{L^2},$$

if one of the functions has compact support in Ω and h is small.

Proof. W.l.o.g. let $\operatorname{supp}(v) \subset \Omega$ and $\Omega' = \operatorname{int}(\operatorname{supp}(v))$. Then we have

$$\begin{aligned}
\langle \Delta_h u, v\rangle &= \int_{\Omega'} \frac{u(x+h)-u(x)}{h} v(x)dx \\
&= \frac{1}{h}\int_{\Omega'} u(x+h)v(x)dx - \frac{1}{h}\int_{\Omega'} u(x)v(x)dx \\
&= -\int_{\Omega} u(y)\frac{v(y)-v(y-h)}{h}dy \\
&= -\int_{\Omega} u(y)\frac{v(y-h)-v(y)}{-h}dy \\
&= -\langle u, \Delta_{-h} v\rangle.
\end{aligned}$$

□

1.3.2 Lemma. *(i) Let $\Omega \Subset \mathbb{R}^n$ be open, $u \in H^{1,p}(\Omega)$, $1 \leq p < \infty$, $\Omega' \Subset \Omega$. Then*

$$\forall |h| < h_0 << 1\colon \|\Delta_h u\|_{p,\Omega'} \leq \|D_n u\|_{p,\Omega} \tag{1.4}$$

and

$$\lim_{h\to 0} \|D_n u - \Delta_h u\|_{p,\Omega'} = 0. \tag{1.5}$$

(ii) For $u \in H^{1,p}(\mathbb{R}^n)$ there hold

$$\|\Delta_h u\|_{p,\mathbb{R}^n} \leq \|D_n u\|_{p,\mathbb{R}^n} \tag{1.6}$$

and

$$\|\Delta_h u\|_{p,\mathbb{R}^n} \to \|D_n u\|_{p,\mathbb{R}^n}. \tag{1.7}$$

Proof. Let $\Omega' \Subset \Omega'' \Subset \Omega$ and $h < \operatorname{dist}(\partial\Omega, \Omega'')$.
(i) Since we can approximate u by $u_\epsilon \in C^1(\Omega) \cap H^{1,p}(\Omega)$ and since $(\Delta_h u)_\epsilon = \Delta_h u_\epsilon$ we have

$$\Delta_h u_\epsilon \to \Delta_h u \text{ in } H^{1,p}(\Omega'),$$

as $\epsilon \to 0$. Thus let $u \in C^1(\Omega) \cap H^{1,p}(\Omega)$. Let $x \in \Omega' \Subset \Omega$, $h > 0$.

$$\Delta_h u(x) = \frac{1}{h} \int_{x_n}^{x_n+h} D_n u(\hat{x}, t) dt,$$

thus

$$\begin{aligned} |\Delta_h u(x)|^p &\le h^{-p} \left| \int_{x_n}^{x_n+h} D_n u(\hat{x}, t) dt \right|^p \\ &\le h^{-p} h^{p-1} \int_{x_n}^{x_n+h} |D_n u(\hat{x}, t)|^p dt \\ &= h^{-1} \int_{x_n}^{x_n+h} |D_n u(\hat{x}, t)|^p dt. \end{aligned}$$

Thus we have

$$\int_{\Omega'} |\Delta_h u(x)|^p dx \le h^{-1} \int_0^h \int_{\Omega'} |D_n u(\hat{x}, x^n + t)|^p dx dt \le \|D_n u\|_{p,\Omega}^p.$$

For $-h$ this holds, since $\Delta_{-h} u(x) = \Delta_h u(x - h)$. Let $\epsilon > 0$. Choose $v \in C^1(\Omega) \cap H^{1,p}(\Omega)$ such that

$$\|v - u\|_{1,p,\Omega'} < \frac{\epsilon}{3}.$$

Then

$$\|D_n u - \Delta_h u\|_{p,\Omega'} \le \|D_n u - D_n v\|_{p,\Omega'} + \|D_n v - \Delta_h v\|_{p,\Omega'} + \|\Delta_h (u - v)\|_{p,\Omega'}.$$

The first and last term are less than $\frac{\epsilon}{3}$. The middle term´s integrand converges to 0 uniformly.
(ii) The proof is exactly the same, but instead of the uniform convergence in the last argument use the decomposition

$$\int_{\mathbb{R}^n} |D_n v - \Delta_h v|^p \le \int_{B_R} |D_n v - \Delta_h v|^p + \int_{|x|>R} |D_n v - \Delta_h v|^p$$

and that the functions are integrable.

$\square$

1.3.3 Lemma. *Let $\Omega \Subset \mathbb{R}^n$ be open, $u \in H^{m,p}(\Omega)$, $1 < p < \infty$, $m \in \mathbb{N}$, $\Omega' \Subset \Omega$ and let*

$$\forall |\alpha| \le m \colon \|\Delta_h D^\alpha u\|_{p,\Omega'} \le c \quad \forall |h| \le h_0.$$

Then

$$D_n u \in H^{m,p}(\Omega')$$

and

$$\|D_n D^\alpha u\|_{p,\Omega'} \leq c.$$

Proof. $1 < p < \infty \Rightarrow L^p(\Omega')$ is reflexive. Thus there exists a sequence h_k such that

$$\Delta_{h_k} D^\alpha u \rightharpoonup v_\alpha \in L^p(\Omega')$$

and

$$\|v_\alpha\|_{p,\Omega'} \leq \liminf_{k\to\infty} \|D_{h_k} D^\alpha u\|_{p,\Omega'} \leq c.$$

Let $\eta \in C_c^\infty(\Omega')$. Then

$$\langle v_\alpha, \eta\rangle = \lim_{k\to\infty} \langle \Delta_{h_k} D^\alpha u, \eta\rangle = (-1)^{|\alpha|+1}\langle u, D_n D^\alpha \eta\rangle.$$

Thus, if $|\alpha| = 0$ we have $D_n u = v_\alpha$.
If $|\alpha| \geq 1$, we have $D_n u \in H^{m,p}(\Omega')$.

□

1.4 Sobolev embedding- and compactness theorems

1.4.1 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open with $H^{1,p}$-extension property, $1 \leq p < n$. Then there holds*

$$H^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Proof. We show

$$\exists c = c(n,p)\ \forall u \in H^{1,p}(\Omega)\colon \|u\|_{p^*} \leq c\|u\|_{1,p}.$$

It suffices to show this for $u \in C_c^\infty(\mathbb{R}^n)$. Let first be $p = 1$ and $x = (\hat{x}_i, x^i)$ for all i.

$$|u(x)| \leq \int_{-\infty}^{x^i} |D_i u(\hat{x}_i, t)|dt$$

$$\Rightarrow |u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{-\infty}^\infty |D_i u(\hat{x}_i, t)|dt\right)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{-\infty}^\infty |u|^{\frac{n}{n-1}} dx^1 \leq \left(\int_{-\infty}^\infty |D_1 u(\hat{x}_1, t)|dt\right)^{\frac{1}{n-1}} \cdot \int_{-\infty}^\infty \prod_{i=2}^n \left(\int_{-\infty}^\infty |D_i u(\hat{x}_i, t)|dt\right)^{\frac{1}{n-1}} dx^1$$

The generalized Hoelder inequality implies

$$\Rightarrow \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx^1 \leq \left(\int_{-\infty}^{\infty} |D_1 u(\hat{x}_1, t)| dt\right)^{\frac{1}{n-1}}$$

$$\cdot \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u(\hat{x}_i, x^i)| dx^i dx^1\right)^{\frac{1}{n-1}}.$$

For $n = 2$ this already implies

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_1 u|\right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_2 u|\right).$$

For $n > 2$ we repeat this argument to obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx^1 dx^2 \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_2 u(\hat{x}_2, x^2)| dx^2 dx^1\right)^{\frac{1}{n-1}}$$

$$\cdot \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_1 u(\hat{x}_1, x^1)| dx^1 dx^2\right)^{\frac{1}{n-1}}$$

$$\cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u(\hat{x}_i, x^i)| dx^i\right)^{\frac{1}{n-1}}$$

Successive integration implies

$$\int_{\mathbb{R}n} |u|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_i u|\right)^{\frac{1}{n-1}} \leq \left(\int_{\mathbb{R}^n} |Du|\right)^{\frac{n}{n-1}}$$

$$\Rightarrow \forall u \in C_c^{\infty}(\mathbb{R}^n)\colon \|u\|_{\frac{n}{n-1}} \leq \|Du\|_1.$$

Let now $1 < p < n$: Define

$$t := \frac{p(n-1)}{n-p} > 1,\ u \in C_c^{\infty}(\mathbb{R}^n)$$

$$\Rightarrow v := |u|^t \in C_c^1(\mathbb{R}^n)$$

$$\Rightarrow \int_{\mathbb{R}^n} |v|^{\frac{n}{n-1}} \leq \left(\int_{\mathbb{R}^n} |Dv|\right)^{\frac{n}{n-1}}.$$

$$|Dv| \leq t|u|^{\frac{n(p-1)}{n-p}} |Du|$$

$$\Rightarrow \|v\|_{\frac{n}{n-1}} \leq t \int_{\mathbb{R}^n} |u|^{\frac{n(p-1)}{n-p}} |Du| \leq t\|Du\|_p (\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}})^{\frac{p-1}{p}}$$

$$\Rightarrow \|u\|_{p^*} \leq t\|Du\|_p.$$

□

1.4.2 Corollary. *For $u \in H_0^{1,p}(\Omega)$ there even holds*

$$\|u\|_{p^*} \leq c\|Du\|_p,$$

which also means, that $\|Du\|_{p,\Omega}$ is a norm on $H_0^{1,p}(\Omega)$.

Proof. This follows from the extension property, i.e.

$$H^{1,p}(\Omega) \hookrightarrow H_0^{1,p}(\Omega_0) \hookrightarrow H_c^{1,p}(\mathbb{R}^n)$$

and the previous proof. $\square$

1.4.3 Theorem. *Suppose Ω has the $H^{m,p}$- extension property. Then*

$$H^{m,p}(\Omega) \hookrightarrow L^q(\Omega),$$

$\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$, if $mp < n$.

Proof. Exercise. $\square$

1.4.4 Proposition. *Let Ω have the $H^{m,p}$- extension property and $|\Omega| < \infty$. Let $mp = n \geq 2$. Then*

$$\forall 1 \leq q < \infty \colon H^{m,p}(\Omega) \hookrightarrow L^q(\Omega).$$

Proof. (i) $p > 1$: Let $p - \epsilon > 1$. Then

$$H^{m,p}(\Omega) \hookrightarrow H^{m,p-\epsilon}(\Omega)$$

and

$$m(p-\epsilon) < n.$$

Thus

$$H^{m,p-\epsilon}(\Omega) \hookrightarrow L^{q_\epsilon}(\Omega),$$

where $q_\epsilon \to \infty$.
(ii) $p = 1$: Then $m \geq 2$ and for $u \in H^{m,1}(\Omega)$ we have $D^{m-1}u \in H^{1,1}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega)$. Thus

$$H^{n,1}(\Omega) \hookrightarrow H^{n-1,\frac{n}{n-1}}(\Omega).$$

Now (i) is applicable. $\square$

1.4.5 Remark. 1.4.4 does not hold for $q = \infty$.

Proof. Choose $\Omega = B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$, $n \geq 2$ and

$$u(x) = \log(-\log|x|) - \log\log 2.$$

There holds

$$Du = \frac{1}{\log|x|}\frac{1}{|x|}\frac{x}{|x|}$$

and

$$\begin{aligned}\int_\Omega |Du|^n &= |\mathbb{S}^{n-1}| \int_0^{\frac{1}{2}} -\frac{1}{\log^n r}\frac{1}{r^n} r^{n-1} \\ &= |\mathbb{S}^{n-1}| \int_0^{\frac{1}{2}} \frac{1}{|\log^n r|} r^{-1} \\ &= c\int_{\log 2}^\infty \frac{1}{t^n} dt < \infty.\end{aligned}$$

□

1.4.6 Theorem.

$$H^{m,p}(\mathbb{R}^n) = H_0^{m,p}(\mathbb{R}^n),$$

if $1 \le p < \infty$.

Proof. We only prove the case $m = 1$, the rest follows from induction. Let $0 \le \eta \le 1$, $\eta \in C_c^\infty(\mathbb{R}^n)$, such that

$$\eta(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| \ge 2 \end{cases}$$

and

$$|D\eta| \le c.$$

Set

$$\eta_k(x) = \eta\left(\frac{x}{k}\right).$$

For $u \in H^{1,p}(\mathbb{R}^n)$ define

$$u_k = u\eta_k \in H_0^{1,p}(\mathbb{R}^n).$$

There clearly holds $u_k \to u$ in $L^p(\mathbb{R}^n)$.
Furthermore $Du_k = Du\eta_k + k^{-1}uD\eta \to Du$ in $L^p(\mathbb{R}^n)$. □

1.4.7 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ have the $H^{1,p}$- extension property. Let $p > n$, then for $\alpha = 1 - \frac{n}{p}$ we have*

$$H^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$$

and

$$\forall u \in H_0^{1,p}(\Omega) \colon [u]_{\alpha,\Omega} \le c\|Du\|_p.$$

Proof. Without loss of generality let $u \in H_0^{1,p}(\Omega_0)$, $\Omega \Subset \Omega_0$, and we will show

$$\forall u \in H_0^{1,p}(\Omega_0) \colon |u|_{0,\alpha,\Omega_0} \le c\|Du\|_p.$$

Let $x_1, x_2 \in \Omega_0$, $0 < \rho = |x_1 - x_2|$, $x \in B_\rho(\frac{x_1+x_2}{2}) \equiv B_\rho(0)$. Then we have for $u \in C_c^1(\Omega_0)$

$$
\begin{aligned}
u(x) - u(x_i) &= \int_0^1 \frac{d}{dt} u(x_i + t(x - x_i))dt \\
&\equiv \int_0^1 D_k u(x_t)(x^k - x_i^k)dt \\
&\le 2\rho \int_0^1 |Du(x_t)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left| \fint_{B_\rho} u - u(x_i) \right| &\le 2c\rho^{1-n} \int_0^1 \int_{B_\rho} |Du(x_i + t(x - x_i))| \\
&\le 2c\rho^{1-n} \int_0^1 t^{-n} \int_{B_{2\rho t}(x_i)} |Du(z)| \\
&\le 2c\rho^{1-n} \int_0^1 t^{-n} \|Du\|_{p,\Omega_0} \rho^{n\frac{p-1}{p}} t^{n\frac{p-1}{p}} \\
&\le c\rho^{1-\frac{n}{p}} \|Du\|_{p,\Omega_0} \int_0^1 t^{-\frac{n}{p}} \\
&\le c(n,p) \|Du\|_{p,\Omega_0} \rho^{1-\frac{n}{p}}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
|u(x_1) - u(x_2)| &\le \left| u(x_1) - \fint_{B_\rho} u \right| + \left| \fint_{B_\rho} u - u(x_2) \right| \\
&\le c\|Du\|_p |x_1 - x_2|^\alpha.
\end{aligned}
$$

Choosing $x_2 \in \partial\Omega_0$ we find $u(x_2) = 0$ and thus

$$
|u|_{0,\Omega_0} \le c\|Du\|_p (\text{diam}\Omega)^\alpha.
$$

$\square$

1.4.8 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ have the $H^{m,p}$- extension property. Then*

$$
H^{m,p}(\Omega) \hookrightarrow C^{j,\alpha}(\bar{\Omega}), \ m \in \mathbb{N}, \ 1 \le p < \infty,
$$

if

$$
\begin{aligned}
m = k + j \quad &\textit{and} \\
(i) \ (k-1)p < n < kp, \ &\alpha = k - \frac{n}{p} \\
(ii) \ (k-1)p = n, \quad &\forall 0 < \alpha < 1.
\end{aligned}
$$

Proof. Exercise. □

1.4.9 Theorem. *(Interpolation theorem)*
Let $1 \le p_1 < p < p_2 < \infty$, $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$, $0 < \alpha < 1$ and Ω be a measure space. Then

$$\forall u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega) \colon \|u\|_p \le \|u\|_{p_1}^{\alpha} \|u\|_{p_2}^{1-\alpha}.$$

Proof. There holds

$$p = \frac{1}{\alpha p_2 + (1-\alpha)p_1}(\alpha p_1 p_2 + (1-\alpha)p_1 p_2).$$

Thus

$$\begin{aligned}\int_\Omega |u|^p &= \int_\Omega |u|^{p_1 \frac{\alpha p_2}{\alpha p_2 + (1-\alpha)p_1}} |u|^{p_2 \frac{(1-\alpha)p_1}{\alpha p_2 + (1-\alpha)p_1}} \\ &\le \left(\int_\Omega |u|^{p_1}\right)^{\frac{\alpha p_2}{\alpha p_2 + (1-\alpha)p_1}} \left(\int_\Omega |u|^{p_2}\right)^{\frac{(1-\alpha)p_1}{\alpha p_2 + (1-\alpha)p_1}}.\end{aligned}$$

□

1.4.10 Theorem. *(Kolmogorov)*
Let $\Omega \Subset \mathbb{R}^n$. A subset $M \subset L^p(\Omega)$, $1 \le p < \infty$, is precompact if and only if

(i) M is bounded and

(ii) M is equicontinuous in the mean,

i.e.

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall u \in M \colon 0 \le h < \delta \Rightarrow \|u - u_h\|_{p,\Omega} < \epsilon.$$

Proof. Let M be precompact. Then M is clearly bounded. Let $\epsilon > 0$. Then there exist $(u_i)_{1 \le i \le N}$ such that

$$M \subset \bigcup_{i=1}^{N} B_\epsilon(u_i).$$

Let $u \in M$, then $u \in B_\epsilon(u_{i_0})$.

$$\begin{aligned}\Rightarrow \|u(\cdot + h) - u\|_{p,\Omega} &\le \|u(\cdot + h) - u_{i_0}(\cdot + h)\| \\ &\quad + \|u_{i_0}(\cdot + h) - u_{i_0}\| + \|u_{i_0} - u\| < 3\epsilon,\end{aligned}$$

if we choose h small enough. Note that a finite collection of functions is equicontinuous.
Now let (i) and (ii) hold. Let $\epsilon > 0$ and for $\delta > 0$ let η_δ be a Dirac sequence. Let

$$u_\delta = u * \eta_\delta.$$

$$
\begin{aligned}
|u_\delta(x) - u(x)|^p &= \left| \int_{B_\delta(0)} \eta_\delta(y)(u(x-y) - u(x)) \right|^p dy \\
&\le \int_{B_\delta(0)} \eta_\delta(y)|u(x-y) - u(x)|^p dy
\end{aligned}
$$

$$
\Rightarrow \int_{\mathbb{R}^n} |u_\delta - u|^p \le \int_{B_\delta(0)} \eta_\delta(y) \int_{\mathbb{R}^n} |u(x-y) - u(x)|^p dx dy
$$

$$
(ii) \Rightarrow \|u_\delta - u\|_p \le \sup_{|y|<\delta} \|u(x-y) - u(x)\|_p < \epsilon,
$$

if δ is small.
We now claim that $M_\delta := \{u_\delta : u \in M\} \subset C^0(\overline{\Omega + \delta}) =: E$ is precompact in E. We have

$$
\begin{aligned}
|u_\delta(x)| &\le \int_{B_\delta(0)} \eta_\delta^{1-\frac{1}{p}}(y)\eta_\delta^{\frac{1}{p}}(y)|u(x-y)|dy \\
&\le \left(\int_{B_\delta(0)} \eta_\delta(y)|u(x-y)|^p \right)^{\frac{1}{p}} \\
&\le \sup_{B_\delta} |\eta_\delta|^{\frac{1}{p}} \|u\|_p \le c
\end{aligned}
$$

Thus M_δ is bounded.
Furthermore

$$
\begin{aligned}
|u_\delta(x+h) - u_\delta(x)| &\le \int_{B_\delta(0)} \eta_\delta^{1-\frac{1}{p}} \eta_\delta^{\frac{1}{p}} |u(x+h-y) - u(x-y)|dy \\
&\le \sup_{B_\delta(0)} |\eta_\delta|^{\frac{1}{p}} \|u(y+h) - u(y)\|_p.
\end{aligned}
$$

Thus M_δ is equicontinuous and by Arzela-Ascoli there exists an ϵ-net $(u_\delta^i)_{1\le i\le N}$ in E. We now claim, that this net is also an ϵ-net in $L^p(\Omega)$. Let $u \in M$ and $1 \le i \le N$. Then

$$
\int_{\mathbb{R}^n} |u - u_\delta^i|^p \le 2^p \int_{\mathbb{R}^n} |u - u_\delta|^p + 2^p \int_{\mathbb{R}^n} |u_\delta - u_\delta^i|^p \le c\epsilon^p.
$$

□

1.4.11 Proposition. *(Kondrašov)*
Let $\Omega \Subset \mathbb{R}^n$ have the $H^{1,p}$- extension property, $1 \le p < \infty$. Let $\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}$, then for $q < p^*$

$$
H^{1,p}(\Omega) \hookrightarrow L^q(\Omega)
$$

is compact.

Proof. Let $u_\epsilon \in H^{1,p}(\Omega)$ be bounded. Suppose

$$\forall \epsilon \colon u_\epsilon \in H^{1,p}(\Omega_0)$$

and

$$\|u_\epsilon\|_{1,p,\Omega_0} \le c.$$

$$\Rightarrow \forall \epsilon > 0 \; \exists v_\epsilon \in C_c^\infty(\Omega_0) : \|v_\epsilon - u_\epsilon\| < \epsilon.$$

Thus it suffices to show, that the v_ϵ are precompact in $L^q(\Omega_0)$. By the interpolation theorem this will follow from the case $q = 1$. We use the Kolmogorov characterization. The boundedness is clear.

$$\begin{aligned} v_\epsilon(x+h) - v_\epsilon(x) &= \int_0^1 \frac{d}{dt} v_\epsilon(x+th)dt \\ &= \int_0^1 D_i v_\epsilon(x+th)h^i dt \end{aligned}$$

and thus

$$\int_{\mathbb{R}^n} |v_\epsilon(x+h) - v_\epsilon(x)| \le |h| \int_0^1 \int_{\mathbb{R}^n} |Dv_\epsilon| \le |h| \|Dv_\epsilon\|_1.$$

□

1.4.12 Corollary. *Let Ω have the $H^{m,p}$- extension property, $\frac{1}{q} > \frac{1}{p} - \frac{m}{n}$, $q \ge 1$. Then*

$$H^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact. In cases $mp = n$ this holds for all $1 \le q < \infty$.

Proof. The case $m = 1$ has been proven. There holds

$$u, Du \in H^{m-1,p}(\Omega) \hookrightarrow L^r(\Omega),$$

where

$$\frac{1}{r} = \frac{1}{p} - \frac{m-1}{n}.$$

Thus $u \in H^{1,r}(\Omega) \hookrightarrow L^q(\Omega)$, being compact, if

$$\frac{1}{q} > \frac{1}{r^*} = \frac{1}{q} - \frac{1}{n} = \frac{1}{p} - \frac{m}{n}.$$

The second claim follows by interpolation. □

1.4.13 Lemma. *(Interpolation of Hoelder spaces)*
Let $\Omega \Subset \mathbb{R}^n$ be open and $0 < \beta < \alpha \le 1$. Then there holds

$$\begin{aligned} [u]_{\beta,\Omega} &\le [u]_\alpha^{\frac{\beta}{\alpha}} \cdot (\operatorname{osc}(u))^{1-\frac{\beta}{\alpha}} \\ &\le [u]_\alpha^{\frac{\beta}{\alpha}} \cdot 2^{1-\frac{\beta}{\alpha}} |u|_0^{1-\frac{\beta}{\alpha}}. \end{aligned}$$

Proof.

$$\begin{aligned}
\frac{|u(x)-u(y)|}{|x-y|^\beta} &= \left(\frac{|u(x)-u(y)|^{\frac{\alpha}{\beta}}}{|x-y|^\alpha}\right)^{\frac{\beta}{\alpha}} \\
&= \left(\frac{|u(x)-u(y)|}{|x-y|^\beta}|u(x)-u(y)|^{\frac{\alpha}{\beta}-1}\right)^{\frac{\beta}{\alpha}} \\
&\le [u]_{\alpha,\Omega}^{\frac{\beta}{\alpha}}(\operatorname{osc}(u))^{1-\frac{\beta}{\alpha}}.
\end{aligned}$$

□

1.4.14 Corollary. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$, $0 < \beta < \alpha$. Then the embedding*

$$C^{k,\alpha}(\bar{\Omega}) \hookrightarrow C^{k,\beta}(\bar{\Omega})$$

is compact.

Proof. Let $u_\epsilon \in C^{k,\alpha}(\bar{\Omega})$ be bounded. By Arzela-Ascoli there exists a subsequence

$$u_\epsilon \to u \in C^{k,\alpha}(\bar{\Omega}) \text{ in } C^k(\bar{\Omega}).$$

Set

$$v_\epsilon := D^\gamma u_\epsilon \to D^\gamma u = v$$

for some multiindex γ. Inserting this into the interpolation theorem yields the result. □

1.4.15 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$, $mp > n$. Then*

$$H^{m,p}(\Omega) \hookrightarrow C^{j,\beta}(\bar{\Omega}), \quad 0 \le \beta < \alpha,$$

is compact, where j, α are as in the Sobolev embedding theorem.

1.4.16 Lemma. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$. Then*

$$C^{0,1}(\bar{\Omega}) = H^{1,\infty}(\Omega).$$

Proof. Let $u \in C^{0,1}(\bar{\Omega})$. Then a mollification u_ϵ converges in $C^{0,1}(\bar{\Omega}')$ to u for all $\Omega' \Subset \Omega$. Thus $u \in H^{1,\infty}(\Omega)$. Let $u \in H^{1,\infty}(\Omega)$. Since

$$|u(x)-u(y)| \le \|Du\|_{\infty,\Omega}|x-y|,$$

we obtain the result locally. For $x, y \in B_\delta(x_0) \cap \Omega$, $x_0 \in \partial\Omega$, we can use a coordinate transformation to convert the problem into the convex set $B_+^1(0)$. □

1.4.17 Proposition. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$*

$$\Rightarrow H^{m,p}(\Omega) \hookrightarrow H^{m-1,p}(\Omega), \quad 1 \le p < \infty, \quad m \ge 1,$$

is compact.

Proof. Follows immediately from the other embedding theorems. □

1.4.18 Proposition. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$, $m \geq 1$ and $1 \leq p < \infty$. Then*

$$\forall \epsilon > 0 \ \exists c_\epsilon \in \mathbb{R} \ \forall u \in H^{m,p}(\Omega)\colon \|u\|_{m-1,p,\Omega} \leq \epsilon \sum_{|\alpha|=m} \|D^\alpha u\|_{p,\Omega} + c_\epsilon \|u\|_{p,\Omega}.$$

Proof. Use the compactness lemma for Banach spaces and absorb the lower order norm in the left hand side. □

1.4.19 Corollary. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$. Then the norm*

$$\|u\| = \sum_{|\alpha|=m} \|D^\alpha u\|_{p,\Omega} + \|u\|_{p,\Omega}, \quad 1 \leq p < \infty,$$

is an equivalent norm on $H^{m,p}(\Omega)$.

1.4.20 Lemma. *Let $\Omega \Subset \mathbb{R}^n$. Then*

$$\|u\| = \|D^m u\|_{p,\Omega}$$

is an equivalent norm on $H_0^{m,p}(\Omega)$.

Proof. $\forall |\gamma| \leq m-1\colon D^\gamma u \in H_0^{1,p}(\Omega)$. □

1.4.21 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$. Then the embedding*

$$H^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$$

is compact for $1 < p < n$ and $1 \leq q < \frac{(n-1)p}{n-p}$ and it is continuous for $q = \frac{(n-1)p}{n-p}$.

Proof. Let $\|u_k\|_{1,p,\Omega} \leq c$. Then a subsequence converges in $L^1(\Omega)$,

$$u_k \to u \in L^1(\Omega).$$

Since, by reflexivity, we have $u \in H^{1,p}(\Omega)$ we may assume $u \equiv 0$.
Let $\epsilon > 0$.

$$\begin{aligned}\int_{\partial\Omega} |u_k| &\leq \int_{\Omega_\epsilon} |Du_k| + c_\epsilon \int_\Omega |u_k| \\ &\leq c \left(\int_\Omega |Du_k|^p \right)^{\frac{1}{p}} |\Omega_\epsilon|^{\frac{p-1}{p}} + c_\epsilon \int_\Omega |u_k|\end{aligned}$$

Thus

$$\limsup_{k\to\infty} \int_{\partial\Omega} |u_k| \leq c|\Omega_\epsilon|^{\frac{p-1}{p}} \to 0, \quad \epsilon \to 0.$$

$$\Rightarrow H^{1,p}(\Omega) \hookrightarrow L^1(\partial\Omega)$$

is compact. Let $q = \frac{(n-1)p}{n-p}$ and set $v := |u|^q \in H^{1,1}(\Omega)$

$$\Rightarrow |Dv| \leq |Du||u|^{\frac{n(p-1)}{n-p}}$$

$$\Rightarrow \int_\Omega |Dv| \leq \left(\int_\Omega |Du|^p\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{\frac{np}{n-p}}\right)^{\frac{p-1}{p}}$$
$$\leq c\|u\|_{1,p,\Omega}^{p(q-1)}.$$
$$\Rightarrow H^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega).$$

□

1.4.22 Theorem. *(Poincare-inequality)*
Let $\Omega \Subset \mathbb{R}^n$ be connected with $H^{1,p}$- extension property, $1 \leq p < n$. Then for all measurable subsets $E \subset \Omega$, $|E| > 0$, there exists a constant $c_E > 0$, such that

$$\forall u \in H^{1,p}(\Omega)\colon \left(\int_\Omega |u - u_E|^p\right)^{\frac{1}{p}} \leq c_E \left(\int_\Omega |Du|^p\right)^{\frac{1}{p}},$$

where $u_E = \frac{1}{|E|}\int_E u$.

Proof. Set

$$V := \left\{u \in H^{1,p}(\Omega)\colon \int_E u = 0\right\}.$$

Suppose the inequality did not hold, then there existed a sequence $u_k \in V$ such that

$$\|u_k\|_{1,p,\Omega} = 1$$

and

$$\|u_k\|_{p,\Omega} > k\|Du_k\|_{p,\Omega}.$$

By compactness we have a subsequence converging to $u \in L^p(\Omega)$. Thus

$$\|Du\| = 0$$

and so $u \equiv \text{const}$, which is a contradiction. □

1.4.23 Theorem. *For $\Omega \subset \mathbb{R}^n$ open, the spaces $H^{m,p}(\Omega)$ are reflexive for $1 < p < \infty$.*

Proof. Exercise. □

1.4.24 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p < \infty$. Then*

$$H_0^{m,p}(\Omega)^* \equiv H^{-m,p}(\Omega) = \left\{\sum_{|\gamma|\leq m} D^\gamma f_\gamma : f_\gamma \in L^{p'}(\Omega)\right\} \subset \mathcal{D}(\Omega).$$

Proof. Exercise. □

1.5 L^2 regularity for weak solutions

1.5.1 Theorem. *(Interior estimates)*
Let $\Omega \Subset \mathbb{R}^n$ and let $a^i \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfy

$$\forall (x,u,p)\colon \left|\frac{\partial a^i}{\partial x}(x,u,p)\right| \le c_A(1+|u|+|p|) \tag{1.8}$$

$$\left|\frac{\partial a^i}{\partial u}\right| + \left|\frac{\partial a^i}{\partial p_j}\right| \le c \tag{1.9}$$

and

$$a^{ij} = \frac{\partial a^i}{\partial p_j} \Rightarrow \exists \lambda > 0\ \forall \xi \in \mathbb{R}^n\colon \lambda|\xi|^2 \le a^{ij}\xi_i\xi_j. \tag{1.10}$$

Let $u \in H^{1,2}_{loc}(\Omega)$ be a weak solution of the equation

$$Au = -(a^i(x,u,Du))_i = f \in L^2(\Omega),$$

i.e. we have equality in $H^{-1,2}(\Omega)$. Then we have

$$u \in H^{2,2}_{loc}(\Omega)$$

and for all $\Omega' \Subset \Omega'' \Subset \Omega$

$$\|u\|_{2,2,\Omega'} \le c(\|f\|_{2,\Omega}, \|u\|_{1,2,\Omega''}, c_A, \lambda).$$

Proof. We use the method of difference quotients. Let $h = he_k$ for a fixed $1 \le k \le n$. Let h_0 be small enough to ensure $\Omega' + h \Subset \Omega''$ for all $|h| \le h_0$. Let $\eta \in C^\infty_c(\Omega'')$, such that

$$\eta_{|\Omega'} = 1.$$

Multiply the equation by

$$-\Delta_{-h}(\Delta_h u\eta^2) \in H^{1,2}_0(\Omega)$$

to obtain

$$\int_\Omega \Delta_h(a^i(x,u,Du))(\Delta_h u\eta^2)_i = -\int_\Omega f\Delta_{-h}(\Delta_h u\eta^2).$$

We have

$$\begin{aligned}\Delta_h a^i(x,u,Du) &= h^{-1}(a^i(x+h,u(x+h),Du(x+h)) - a^i(x,u(x),Du(x)))\\ &= h^{-1}\int_0^1 \frac{d}{dt}a^i(x+th, tu(x+h)+(1-t)u(x),...)dt\\ &= h^{-1}(a^{ij}(u(x+h)-u(x))_j + b^i(u(x+h)-u(x)) + c^i h),\end{aligned}$$

where

$$a^{ij} = \int_0^1 \frac{\partial a^i}{\partial p_j}, \quad b^i = \int_0^1 \frac{\partial a^i}{\partial u}, \quad c_i = \sum_{k=1}^n \int_0^1 \frac{\partial a^i}{\partial x^k}.$$

By the assumptions we have

$$|c^i| \leq c_A(1 + |u(x)| + |Du(x)| + |h||\Delta_h u| + |h||\Delta_h Du|),$$

$$|a^{ij}| + |b^i| \leq c$$

as well as the uniform ellipticity of a^{ij}. There holds

$$\begin{aligned}\int_\Omega (a^{ij}(\Delta_h u)_j + b^i \Delta_h u + c^i)(\Delta_h u \eta^2)_i &= -\int_\Omega f \Delta_{-h}(\Delta_h u \eta^2)\\ &\leq \frac{\epsilon}{2}\int_{\Omega''} f^2 + \frac{1}{2\epsilon}\int_\Omega |D(\Delta_h u \eta^2)|^2,\end{aligned}$$

$$\begin{aligned}\int_\Omega a^{ij}(\Delta_h u)_j(\Delta_h u \eta^2)_i &= \int_\Omega a^{ij}(\Delta_h u)_j(\Delta_h u)_i \eta^2\\ &\quad + 2\int_\Omega a^{ij}(\Delta_h u)_j \eta_i \Delta_h u \eta.\end{aligned}$$

We have

$$\begin{aligned}\left|\int_\Omega a^{ij}(\Delta_h u)_j \eta \eta_i \Delta_h u\right| &\leq \frac{\epsilon}{2}\int_\Omega a^{ij}(\Delta_h u)_j(\Delta_h u)_i \eta^2\\ &\quad + \frac{1}{2\epsilon}\int_\Omega a^{ij}\eta_i\eta_j|\Delta_h u|^2.\end{aligned} \tag{1.11}$$

But

$$\int_\Omega a^{ij}\eta_i\eta_j|\Delta_h u|^2 \leq c(D\eta)\int_{\Omega''}|D_k u|^2,$$

$$|\int_\Omega b^i \Delta_h u(\Delta_h u \eta^2)_i| \leq \int_\Omega |b^i||\Delta_h u|(|D\Delta_h u|\eta^2 + 2|\Delta_h u||D\eta|\eta) \tag{1.12}$$

and

$$\begin{aligned}|\int_\Omega c^i(\Delta_h u \eta^2)_i| \leq \int_\Omega &(1 + |u| + |Du| + |h||\Delta_h u| + |D\Delta_h u||h|)\\ &\cdot (|D\Delta_h u|\eta^2 + 2|\Delta_h u||D\eta|\eta).\end{aligned} \tag{1.13}$$

For small ϵ we obtain, also absorbing the $|D\Delta_h u|$ in (1.12) and (1.13),

$$\begin{aligned}\frac{\lambda}{2}\int_{\Omega'}|D\Delta_h u|^2 &\leq \frac{1}{2}\int_\Omega a^{ij}(\Delta_h u)_i(\Delta_h u)_j\eta^2\\ &\leq c\int_{\Omega''}(|f|^2 + |Du|^2 + |u|^2 + 1) \quad \forall |h| < h_0.\end{aligned}$$

$$\Rightarrow \int_{\Omega'} |DD_k u|^2 \le \frac{2}{\lambda} c \int_{\Omega''} (f^2 + |Du|^2 + |u|^2 + 1)$$

$$\Rightarrow u \in H^{2,2}_{loc}(\Omega).$$

$\square$

1.5.2 Remark. Now we want to prove boundary estimates. Since a divergence writes in coordinates

$$-a^i_i = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} a^i)$$

we even may suppose that the differential operator is given in terms of covariant derivatives, after possibly multiplying the right hand side by $\sqrt{g}$ and the vector filed by $\sqrt{g}^{-1}$. Thus we are given a function on both sides and are free to consider the equation on $B^+_1(0)$ without loss of generality.

1.5.3 Theorem. *(Local boundary estimates)*
Let $0 < \rho_1 < \rho_2 < \rho$, $x_0 \in \partial\Omega$ and $B_\rho(x_0) \cap \partial\Omega = \Gamma \in C^2$. Let $u \in H^{1,2}(\Omega)$ be a solution of

$$-(a^i(x, u, Du))_i = f, \quad u_{|\partial\Omega} = \phi \in H^{2,2}(\Omega),$$

where a^i satisfies (1.8), (1.9) and (1.10). Then

$$u \in H^{2,2}(\Omega \cap B_{\rho_1}(x_0))$$

and

$$\|u\|_{2,2,\Omega_{\rho_1}} \le c(\|u\|_{1,2,\Omega_{\rho_2}}, \|f\|_{2,\Omega}, \|\phi\|_{2,2,\Omega_{\rho_2}}, c_A, \rho_1, \rho_2, |\Gamma|_2),$$

where $\Omega_{\rho_i} = \Omega \cap B_{\rho_i}(x_0)$.

Proof. Without loss of generality the equation holds in $\Omega = B^+_1(0)$ with $x_0 = 0$. Choose

$$0 \le \eta \in C^\infty_c(B_{\rho_2}), \ \eta_{|B_{\rho_1}} \equiv 1.$$

Define with abuse of notation

$$h = h \cdot e_k, \quad 1 \le k \le n-1.$$

Multiply the equation with

$$-\Delta_{-h}(\Delta_h(u - \phi)\eta^2) \in H^{1,2}_0(\Omega).$$

Then

$$\int_\Omega \Delta_h a^i D_i(\Delta_h(u - \phi)\eta^2) = -\int_\Omega f \Delta_{-h}(\Delta_h(u - \phi)\eta^2).$$

As in the proof of 1.5.1 we obtain

$$\lambda \int_\Omega |D\Delta_h u|^2 \eta^2 \leq c\left(\int_{\Omega_{\rho_2}} f^2 + \int_{\Omega_{\rho_2}} |D\Delta_h \phi|^2 + 1 + \int_{\Omega_{\rho_2}} (|Du|^2 + u^2)\right)$$

$$\Rightarrow \int_{\Omega_{\rho_1}} \sum_{i+j<2n} |D_i D_j u|^2 \leq c\left(\int_{\Omega_{\rho_2}} f^2 + \int_{\Omega_{\rho_2}} |D\Delta_h \phi|^2 + 1 + \int_{\Omega_{\rho_2}} (|Du|^2 + u^2)\right).$$

$$-D_i a^i(x, u, Du) = f$$

$$\Rightarrow -a^{ij} u_{ij} - \frac{\partial a^i}{\partial x^i} - \frac{\partial a^i}{\partial u} u_i = f.$$

Using $a^{nn} \geq \lambda$, we obtain the claim. $\square$

1.5.4 Theorem. *Let $a^{ij}, b^i, c \in C^{m,1}(\Omega)$, $f \in H^{m,2}_{loc}(\Omega)$ and $u \in H^{1,2}_{loc}(\Omega)$ be a weak solution of*

$$-(a^{ij} u_j)_i + b^i u_i + cu = f, \tag{1.14}$$

then

$$u \in H^{m+2,2}_{loc}(\Omega)$$

and for all $\Omega' \Subset \Omega'' \Subset \Omega$ we have

$$\|u\|_{m+2,\Omega'} \leq c(\|f\|_{m,2,\Omega''} + \|u\|_{1,2,\Omega''}),$$

where $c = c(|a^{ij}|_{m,1,\Omega''}, |b^i|_{m,1,\Omega''}, |c|_{m,1,\Omega''}, \Omega', \Omega'')$.

Proof. By induction. For $m = 0$ this is theorem 1.5.1 So let $m > 0$ and suppose the claim holds for $m - 1$. For $1 \leq k \leq n$ choose $v = u_k \in H^{1,2}_{loc}(\Omega)$.

$$\Rightarrow -(a^{ij} v_j)_i + b^i v_i + cv = f_k + \left(\frac{\partial a^{ij}}{\partial x^k}\right)_i u_j - b^j_k u_j + c_k u \equiv F \in H^{m-1,2}_{loc}(\Omega).$$

Let $\Omega' \Subset \tilde{\Omega} \Subset \Omega''$.

$$\Rightarrow \|v\|_{m+1,2,\Omega'} \leq c(\|F\|_{m-1,2,\tilde{\Omega}} + \|v\|_{1,2,\Omega''})$$

$$\|v\|_{1,2,\tilde{\Omega}} \leq \|u\|_{2,2,\tilde{\Omega}} \leq c(\|f\|_{2,\Omega''} + \|u\|_{1,2,\Omega''})$$

and

$$\|F\|_{m-1,2,\tilde{\Omega}} \leq c(\|f\|_{m,2,\Omega''} + \|u\|_{m+1,2,\tilde{\Omega}}) \leq c(\|f\|_{m,2,\Omega''} + \|u\|_{1,2,\Omega''}).$$

$\square$

1.5.5 Theorem. *(Local boundary estimates of higher order)* *Let* $0 < \rho_1 < \rho_2 < \rho, \ , x_0 \in \partial\Omega, \ B_\rho(x_0) \cap \Omega = \Gamma \in C^{m+2}$. *Let* $u \in H^{1,2}(\Omega)$ *be a solution of*

$$-(a^{ij}u_j)_i + b^i u_i + cu = f, \ u_{|\partial\Omega} = \phi \in H^{m+2,2}(\Omega),$$

$f \in H^{m,2}(\Omega)$, $a^{ij}, b^i, c \in C^{m,1}(\Omega \cap B_\rho(x_0))$. *Then*

$$\|u\|_{m+2,2,\Omega_{\rho_1}} \leq c(\|f\|_{m,2,\Omega_{\rho_2}} + \|u\|_{1,2,\Omega_{\rho_2}} + \|\phi\|_{m+2,2,\Omega_{\rho_2}}),$$

where $c = c(|a^{ij}|_{m,1,\Omega''}, |b^i|_{m,1,\Omega''}, |c|_{m,1,\Omega''}, \Omega', \Omega'')$.

Proof. By induction, where $m = 0$ has already been proven. Let $m > 0$ and suppose without loss of generality $\Omega = B_1^+(0)$, $x_0 = 0$. Set

$$\Gamma = B_1(0) \cap \{x^n = 0\}.$$

Let $1 \leq k \leq n-1$ and

$$v = u_k \in H^{1,2}(\Omega_\rho), \ v_{|\Gamma} = \phi_k \in H^{m+1,2}(\Omega_\rho).$$

Then

$$-(a^{ij}v_j)_i + b^i v_i + cv = f_k + \frac{\partial a^{ij}}{\partial x^k} u_j - (b^j u_j)_k - c_k u \equiv F \in H^{m-1,2}(\Omega_{\rho_2}).$$

Let $0 < \rho_1 < \tilde{\rho} < \rho_2$

$$\Rightarrow \|v\|_{m+1,2,\Omega_{\rho_1}} \leq c(\|F\|_{m-1,2,\Omega_{\tilde{\rho}}} + \|v\|_{1,2,\Omega_{\tilde{\rho}}} + \|\phi\|_{m+2,2,\Omega_{\tilde{\rho}}}).$$

For $k = n$ we again use the differential equation to obtain

$$\|u_{nn}\|_{m,2,\Omega_{\rho_1}} \leq c(\|u\|_{m+1,2,\Omega_{\tilde{\rho}}} + \sum_{k=1}^{n-1} \|u_k\|_{m+1,2,\Omega_{\tilde{\rho}}} + \|\phi\|_{m+2,2,\Omega_{\tilde{\rho}}} + \|f\|_{m,2,\Omega_{\tilde{\rho}}}).$$

□

We now consider L^2-estimates for the Neumann boundary value problem.

1.5.6 Theorem. *Let* $\Omega \Subset \mathbb{R}^n$ *be open,* $\partial\Omega \in C^2$ *and let* $u \in H^{1,2}(\Omega)$ *be a weak solution of*

$$-(a^i(x, u, Du))_i = f \ \text{in } \Omega, \ -a^i \nu_i = \phi \ \text{on } \partial\Omega,$$

where $f \in L^2(\Omega)$, $\phi \in H^{2,2}(\Omega)$ *or* $\phi \in C^{0,1}(\partial\Omega)$, $a^i \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ *and let (1.8), (1.9) as well as (1.10). Then we have* $u \in H^{2,2}(\Omega)$ *and*

$$\|u\|_{2,2,\Omega} \leq c(\|\phi\|_{2,2,\Omega} + \|f\|_{2,\Omega} + \|u\|_{1,2,\Omega} + 1)$$

in case $\phi \in H^{2,2}(\Omega)$ *and*

$$\|u\|_{2,2,\Omega} \leq c(\|f\|_{2,\Omega} + \|u\|_{1,2,\Omega} + 1),$$

where c *now also depends on* $|\phi|_{0,1,\partial\Omega}$.

Proof. We only prove the boundary estimates, since the interior estimates are theorem 1.5.1. Let $\Omega = B_1^+(0)$, $\Gamma = \{x^n = 0\} \cap \partial\Omega$. Then the weak formulation of the equation reads

$$\forall \eta \in H^{1,2}(\Omega) \cap H_c^{1,2}(B_1(0)) \colon \int_\Omega a^i \eta_i + \int_\Gamma \phi\eta = \int_\Omega f\eta.$$

Let $1 \leq k \leq n-1$, $h = h \cdot e_k$ be small and

$$\tilde{\eta} = -\Delta_{-h}(\Delta_h u\eta^2), \ \eta \in C_c^1(B_1(0)).$$

Then

$$\int_\Omega \Delta_h a^i (\Delta_h u\eta^2)_i + \int_{\partial\Omega} \Delta_h \phi \Delta_h u\eta^2 = -\int_\Omega f \Delta_{-h}(\Delta_h u\eta^2).$$

(i) If $\phi \in C^{0,1}(\partial\Omega)$, we have

$$\begin{aligned}\int_\Gamma |\Delta_h\phi\Delta_h u\eta^2| &\leq L\int_\Gamma |\Delta_h u\eta^2| \\ &\leq L\int_\Omega |D(\Delta_h u\eta^2)| + c\int_\Omega |\Delta_h u\eta^2|,\end{aligned}$$

which can be absorbed by ϵ in the left hand side.
(ii) If $\phi \in H^{2,2}(\Omega)$, we have

$$\int_\Gamma |\Delta_h\phi\Delta_h u\eta^2| \leq \int_\Omega |D(\Delta_h\phi\Delta_h u\eta^2)| + c\int_\Omega |\Delta_h\phi\Delta_h u\eta^2|.$$

□

1.5.7 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open, $\partial\Omega \in C^2$ and let $a^{ij}, b^i, c \in L^\infty(\Omega)$, $c \geq c_0 > 0$, a^{ij} uniformly elliptic, $f \in L^2(\Omega)$ and $\phi \in H^{1,2}(\Omega)$. Then*

$$\begin{aligned}-(a^{ij}u_j)_i + b^i u_i + cu &= f \ \text{in } \Omega \\ -a^{ij}u_j\nu_i &= \phi \ \text{on } \partial\Omega\end{aligned}$$

has a weak solution $u \in H^{1,2}(\Omega)$.
If additionally $\partial\Omega \in C^{m+2}$, $a^{ij} \in C^{m+1}(\Omega)$, $b^i, c \in C^m(\Omega)$, $f \in H^{m,2}(\Omega)$ and $\phi \in C^{m,1}(\partial\Omega)$ or $\phi \in H^{m+2,2}(\Omega)$, then we have

$$u \in H^{m+2,2}(\Omega)$$

and

$$\|u\|_{m+2,2,\Omega} \leq c(\|\phi\|_{m+2,2,\Omega} + \|f\|_{m,2,\Omega} + \|u\|_{1,2,\Omega}),$$

if $\phi \in H^{m+2,2}(\Omega)$. If $\phi \in C^{m,1}(\partial\Omega)$, then the constant also depends on $|\phi|_{m,1,\partial\Omega}$.

Proof. Exercise. □

1.6 Eigenvalueproblems for the Laplacian

In this section we want to solve the eigenvalue problems

$$\begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega \\ u_{|\partial\Omega} &= 0, \end{aligned} \tag{1.15}$$

$$\begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \end{aligned} \tag{1.16}$$

and

$$-\Delta u = \lambda u \text{ in } M, \tag{1.17}$$

where M is a compact Riemannian manifold.
We will reduce each of these problems to an abstract eigenvalue problem in a suitable Hilbert space.

1.6.1 Assumptions of this section. In this section we use the following assumptions:
(1) H is a real, separable Hilbert space.
(2) K is a symmetric, continuous and compact bilinear form on H, such that

$$\forall u \neq 0 \colon K(u) = K(u, u) > 0.$$

(3) B is a symmetric, continuous bilinear form on H, which is coercive relative K, i.e.

$$\exists c_0, c > 0 \; \forall u \in H \colon B(u) = B(u, u) \geq c\|u\|^2 - c_0 K(u).$$

We will solve the *abstract eigenvalue problem*

$$\exists 0 \neq u \in H, \lambda \in \mathbb{R} \; \forall v \in H \colon B(u, v) = \lambda K(u, v).$$

1.6.2 Lemma. *Let $\{0\} \neq V \subset H$ be a closed subspace. Then the variational problem*

$$B(v) \to \min, \; v \in W := V \cap \{K(v) = 1\}$$

has a solution u, which is also a solution of

$$\frac{B(v)}{K(v)} \to \min, \; 0 \neq v \in V.$$

Setting

$$\lambda = \inf_{0 \neq v \in V} \frac{B(v)}{K(v)},$$

then we have

$$\forall v \in V \colon B(u, v) = \lambda K(u, v).$$

Proof. By coercivity we see, that B is bounded below in W and that a minimal sequence u_ϵ is bounded above. Thus we suppose

$$u_\epsilon \rightharpoonup u \in V.$$

$$\Rightarrow K(u_\epsilon) \to K(u) = 1.$$

B is lower semicontinuous, because $B + c_0 K$ is an equivalent norm on H. Thus the first two claims follow. The eigenvalue problem is the first variation of

$$v \mapsto \frac{B(v)}{K(v)}.$$

□

1.6.3 Theorem. *The eigenvalue problem*

$$\forall v \in H\colon B(u_i, v) = \lambda_i K(u_i, v)$$

has countably many eigenvalues of finite multiplicity. If we write

$$\lambda_1 \le \lambda_2 \le ...,$$

we find

$$\lim_{i\to\infty} \lambda_i = \infty.$$

The eigenvectors (u_i) *are complete in* H*. They fulfill the orthogonality relations*

$$K(u_i, u_j) = \delta_{ij}$$

and

$$B(u_i, u_j) = \lambda_i K(u_i, u_j),$$

as well as the expansions

$$B(u, v) = \sum_i \lambda_i K(u_i, u) K(u_i, v)$$

and

$$K(u, v) = \sum_i K(u_i, u) K(u_i, v).$$

The pairs (λ_i, u_i) *are defined by the variational problem*

$$\lambda_i = B(u_i, u_i) = \inf\left\{\frac{B(u)}{K(u)} : 0 \neq u \in H, K(u, u_j) = 0 \ \forall 1 \le j \le i-1\right\}.$$

Proof. 1. Solve the variational problem

$$\frac{B(u)}{K(u)} \to \min,\ 0 \neq u \in H.$$

By the previous theorem there exists a solution u_1 and there holds

$$\forall v \in H\colon B(u_1, v) = \lambda_1 K(u_1, v),\ K(u_1) = 1,$$

such that λ_1 is the infimum.
2. Let $i > 1$ and let there be solutions for $1 \leq j \leq i-1$. Set

$$V_i = \langle u_1, ..., u_{i-1} \rangle$$

and let $V^\perp$ be the orthogonal complement of V relative K. Again, by the previous theorem

$$\exists u_i \in V^\perp\colon B(u_i) = \lambda_i = \inf\left\{\frac{B(u)}{K(u)} \colon u \in V^\perp\right\}$$

and

$$\forall v \in V^\perp\colon B(u_i, v) = \lambda_i K(u_i, v).$$

For $1 \leq j \leq i-1$ we have

$$B(u_j, u_i) = \lambda_j K(u_j, u_i) = 0.$$

Thus

$$\forall v \in H\colon B(u_i, v) = \lambda_i K(u_i, v),$$

since

$$H = V_i \oplus_K V_i^\perp.$$

Let $u \in H$ and set

$$u_m = \sum_{i=1}^{m} K(u, u_i) u_i \in V_{m+1}.$$

$$\Rightarrow u = u_m + (u - u_m) \in V_{m+1} \oplus V_{m+1}^\perp.$$

The u_i satisfy the orthogonality relation

$$B(u_i, u_j) = \lambda_i K(u_i, u_j) = \lambda \delta_{ij}.$$

3. Suppose now the eigenvalues were bounded. We have

$$B(u_i) = \lambda_i$$

and

$$K(u_i) = 1,$$

and thus

$$c_0 K(u_i) + B(u_i) = \lambda_i + c_0,$$

so that

$$\|u_i\| \le c.$$

$$\Rightarrow 2 = K(u_i - u_{i+1}) \to 0$$

for a subsequence, which is a contradiction. By the same reasoning the multiplicity must be finite.
4. We prove the completeness. Let $u \in H$.

$$\tilde{u}_m = \sum_{i=1}^{m} K(u, u_i) u_i \equiv \sum_{i=1}^{m} c_i u_i.$$

Set

$$v_m = u - \tilde{u}_m.$$

$$v_m \in V_{m+1}^{\perp}$$

and thus

$$\lambda_{m+1} K(v_m) \le B(v_m).$$

$$K(v_m) = K(u) - \sum_{i=1}^{m} c_i^2$$

and

$$B(v_m) = B(u) - \sum_{i=1}^{m} \lambda_i c_i^2$$

imply

$$B(v_m) \le c$$

and thus

$$K(v_m) \to 0.$$

Furthermore there holds

$$\sum_{i=1}^{\infty} \lambda_i c_i^2 < \infty.$$

Let $m < n$.

$$B(v_n - v_m) = \sum_{i=m+1}^{n} \lambda_i c_i^2 < \epsilon.$$

Thus the (v_n) form a Cauchy sequence in H and by $K(v_m) \to 0$ we find

$$v_m \to 0.$$

Thus the (u_i) are complete and

$$B(u) = \sum_{i=1}^{\infty} \lambda_i c_i^2.$$

$\square$

1.6.4 Theorem. *(Minimax principle)*
For a subspace $V \subset H$ define

$$d(V) = \inf \left\{ \frac{B(u)}{K(u)} \colon 0 \neq u \in V^{\perp} \right\}.$$

Then λ_i is characterized by

$$\lambda_i = \max\{d(V) \colon V \subset H, \dim V \leq i-1\}$$

where the maximum is attained at

$$\langle u_1, ..., u_{i-1} \rangle,$$

where the u_i are defined as in 1.6.3.

Proof. For $i \geq 2$ let

$$V_i = \langle v_1, ..., v_{i-1} \rangle.$$

For $i = 1$ the claim has already been proven. We show

$$d(V_i) \leq \lambda_i = d(\langle u_1, ..., u_{i-1} \rangle).$$

Set

$$u = \sum_{j=1}^{i} c_j u_j, \;\; c_j \in \mathbb{R}$$

and solve

$$K(u, v_j) = 0 \;\; 1 \leq j \leq i-1.$$

Let u be a solution with $K(u) = \sum_{j=1}^{i} c_j^2 = 1$.

$$d(V_i) \leq \frac{B(u)}{K(u)} = \sum_{j=1}^{i} \lambda_j c_j^2 \leq \lambda_i.$$

□

1.6.5 Example. Let $\Omega \Subset \mathbb{R}^n$ and consider (1.15). This eigenvalue problem is realized in the above setting by

$$H = H_0^{1,2}(\Omega),$$

$$B(u, v) = \int_{\Omega} D_i u D^i v$$

and

$$K(u, v) = \int_{\Omega} uv.$$

Those bilinear forms obviously satisfy the assumptions of the abstract eigenvalue problem. Furthermore we have:

1.6.6 Theorem. *The smallest eigenvalue, λ_1, has multiplicity* 1 *and a corresponding eigenfunction u_1 has a strict sign.*

Proof. Exercise. □

1.6.7 Example. Let $\Omega \Subset \mathbb{R}^n$ be open and $\partial\Omega \in C^{0,1}$, $H = H^{1,2}(\Omega)$. Consider (1.16). This eigenvalue problem is realized by setting

$$B(u, v) = \int_\Omega D_i u D^i v$$

and

$$K(u, v) = \int_\Omega uv.$$

1.6.8 Example. To solve (1.17) we define the bilinear forms as in the previous examples on the space $H = H^{1,2}(M)$.

1.6.9 Definition. Let $f \colon M \to \mathbb{R}$ be a function.
(a) f is called *measurable* on M, if f is measurable in coordinates.
(b) We say $f \in L^p(M)$, if f is measurable and

$$\int_M |f|^p < \infty.$$

(c) Let $u \in L^p(M)$ and $(\eta^i) \in C_c^\infty(M, \mathbb{R}^n)$. Define the *weak derivative* of first order of u, $(D_i u)$, to be a tensor satisfying

$$\int_M D_i u \eta^i = -\int_M u \operatorname{div} \eta.$$

(d) Let

$$H^{m,p}(M) = \left\{ u \in L^p(M) \colon \int_M \sum_{k=0}^m \left(\sum_{|\alpha|=k} |D_\alpha u D^\alpha u|^{\frac{p}{2}} \right) < \infty \right\}.$$

1.6.10 Lemma. *Let $u \in C^2(M)$. Then $-\Delta$ is the Euler-Lagrange operator of the functional*

$$J(v) = \frac{1}{2} \int_M |Dv|^2.$$

Proof.

$$\forall \eta \in C_c^\infty(M) \colon 0 = \delta J(u; \eta) = \int_M u_i \eta^i.$$

□

1.6.11 Theorem. *Let $\Omega \Subset M$, then the Sobolev embedding theorems also hold for $H^{m,p}(\Omega)$ and $H_0^{m,p}(\Omega)$.*

Proof. The case $m = 1$ is an exercise and the rest follows by induction. $\square$

1.6.12 Theorem. *Let M be compact. Then there are countably many eigenvalues λ_i of $-\Delta$,*

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ... \to \infty.$$

The eigenfunctions are complete in $L^2(M)$ as well as in $H^{1,2}(M)$. The kernel of $-\Delta$ is spanned by a nonzero constant function.

Proof. The claim follows from the above examples and by 11.8.16, Analysis II. $\square$

1.6.13 Theorem. *Let u be harmonic and homogeneous of degree k in a neighborhood of $\mathbb{S}^n$. Then $u_{|\mathbb{S}^n}$ is an eigenfunction with eigenvalue $\lambda = k(k+n-1)$ of $-\Delta_{\mathbb{S}^n}$.*
Let u be an eigenfunction with eigenvalue $\lambda = k(k+n-1)$ on $\mathbb{S}^n$ of $-\Delta_{\mathbb{S}^n}$, then we have

$$u \in C^\infty(\mathbb{S}^n).$$

In $\mathbb{R}^{n+1}$ define

$$u(x) = u\left(\frac{x}{|x|}\right)|x|^k,$$

then

$$\Delta_{\mathbb{R}^{n+1}} u = 0.$$

Proof. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface, $u \in C^2(\Omega)$ and $M \subset \Omega \subset \mathbb{R}^{n+1}$ open. Let

$$\Delta = \Delta_M \ \wedge \ \bar{\Delta} = \Delta_{\mathbb{R}^{n+1}}$$

and

$$(x^\alpha), (\xi^i)$$

coordinates for the ambient space and the hypersurface respectively. Then we have

$$\begin{aligned} u_{ij} &= u_{\alpha\beta} x_i^\alpha x_j^\beta + u_\alpha x_{ij}^\alpha \\ &= u_{\alpha\beta} x_i^\alpha x_j^\beta - h_{ij} u_\alpha \nu^\alpha. \end{aligned}$$

$$\Rightarrow \Delta u = g^{ij} u_{ij} = u_{\alpha\beta} x_i^\alpha x_j^\beta g^{ij} - H u_\alpha \nu^\alpha.$$

Choose, in a given point, coordinates such that

$$g_{ij} = \delta_{ij},$$

such that in this point we have

$$\begin{aligned} u_{\alpha\beta} x_i^\alpha x_j^\beta g^{ij} &= u_{\alpha\beta} \delta_i^\alpha \delta_j^\beta g^{ij} \\ &= \bar{g}^{\alpha\beta} u_{\alpha\beta} - u_{00} \\ &= \bar{g}^{\alpha\beta} u_{\alpha\beta} - u_{\alpha\beta} \nu^\alpha \nu^\beta. \end{aligned}$$

$$\Rightarrow \Delta u = \bar{\Delta} u - u_{\alpha\beta}\nu^\alpha\nu^\beta - H u_\alpha \nu^\alpha.$$

Set $\lambda = k(k+1-1)$. On $M = \mathbb{S}^n$ we have $H = n$. Let u be homogeneous of degree k in a neighborhood of $\mathbb{S}^n$, then

$$u(x) = |x|^k u\left(\frac{x}{|x|}\right).$$

Let (x^α) be euclidian coordinates, then

$$u_\alpha \nu^\alpha = u_\alpha x^\alpha = ku.$$

$$\Rightarrow k u_\beta x^\beta = u_{\alpha\beta} x^\alpha x^\beta + u_\beta x^\beta$$

$$\Rightarrow k(k-1)u = (k-1)u_\beta x^\beta = u_{\alpha\beta} x^\alpha x^\beta$$

$$\begin{aligned} -\Delta u &= -\bar{\Delta} u + k(k-1)u + nku \\ &= -\bar{\Delta} u + k(k+n-1)u \end{aligned}$$

□

1.7 The Harnack inequality

1.7.1 Assumptions of this section. Let $\Omega \Subset \mathbb{R}^n$ be open, $n \geq 2$. In this section we investigate the linear divergence form equation

$$Lu = -(a^{ij}u_j)_i + b^i u_i + cu = 0,$$

where

$$a^{ij}, b^i, c \in L^\infty(\Omega),$$

$$\|a^{ij}\|_\infty + \|b^i\|_\infty + \|c\|_\infty \leq M$$

and

$$\exists \lambda > 0\ \forall \xi \in \mathbb{R}^n \colon a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2.$$

1.7.2 Theorem. *Let $0 \leq u \in H^{1,2}_{loc}(\Omega)$ and $Lu \leq 0$, then for all $B_{2R}(x_0) \subset \Omega'$, $q > 1$, we have*

$$\sup_{B_R(x_0)} u \leq c\left(\frac{1}{R^n}\int_{B_{2R}(x_0)} u^q\right)^{\frac{1}{q}},$$

where $c = c(\Omega', n, q, \lambda, M)$.

Proof. In this proof we use the so called *Moser iteration technique.*
(1) Suppose first that

$$u \in L^\infty(B_{2R}(x_0)).$$

Let $p > 1$,

$$\eta \in C_c^{0,1}(B_{2R}(x_0)),\ 0 \le \eta \le 1,$$

$u_\delta = u + \delta$ and use $u_\delta^{p-1}\eta^2$ as a test function. Then

$$\begin{aligned}
(p-1)\int_\Omega |Du|^2 u_\delta^{p-2}\eta^2 &\le c\int_\Omega |Du||D\eta|u_\delta^{-1}\eta(u_\delta^p dx) \\
&\quad + c\int_\Omega |Du|u_\delta^{-1}(\eta^2 u_\delta^p dx) + c\int_\Omega u_\delta^p\eta^2 \\
&\le \frac{c\epsilon}{2}\int_\Omega |Du|^2 u_\delta^{-2}\eta^2 u_\delta^p + \frac{c}{2\epsilon}\int_\Omega |D\eta|^2 u_\delta^p \\
&\quad + \frac{c\epsilon}{2}\int_\Omega |Du|^2 u_\delta^{-2}\eta^2 u_\delta^p + \frac{c}{2\epsilon}\int_\Omega \eta^2 u_\delta^p + c\int_\Omega u_\delta^p\eta^2.
\end{aligned}$$

Setting $\epsilon = \frac{p-1}{2c}$ implies

$$(p-1)\int_\Omega |Du|^2 u^{p-2}\eta^2 \le \frac{c}{p-1}\int_\Omega (|D\eta|^2 + \eta^2)u_\delta^p. \tag{1.18}$$

Set

$$v = u_\delta^p \eta^2.$$

$$\begin{aligned}
\int_\Omega |Dv| &\le p\int_{-\infty}^{\infty} |Du||u_\delta^{p-1}|\eta^2 + 2\int_\Omega u_\delta^p|D\eta|\eta \\
&\le \epsilon(p-1)\int_\Omega |Du|^2 u_\delta^{p-2}\eta^2 + \frac{p^2}{p-1}\frac{1}{4\epsilon}\int_\Omega u_\delta^p\eta^2 \\
&\quad + 2\int_\Omega u_\delta^p|D\eta|\eta.
\end{aligned}$$

Setting $\epsilon = R$ and observing

$$H^{1,1}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega)$$

we conclude

$$\left(\int_\Omega u_\delta^{p\frac{n}{n-1}}\eta^{2\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le c\left(\frac{p^2}{p-1}+1\right)\int_\Omega (R|D\eta|^2 + \eta^2 R + \frac{1}{R}\eta^2)u_\delta^p.$$

For $r \in \mathbb{N}$ we set $p = q\kappa^r$, $\kappa = \frac{n}{n-1}$ and

$$\rho_r = R + \frac{R}{2^r}.$$

Choose
$$\eta = \begin{cases} 1, & x \in B_{\rho_{r+1}} \\ 0, & x \notin B_{\rho_r}, \end{cases}$$
such that
$$|D\eta| \leq \frac{1}{\rho_r - \rho_{r+1}} = \frac{2^{r+1}}{R}.$$
$$\Rightarrow \left(\int_{B_{\rho_{r+1}}} u_\delta^{q\kappa^{r+1}} \right)^{\frac{1}{\kappa}} \leq c8^r \frac{1}{R} \int_{B_{\rho_r}} u_\delta^{q\kappa^r}$$
$$\Rightarrow \left(\frac{1}{R^n} \int_{B_{\rho_{r+1}}} u_\delta^{q\kappa^{r+1}} \right)^{\frac{1}{\kappa}} \leq c8^r \frac{1}{R^n} \int_{B_{\rho_r}} u_\delta^{q\kappa^r}$$
$$\Rightarrow \left(\frac{1}{R^n} \int_{B_{\rho_{r+1}}} u_\delta^{q\kappa^{r+1}} \right)^{\frac{1}{\kappa^{r+1}}} \leq c^{\frac{1}{\kappa^r}} 8^{\frac{r}{\kappa^r}} \left(\frac{1}{R^n} \int_{B_{\rho_r}} u_\delta^{q\kappa^r} \right)^{\frac{1}{\kappa^r}}.$$
This inequality is of the form
$$\forall r \in \mathbb{N} \colon I_{r+1} \leq c^{\frac{1}{\kappa^r}} 8^{\frac{r}{\kappa^r}} I_r,$$
which implies
$$I_{r+1} \leq c^{\sum_{i=0}^r \frac{1}{\kappa^i}} 8^{\sum_{i=0}^r \frac{i}{\kappa^i}} I_0$$
and thus
$$\sup_{B_R} u_\delta^q \leq c \frac{1}{R^n} \int_{B_{2R}} u_\delta^q.$$
$\delta \to 0$ implies the claim.

(2) We now prove that $u \in L^\infty_{loc}(\Omega)$.
Define
$$\forall 1 \leq p < \infty \colon v = \log(u+1) \in L^p_{loc}(\Omega).$$
Let $p \geq 2$, then for $\eta \in C_c^{0,1}(\Omega)$ we have the test function
$$v^{p-1}\eta^2 \in H_0^{1,2}(\Omega).$$

$$\begin{aligned}(p-1) \int_\Omega Du \cdot Dv v^{p-2} \eta^2 \leq c \int_\Omega |Du||D\eta| v^{p-1} \eta + c \int_\Omega |Du| v^{p-1} \eta^2 \\ + c \int_\Omega v^{p-1} \eta^2 u\end{aligned}$$
As in (1.18) we obtain
$$\Rightarrow (p-1) \int_\Omega |Dv|^2 v^{p-2} \eta^2 (u+1) \leq c \int_\Omega (|D\eta|^2 + \eta^2)(v^{p-1} + v^p)(u+1).$$

$H^{1,1}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega)$ implies

$$\left(\int_\Omega (v^{p-1}\eta^2(1+u))^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le c(p-1)\int_\Omega |Dv|v^{p-2}\eta^2(1+u) + c\int_\Omega v^{p-1}|D\eta|\eta(u+1) + c\int_\Omega v^{p-1}\eta^2|Dv|(u+1). \tag{1.19}$$

Thus

$$\left(\int_\Omega (v^{p-1}\eta^2(1+u))^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le c(p-1)\int_\Omega (|D\eta|^2+\eta^2)(v^{p-2}+v^{p-1}+v^p)(u+1).$$

Note that

$$v^p(u+1) \le cv^{p-1}(u+1)^{\frac{n}{n-1}}$$

and

$$v^{p-2} \le v^{p-1}+1,$$

since $p \ge 2$. Thus

$$\left(\int_\Omega (v^{p-1}\eta^2(1+u))^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le c(p-1)\int_\Omega (|D\eta|^2+\eta^2)(1+v^{p-1})(u+1)^{\frac{n}{n-1}}.$$

Choose η as in part (1), $\rho_r = R + \frac{R}{2^r}$, $\kappa = \frac{n}{n-1}$.

$$\Rightarrow \left(\int_{B_{\rho_{r+1}}} v^{(p-1)\kappa}(1+u)^\kappa\right)^{\frac{1}{\kappa}} \le c(p-1)8^r\frac{1}{R^2}\int_{B_{\rho_r}} (1+v^{p-1})(1+u)^\kappa.$$

There holds

$$\left(R^{-n}\int_{B_{\rho_{r+1}}} (v^{(p-1)\kappa}+1)(1+u)^\kappa\right)^{\frac{1}{\kappa}} \le \left(R^{-n}\int_{B_{\rho_{r+1}}} v^{(p-1)\kappa}(1+u)^\kappa\right)^{\frac{1}{\kappa}} + \left(R^{-n}\int_{B_{\rho_{r+1}}} (1+u)^\kappa\right)^{\frac{1}{\kappa}}.$$

Then

$$\left(\frac{1}{R^n}\int_{B_{\rho_{r+1}}} (v^{(p-1)\kappa}+1)(1+u)^\kappa\right)^{\frac{1}{\kappa}} \le c(p-1)8^r\left(\frac{1}{R^{n+1}}\int_{B_{\rho_r}} (1+v^{p-1})(1+u)^\kappa\right).$$

Set $p-1=\kappa^r$. For

$$I_r = \left(\frac{1}{R^n}\int_{B_{\rho_r}} (v^{\kappa^r}+1)(1+u)^\kappa\right)^{\frac{1}{\kappa^r}}$$

we find

$$I_{r+1} \leq \left(\frac{c}{R}\right)^{\frac{1}{\kappa^r}} \kappa^{\frac{r}{\kappa^r}} 8^{\frac{r}{\kappa^r}} I_r.$$

As is part (1) we conclude, using (1.19) with $p = \frac{n}{n-1} + 1$,

$$\begin{aligned}\sup_{B_R} v &\leq c\frac{1}{R^n} \int_{B_{2R}} ((v+1)^{\frac{1}{\kappa}}(1+u))^{\kappa} \\ &\leq c\frac{1}{R^n} \int_{B_{2R}} (1+u)^p \\ &\leq c\frac{1}{R} \|u\|_{1,2,B_{2R}}.\end{aligned}$$

□

1.7.3 Theorem. *Let $0 \leq u \in H^{1,2}_{loc}(\Omega)$ and $Lu \geq 0$. Then for all $B_{2R} \subset \Omega$ and for all $q < 0$ we have*

$$\inf_{B_R} u \geq c \left(\frac{1}{R^n} \int_{B_{2R}} u^q \right)^{\frac{1}{q}},$$

where $c = c(L, q, n)$.

Proof. Let $\delta > 0$, $u_\delta = u + \delta$ and $p < 1$. Let $0 \leq \eta \in C^{0,1}_c(B_{2R})$ and multiply the inequality by

$$u_\delta^{p-1} \eta^2.$$

As in the previous theorem we conclude

$$|p-1| \int_\Omega |Du_\delta|^2 u_\delta^{p-2} \eta^2 \leq \frac{c}{|p-1|} \int_\Omega \left(\frac{|D\eta|^2}{|p-1|} + \eta^2\right) u_\delta^p.$$

As in the proof of the previous theorem we obtain, using the ϵ-trick, that

$$\left(\int_\Omega u_\delta^{p\frac{n}{n-1}} \eta^{2\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq c \left(\frac{p^2}{|p-1|} + 1\right) \int_\Omega \left(R|D\eta|^2 + \eta^2 R + \frac{1}{R}\eta^2\right) u_\delta^p. \tag{1.20}$$

Choose $q < 0$, $\kappa = \frac{n}{n-1}$, $p = q\kappa^r$, $r \in \mathbb{N}$. Using Moser iteration we obtain

$$\sup_{B_R} u_\delta^q \leq c \left(\frac{1}{R^n} \int_{B_{2R}} u_\delta^q \right)$$

and since $q < 0$ we have

$$\inf_{B_R} u_\delta \geq c \left(\frac{1}{R^n} \int_{B_{2R}} u_\delta^q \right)^{\frac{1}{q}}.$$

For $\delta \to 0$ we obtain the claim. □

1.7.4 Lemma. *Let* $0 \le u \in H^{1,2}_{loc}(\Omega)$ *and* $Lu = 0$. *Let* $B_{4R} \subset \Omega$. *Then for all* $0 < q < 1$ *with the property*

$$\forall r \in N\colon q\left(\frac{n}{n-1}\right)^r \neq 1$$

we have

$$\sup_{B_R} u \le c\left(\frac{1}{R^n}\int_{B_{4R}} u^q\right)^{\frac{1}{q}},$$

where $c = c(L, n, q)$.

Proof. Set $\kappa = \frac{n}{n-1}$. Let r_0 be minimal, such that

$$q\kappa^{r_0} > 1, \;\; \tilde{R} = 2R.$$

Let

$$\rho_r = \tilde{R} + \frac{\tilde{R}}{2^r}$$

and

$$p = q\kappa^r, \quad 0 \le r \le r_0 - 1.$$

Let η be as in the proof of 1.7.2. Using (1.20) we obtain, using $\tilde{R}$ instead of R, as well as $Lu \ge 0$,

$$I_{r+1} \le cI_r = c\left(\frac{1}{R^n}\int_{B_{\rho_r}} u^{q\kappa^r}\right)^{\frac{1}{\kappa^r}}.$$

Thus

$$\left(\frac{1}{R^n}\int_{B_{2R}} u^{q\kappa^{r_0}}\right)^{\frac{1}{q\kappa^{r_0}}} \le c\left(\frac{1}{R^n}\int_{B_{4R}} u^q\right)^{\frac{1}{q}}.$$

By 1.7.2 We obtain, using $Lu \le 0$,

$$\sup_{B_R} u \le c\left(\frac{1}{R^n}\int_{B_{4R}} u^q\right)^{\frac{1}{q}}.$$

$\square$

1.7.5 Corollary. *Let* $0 \le u \in H^{1,2}_{loc}(\Omega)$, $Lu = 0$ *and* $B_{4R} \subset \Omega$. *Then for all* $0 < q \in \mathbb{R}$ *we have*

$$\sup_{B_R} u \le c\left(\frac{1}{R^n}\int_{B_{4R}} u^q\right)^{\frac{1}{q}},$$

where $c = c(L, n, q)$.

Proof. Since the estimate holds for all $q > 1$ and for a dense subset of $\{0 < q < 1\}$, we obtain the claim using the Hölder inequality. $\square$

1.7.6 Theorem. *Let $B = B_R$ and suppose that $u \in H^{1,1}(B)$ satisfies*

$$\forall B_\rho(x_0), x_0 \in B, 0 < \rho < 2R\colon \int_{B \cap B_\rho(x_0)} |Du| \le A\rho^{n-1}. \tag{1.21}$$

Then there exists $c = c(n)$, such that

$$\forall 0 < b \le \frac{1}{cA}\colon \int_B e^{b|u-u_B|} \le c|B|,$$

where $u_B = \frac{1}{|B|}\int_B u$.

Proof. Let $u \in C_c^1(\mathbb{R}^n)$, $x, y \in B$. Without loss of generality suppose $x = 0$ and choose polar coordinates around x to obtain

$$u(x) - u(y) = -\int_0^{|y|} u_r \; dr.$$

$$\begin{aligned}
|u(x) - \frac{1}{|B|}\int_B u| &\le \frac{c}{R^n}\int_{B_{2R}(x_0)}\int_0^{|y|} |Du(r,\xi)|\chi_B dr dy \\
&\le cR^{-n}\int_{\mathbb{S}^{n-1}}\int_0^{2R} t^{n-1}\int_0^{2R} |Du(r,\xi)|\chi_B dr dt dH_{n-1} \\
&= cR^{-n}\int_{\mathbb{S}^{n-1}}\int_0^{2R} t^{n-1}\int_0^{2R} r^{n-1}\frac{|Du(r,\xi)|}{r^{n-1}}\chi_B \\
&= cR^{-n}\int_0^{2R} t^{n-1}\int_{B \cap B_{2R}(x_0)} \frac{|Du(y)|}{|x-y|^{n-1}} \\
&= c\int_B \frac{|Du(y)|}{|x-y|^{n-1}}.
\end{aligned}$$

Thus we have

$$\forall u \in H^{1,1}(B)\colon \int_B |u - u_B|^p \le c^p \int_B \left(\int_B \frac{|Du(y)|}{|x-y|^{n-1}}\right)^p. \tag{1.22}$$

We have

$$\frac{|Du(y)|}{|x-y|^{n-1}} = \frac{|Du(y)|^{\frac{1}{p}}}{|x-y|^{\frac{n-1}{p}+\frac{1}{2p}}}\frac{|Du(y)|^{\frac{1}{p'}}}{|x-y|^{\frac{n-1}{p'}-\frac{1}{2p}}}, \; p \ge 2.$$

Thus

$$\int_B |u - u_B|^p \le c^p \int_B \left(\int_B \frac{|Du|}{|x-y|^{n-1+\frac{1}{2}}}\right)\left(\int_B \frac{|Du|}{|x-y|^{n-1-\frac{1}{2(p-1)}}}\right)^{p-1} \tag{1.23}$$

Set $Du(y) = 0$ for $y \neq B$ define for $x \in B, \alpha > 0$

$$
\begin{aligned}
I_\alpha(u) &= \int_B \frac{|Du(y)|}{|x-y|^{n-1-\alpha}} \\
&= \int_{|x-y|<2R} \frac{|Du(y)|}{|x-y|^{n-1-\alpha}} \\
&= \sum_{t=0}^{\infty} \int_{\frac{R}{2^t}<|x-y|<\frac{R}{2^{t-1}}} \frac{|Du(y)|}{|x-y|^{n-1-\alpha}} \\
&\leq \sum_{t=0}^{\infty} (2^t R^{-1})^{n-1-\alpha} \int_{|x-y|<2^{1-t}R} |Du(y)| \\
&\leq A \sum_{t=0}^{\infty} (2^t R^{-1})^{n-1-\alpha} (2^{1-t}R)^{n-1} \\
&= A R^\alpha 2^{n-1} \sum_{t=0}^{\infty} 2^{-\alpha t} \\
&= A R^\alpha 2^{n-1} \frac{1}{1-2^{-\alpha}}.
\end{aligned}
$$

The last integral in (1.23) is $I_{\frac{1}{2(p-1)}}(u), p \geq 2$. There holds

$$
\forall p \geq 2\colon \frac{1}{1-2^{-\frac{1}{2(p-1)}}} \leq c_0 p,
$$

because:
Set $t = \frac{1}{p-1}$. We have $1 - 2^{-\frac{1}{2}t} = 1 - e^{-at}, a > 0$. Since

$$
\frac{1-e^{-at}}{t} \to a,
$$

we have

$$
1 - e^{-at} \geq \frac{a}{2} t,
$$

from which the claim follows. Thus

$$
I_{\frac{1}{2(p-1)}} \leq A R^\alpha 2^{n-1} c_0 p
$$

In (1.23) this reads

$$
\int_B |u - u_B|^p \leq c_p c_0^{p-1} p^{p-1} A^{p-1} R^{\frac{1}{2}} \int_B |Du(y)| \left(\int_B \frac{dx}{|x-y|^{n-1+\frac{1}{2}}} \right).
$$

$$
\forall p \geq 2\colon \int_B |u - u_B|^p \leq c_1 R^n (c c_0 A p)^p. \tag{1.24}
$$

Using the potential estimates, (1.21), (1.22) to handle the case $p = 1$ and (1.24) that

$$\begin{aligned}\int_B e^{b|u-u_B|} &= \sum_{p=0}^{\infty} \int_B \frac{b^p}{p!}|u-u_B|^p \\ &\le \sum_{p=1}^{\infty} c_1 R^n \frac{(bc c_0 Ap)^p}{p!} + |B|\end{aligned}$$

Let $bcc_0A \le \frac{\kappa}{e}$, then the series converges by the quotient criterion and

$$\forall 0 < b \le b_0 \colon \int_B e^{b|u-u_B|} \le cR^n.$$

□

1.7.7 Lemma. *Let $0 \le u \in H^{1,2}_{loc}(\Omega)$, $Lu \ge 0$ and $v = \log u$. Then there holds for all $B_{2\rho} \subset \Omega$ and $\rho < 1$, that*

$$\int_{B_\rho} |Dv| \le A\rho^{n-1}.$$

Proof. Let $\epsilon > 0$, $v_\epsilon = \log(u+\epsilon)$, $0 \le \eta \in C^{0,1}_0(B_{2\rho})$ such that

$$\eta_{|B_\rho} = 1 \ \wedge \ |D\eta| \le \frac{1}{\rho}.$$

Multiply $Lu \ge 0$ by $(u+\epsilon)^{-1}\eta^2$ and set $u_\epsilon = u + \epsilon$.

$$\begin{aligned}\int_\Omega |Du_\epsilon|^2 u_\epsilon^{-2}\eta^2 &\le c\int_\Omega |Du_\epsilon| u_\epsilon^{-1}|D\eta|\eta \\ &\quad + c\int_\Omega |Du_\epsilon|u_\epsilon^{-1}\eta^2 + c\int_\Omega \eta^2\end{aligned}$$

$$\Rightarrow \int_{B_\rho} |Dv_\epsilon|^2 \le c\int_{B_{2\rho}} \frac{1}{\rho^2} + c\int_{B_{2\rho}} 1 \le c\rho^{n-2}.$$

$$\Rightarrow \int_{B_\rho} |Dv_\epsilon| \le c\left(\int_{B_\rho} |Dv_\epsilon|^2\right)^{\frac{1}{2}} \rho^{\frac{n}{2}} \le c\rho^{n-1}.$$

For $\epsilon \to 0$ we obtain the claim. □

1.7.8 Lemma. *Let $0 \le u \in H^{1,2}_{loc}(\Omega)$ and $Lu \ge 0$. Then there exist $\alpha, c > 0$ such that for all $B_{2\rho} \subset \Omega$ and $\rho < 1$*

$$\left(\frac{1}{\rho^n}\int_{B_\rho} |u|^\alpha\right)^{\frac{1}{\alpha}} \le c\left(\frac{1}{\rho^n}\int_{B_\rho} |u|^{-\alpha}\right)^{-\frac{1}{\alpha}}.$$

Proof. Set $v = \log u$. Then by 1.7.6 and 1.7.7 we have

$$\int_{B_\rho} e^{b|v - v_B|} \leq c\rho^n$$

for small b. Thus

$$\int_{B_\rho} e^{b(v-v_B)} \leq c\rho^n \ \wedge \ \int_{B_\rho} e^{-b(v-v_B)} \leq c\rho^n.$$

Multiplying those inequalities we obtain

$$\frac{1}{\rho^n}\int_{B_\rho} u^b \leq c\left(\frac{1}{\rho^n}\int_{B_\rho} u^{-b}\right)^{-1}.$$

□

1.7.9 Theorem. *(Weak Harnack inequality)*
Let $0 \leq u \in H^{1,2}_{loc}(\Omega)$ and $Lu \geq 0$. Let $B_{4\rho} \subset \Omega$, $\rho < 1$. Then there is $p > 0$, such that

$$\left(\frac{1}{\rho^n}\int_{B_\rho} u^p\right)^{\frac{1}{p}} \leq c \inf_{B_\rho} u,$$

where $c = c(n, L, p)$.

Furthermore there holds

1.7.10 Theorem. *Let $0 \leq u \in H^{1,2}_{loc}(\Omega)$, $Lu = 0$. Then for all $B_{4\rho} \subset \Omega$ and $0 < \rho < 1$, there holds*

$$\sup_{B_\rho} u \leq c \inf_{B_\rho} u, \quad c = c(n, L)$$

and for connected $\Omega' \Subset \Omega$ we have

$$\sup_{\Omega'} u \leq c \inf_{\Omega'} u, \quad c = c(n, L, \Omega').$$

1.7.11 Theorem. *Let $0 \leq u \in H^{1,2}_{loc}(\Omega)$, $Lu \geq 0$ and Ω connected. If for $B \subset \Omega$ we have $\inf_B u = 0$, then $u \equiv 0$ in Ω.*

Proof. Follows immediately from the previous theorems. □

1.7.12 Theorem. *(Strong maximum principle)*
Let Ω be connected and $u \in H^{1,2}_{loc}(\Omega)$, $Lu \leq 0$ and $c \geq 0$. If for a ball $B \subset \Omega$ we have $\sup_B u = \sup_\Omega u \geq 0$, then $u \equiv const.$

Proof. Set $M = \sup_\Omega u \geq 0$, $v = M - u \geq 0$. Then $Lv \geq 0$, since $c \geq 0$. Then the previous theorem implies the claim. □

Chapter 2

Hölder continuity of weak solutions

2.1 Solution of the homogeneous equation

2.1.1 Lemma. *Let $\omega \in L^\infty_{loc}(0, \rho_0)$ suffice*

$$\omega(\rho) \leq a\omega(4\rho) + k\rho^\alpha$$

for $0 < 4\rho < \rho_0 < 1$, $0 < a < 1$, $k \geq 0$, $0 < \alpha < 1$. Then we have

$$\forall 0 < R < \rho_0 \ \exists c > 0 \ \exists 0 < \lambda \leq \alpha \ \forall 0 \leq \rho \leq R \colon \omega(\rho) \leq c\rho^\lambda, \ \ \lambda = \lambda(a, \alpha).$$

Proof. Choose $0 < \beta < 1$ and a_0, such that $a4^\beta = a_0 < 1$. Set $\lambda = \min(\alpha, \beta)$. Let $\frac{R}{4} \leq \rho < R$ and

$$M = \sup_{\frac{R}{4} \leq \rho < R} \frac{\omega(\rho)}{\rho^\lambda}.$$

Then

$$\forall \frac{R}{4} \leq \rho < R \colon \omega(\rho) \leq M\rho^\lambda.$$

Let $\frac{R}{4^2} \leq \rho < \frac{R}{4}$. Then

$$\begin{aligned}
\Rightarrow \omega(\rho) &\leq a\omega(4\rho) + k\rho^\lambda \\
&\leq aM(4\rho)^\lambda + k\rho^\lambda \\
&= (aM4^\lambda + k)\rho^\lambda
\end{aligned}$$

By induction we then have in $\frac{R}{4^{i+1}} \leq \rho < \frac{R}{4^i}$

$$\omega(\rho) \leq (M(4^\lambda a)^i + k\sum_{j=0}^{i} 4^{j\lambda}a^j)\rho^\lambda,$$

since it holds for $i = 0$ and if it holds for $i-1$, then for $\frac{R}{4^{i+1}} \leq \rho < \frac{R}{4^i}$ we have

$$\begin{aligned}
\omega(\rho) &\leq a\omega(4\rho) + k\rho^\lambda \\
&\leq a(M(4^\lambda a)^{i-1} + k\sum_{j=0}^{i-1} 4^{j\lambda} a^j)(4\rho)^\lambda + k\rho^\lambda
\end{aligned}$$

$$\begin{aligned}
\omega(\rho) &\leq (M(4^\lambda a)^i + k\sum_{j=0}^{i} 4^{j\lambda} a^j)\rho^\lambda \\
&\leq (M + k\sum_{j=0}^{\infty} a_0^j)\rho^\lambda \\
&= (M + k\frac{1}{1-a_0})\rho^\lambda \quad \forall 0 < \rho < R,
\end{aligned}$$

since every ρ lies in a $\frac{R}{4^i} \leq \rho < \frac{R}{4^{i-1}}$. □

2.1.2 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open, $n \geq 2$, and let $u \in H^{1,2}_{loc}(\Omega)$ be a solution of the equation*

$$Lu = -(a^{ij}u_j)_i + b^i u_i = 0,$$

where $a^{ij}, b^i \in L^\infty(\Omega)$ and a^{ij} is uniformly elliptic. Then $u \in C^{0,\alpha}(\Omega)$, $\alpha = \alpha(n, L)$.

Proof. By the previous lemma it suffices to derive an estimate for the oscillation of u,

$$\omega_\rho = \sup_{B_\rho} u - \inf_{B_\rho} u,$$

for every ball $B_{4\rho}(x) \subset \Omega$, such that

$$\forall x \in \Omega\ \exists 0 \leq a < 1\ \forall \rho \leq 1\colon \omega_\rho \leq a\omega_{4\rho}.$$

So let $B_{4\rho} \Subset \Omega$ and define

$$m(\rho) = \inf_{B_\rho} u \ \wedge \ M(\rho) = \sup_{B_\rho} u.$$

$$\Rightarrow v = M(4\rho) - u \geq 0 \text{ in } B_{4\rho}.$$

Thus v is a nonnegative solution and by the Harnack inequality we obtain

$$\sup_{B_\rho} v = M(4\rho) - m(\rho) \leq c \inf_{B_\rho} v = c(M(4\rho) - M(\rho)).$$

Similarly for $w = u - m(4\rho) \geq 0$ we find

$$\sup_{B_\rho} w = M(\rho) - m(4\rho) \leq c(m(\rho) - m(4\rho))$$

and thus

$$\omega_{4\rho} + \omega_\rho \leq c(\omega_{4\rho} - \omega_\rho)$$
$$\Rightarrow \omega_\rho \leq \frac{c-1}{c+1}\omega_{4\rho}.$$

□

2.2 Local Hoelder continuity

2.2.1 Assumptions of this section. Let $\Omega \Subset \mathbb{R}^n$ be open, $n \geq 2$. We consider solutions $u \in H^{1,2}_{loc}(\Omega)$ of

$$Lu = -(a^{ij}u_j)_i + b^i u_i + cu = -(f^i)_i,$$

where

$$a^{ij}, b^i, c \in L^\infty(\Omega),$$
$$(f^i) \in L^p(\Omega, \mathbb{R}^n),\ n < p < \infty$$

and a^{ij} is uniformly elliptic. Furthermore we define the operator

$$\tilde{L} = L - c.$$

2.2.2 Theorem. *Let $u \in H^{1,2}_{loc}(\Omega)$ be a solution of $Lu = -(f^i)_i$. Then u is locally bounded.*

Proof. Will be proven more generally in a later theorem. □

2.2.3 Lemma. *(Stampacchia)*
Let $0 \leq \phi\colon [k_0, \infty) \to \mathbb{R}$ be a nonincreasing function satisfying

$$\forall h > k \geq k_0\colon \phi(h) \leq \frac{c}{(h-k)^\alpha}\phi(k)^\beta \tag{2.1}$$

with positive constants α, β, c, then there hold
(1) $\beta > 1 \Rightarrow \phi(k_0 + d) = 0$, where $d^\alpha = c\phi(k_0)^{\beta-1} \cdot 2^{\frac{\alpha\beta}{\beta-1}}$,
(2) $\beta = 1 \Rightarrow \forall h > k_0\colon \phi(h) \leq e\phi(k_0)e^{-a(h-k_0)}$, where $a = (ec)^{-\frac{1}{\alpha}}$ and
(3) $\beta < 1 \wedge k_0 \geq 0 \Rightarrow \phi(h) \leq 2^{\frac{\mu}{1-\beta}}(c^{\frac{1}{1-\beta}} + (2k_0)^\mu\phi(k_0))h^{-\mu}$, where $\mu = \frac{\alpha}{1-\beta}$.

Proof. (1) Consider the sequence

$$k_i = k_0 + d - d2^{-i},\ i \in \mathbb{N}.$$

By (2.1) we obtain

$$\phi(k_{i+1}) \leq \frac{c2^{\alpha(i+1)}}{d^\alpha}\phi(k_i)^\beta \tag{2.2}$$

$$\Rightarrow \phi(k_i) \leq \frac{\phi(k_0)}{2^{i\mu}},\ \ \mu = \frac{\alpha}{\beta-1}, \tag{2.3}$$

since it holds for $i = 0$ and

$$\phi(k_{i+1}) \leq \frac{c2^{\alpha(i+1)}}{d^\alpha}\frac{\phi(k_0)^\beta}{2^{i\mu\beta}} = \frac{\phi(k_0)}{2^{(i+1)\mu}}.$$

(2) Consider

$$k_i = k_0 + i(ec)^{\frac{1}{\alpha}}.$$

By (2.1) we have

$$\phi(k_i) \leq \frac{1}{e}\phi(k_{i-1}).$$

Let $h > k_0$. Then there exists an $i \in \mathbb{N}$, such that

$$k_0 + (i-1)(ec)^{\frac{1}{\alpha}} \leq h \leq k_0 + i(ec)^{\frac{1}{\alpha}}.$$

$\phi(h) \leq \phi(k_0 + (i-1)(ec)^{\frac{1}{\alpha}}) \leq e^{-(i-1)}\phi(k_0) \leq ee^{-a(h-k_0)}\phi(k_0),\ a = (ec)^{-\frac{1}{\alpha}}.$

(3) Let

$$\psi(h) = \phi(h)\frac{h^\mu}{c^{\frac{1}{1-\beta}}}.$$

By (2.1) we have for all $h > k \geq k_0 \geq 0$

$$\begin{aligned}\psi(h) &\leq \frac{h^\mu}{c^{\frac{1}{1-\beta}}}\frac{c}{(h-k)^\alpha}\frac{c^{\frac{\beta}{1-\beta}}}{k^{\mu\beta}}\psi(k)^\beta \\ &= \psi(k)^\beta\left(\frac{h}{(h-k)^{1-\beta}k^\beta}\right)^\mu.\end{aligned}$$

$h := 2k$ implies

$$\psi(2k) \leq 2^\mu\psi(k)^\beta \tag{2.4}$$

$$\psi(2^ik) \leq \psi(k)^{\beta^i}2^{\mu\sum_{j=0}^{i-1}\beta^j}, \tag{2.5}$$

since it holds for $i = 1$ and we have

$$\begin{aligned}\psi(2^{i+1}) &\leq 2^\mu\psi(2^ik)^\beta \\ &\leq \psi(k)^{\beta^{i+1}}2^{\mu\sum_{j=0}^{i-1}\beta^{j+1}} \cdot 2^\mu \\ &= \psi(k)^{\beta^{i+1}}2^{\mu\sum_{j=0}^{i}\beta^j}.\end{aligned}$$

$\beta < 1$ implies

$$\sup_{k_0\leq k\leq 2k_0}\psi(k)^{\beta^i} \leq 1 + \sup_{k_0\leq k\leq 2k_0}\psi(k) \leq 1 + \phi(k_0)(2k_0)^\mu c^{-\frac{1}{1-\beta}}$$

$$\Rightarrow \psi(2^i k) \leq (1 + \phi(k_0)(2k_0)^\mu c^{-\frac{1}{1-\beta}})2^{\mu \frac{1}{1-\beta}}. \tag{2.6}$$

Each number $h \geq 2k_0$ is of the form $h = 2^i k$, $k \in [k_0, 2k_0]$. Thus by (2.6) we have

$$\sup_{h \geq k_0} \psi(h) \leq (1 + \phi(k_0)(2k_0)^\mu c^{-\frac{1}{1-\beta}})2^{\mu \frac{1}{1-\beta}}$$

$$\Rightarrow \phi(h) \leq 2^{\frac{\mu}{1-\beta}}(c^{\frac{1}{1-\beta}} + \phi(k_0)(2k_0)^\mu)h^{-\mu}.$$

□

2.2.4 Theorem. *Let $u \in H^{1,2}(\Omega)$ be a solution of*

$$\tilde{L}u = -(f^i)_i,$$

then there holds
(1) If $b^i = 0$ or $|\Omega| \leq \epsilon_0 = \epsilon_0(n, L) << 1$, then

$$|u| \leq \sup_{\partial\Omega} |u| + c\|f\|_p |\Omega|^{\frac{1}{n} - \frac{1}{p}}.$$

(2) Otherwise there holds

$$|u| \leq c_0 + c\|f\|_p |\Omega|^{\frac{1}{n} - \frac{1}{p}},$$

where $c_0 = c_0(n, \sup_{\partial\Omega} |u|, \|u\|_1)$.

Proof. Let $k \in \mathbb{R}$ and set

$$A(k) = \{u > k\}.$$

Let $k_0 = \max(\sup_{\partial\Omega} u, 0)$. For $k \geq k_0$ define

$$\eta = \max(u - k, 0) \in H_0^{1,2}(\Omega)$$

as test function. Then

$$\int_\Omega |D\eta|^2 \leq c \int_\Omega |f||D\eta| + c \int_\Omega |D\eta|\eta$$

$$\Rightarrow \int_\Omega |D\eta|^2 \leq c \int_{A(k)} |f|^2 + c \int_{A(k)} |\eta|^2$$

$n \geq 3 \Rightarrow 2^* = \frac{2n}{n-2}$.

$$\int_{A(k)} |\eta|^2 \leq c \int_\Omega |D\eta|^2 |A(k)|^{\frac{2}{n}}.$$

For $n = 2$ we have

$$\int_{A(k)} |\eta|^2 \leq c \left(\int_\Omega |D\eta| \right)^2 \leq c \int_\Omega |D\eta|^2 |A(k)|.$$

For small $|\Omega|$ we find

$$\forall k \geq k_0 \colon \int_\Omega |D\eta|^2 \leq c \int_{A(k)} |f|^2.$$

If Ω is arbitrary, k_0 has to be chosen large enough, depending on $\|u\|_1$ and n, since

$$|A(k)| = \int_{A(k)} 1 \leq \int_{A(k)} \frac{u}{k} \leq k^{-1} \int_\Omega |u|.$$

Then

$$\forall k \geq k_0 \colon \int_\Omega |D\eta|^2 \leq c \int_{A(k)} |f|^2 \leq c\|f\|_p^2 |A(k)|^{1-\frac{2}{p}}.$$

$$\left(\int_\Omega \eta^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq c \int_\Omega |D\eta| \leq c \left(\int_\Omega |D\eta|^2\right)^{\frac{1}{2}} |A(k)|^{\frac{1}{2}}$$

$$\Rightarrow \left(\int_\Omega \eta^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq c|A(k)|^{1-\frac{1}{p}} \|f\|_p.$$

$$\int_\Omega \eta \leq \left(\int_\Omega \eta^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} |A(k)|^{\frac{1}{n}} \leq c|A(k)|^{1+\frac{1}{n}-\frac{1}{p}} \|f\|_p.$$

Now for all $h > k \geq k_0$ we have

$$(h-k)|A(h)| \leq \int_{A(h)} (u-k) \leq \int_\Omega \eta \leq c\|f\|_p |A(k)|^{1+\frac{1}{n}-\frac{1}{p}}.$$

For $\beta = 1 + \frac{1}{n} - \frac{1}{p} > 1$ we have by 2.2.3

$$|A(k_0 + d)| = 0,$$

where $d = c\|f\|_p |A(k_0)|^{\frac{1}{n}-\frac{1}{p}}$.

$$\Rightarrow u \leq k_0 + d.$$

Analogously this holds for $-u$, which implies the claim. □

2.2.5 Theorem. *Let $u \in H^{1,2}_{loc}(\Omega)$ be a solution of*

$$Lu = -(f^i)_i, \;\; f^i \in L^p_{loc}(\Omega), \;\; p > n,$$

then $u \in C^{0,\alpha}(\Omega)$.

Proof. If $u \in L^\infty_{loc}(\Omega)$, we may consider

$$\tilde{L}u = -cu - f^i_i \equiv g - f^i_i.$$

Let $\Omega \equiv \Omega' \Subset \Omega$, then we claim:

$$\exists w \in C^{0,1}(\mathbb{R}^n) \colon -\Delta w = g = -(\delta^{ij} w_j)_i.$$

Proof: Extend g identically 0 to $\mathbb{R}^n$ and call the mollification g_ϵ. Set

$$\omega_\epsilon = \gamma * g_\epsilon,$$

where γ is the Newtonian potential. Then we have

$$-\Delta\omega_\epsilon = g_\epsilon$$

and

$$|D\omega_\epsilon| \leq \text{const.}$$

As $\epsilon \to 0$ we obtain a limit

$$\omega_\epsilon \to \omega \in C^{0,1}(\bar{\Omega}):\ -\Delta\omega = g.$$

Thus without loss of generality we may assume $g = 0$.
Now let $B_{4\rho} \Subset \Omega$ and ρ so small that $\tilde{L}$ coercitive in $H_0^{1,2}(B_{4\rho})$, i.e.

$$\forall u \in H_0^{1,2}(B_{4\rho}) \colon \langle \tilde{L}u, u\rangle \geq c\|u\|_{1,2}^2.$$

Then solve

$$\begin{aligned} \tilde{L}w &= -(f^i)_i \text{ in } B_{4\rho} \equiv B \\ w_{|\partial B_{4\rho}} &= 0. \end{aligned}$$

Therefore define

$$a(u,v) = \langle \tilde{L}u, v\rangle$$

and

$$\phi \in H_0^{1,2}(B)^*$$

by

$$v \mapsto \langle -f_i^i, v\rangle = \int_\Omega f^i v_i.$$

Then a induces a linear operator $A \in L(H_0^{1,2}(B))$. Thus the above equation reduces to

$$\langle Aw, v\rangle = \langle \phi, v\rangle \quad \forall v \in H_0^{1,2}(B).$$

There exists a solution by Exercises 13. Thus by the previous theorem we obtain for such a solution

$$|w| \leq c\|f\|_p \rho^{1-\frac{n}{p}}.$$

Set $v = u - w$, then

$$\tilde{L}v = 0 \text{ in } B_{4\rho}.$$

Let $\omega_v = \operatorname{osc}(v)$, then as in the proof of theorem 2.1.1 we obtain

$$\omega_v(\rho) \leq a\omega_v(4\rho), \quad 0 < a < 1.$$

$$\omega_u(\rho) \leq \omega_v(\rho) + \omega_w(\rho)$$

and

$$\omega_v(4\rho) \leq \omega_u(4\rho) + \omega_w(4\rho)$$

$$\Rightarrow \omega_u(\rho) \leq a\omega_u(4\rho) + c\|f\|_p \rho^{1-\frac{n}{p}}$$

$$\Rightarrow \exists 0 < \alpha \leq 1 - \frac{n}{p} \;\forall 0 < \tilde{\rho} \leq 2\rho \colon \omega_u \leq c\tilde{\rho}^\alpha.$$

□

2.2.6 Lemma. *(Stampacchia)*
Let $0 \leq \phi(k, \rho)$ *be a real function,* $k > k_0$, $0 < \rho < R_0$ *such that*
(1) $\phi(\cdot, \rho)$ *is monotonely decreasing,*
(2) $\phi(k, \cdot)$ *is monotonely increasing and*
(3) $\forall k_0 < k < h \;\forall 0 < \rho < R < R_0 \colon \phi(h, \rho) \leq \frac{c}{(h-k)^\alpha} \frac{1}{(R-\rho)^\gamma} |\phi(k, R)|^\beta$,
$c, \alpha, \beta, \gamma > 0, \beta > 1$.
Then there holds

$$\forall 0 < \sigma < 1 \colon \phi(k_0 + d, R_0(1 - \sigma)) = 0$$

with

$$d^\alpha = 2^{(\alpha+\gamma)\frac{\beta}{\beta-1}} c \frac{|\phi(k_0, R_0)|^{\beta-1}}{\sigma^\gamma R_0^\gamma}.$$

Proof. Consider $k_i = k_0 + d - \frac{d}{2^i}$, $\rho_i = R_0 - \sigma R_0 + \frac{\sigma R_0}{2^i}$. Then there holds

$$\phi(k_i, \rho_i) \leq \frac{\phi(k_0, R_0)}{2^{\mu i}}, \;\mu = \frac{\alpha + \gamma}{\beta - 1},$$

since it clearly holds for $i = 0$ and

$$\begin{aligned} \phi(k_{i+1}, \rho_{i+1}) &\leq c\phi(k_0, R_0)^\beta 2^{-\mu\beta i} d^{-\alpha} \cdot 2^{(i+1)(\alpha+\gamma)} \sigma^{-\gamma} R_0^{-\gamma} \\ &= \frac{\phi(k_0, R_0)}{2^{\mu(i+1)}}. \end{aligned}$$

□

2.2.7 Theorem. *Let* $\Omega \Subset \mathbb{R}^n$ *be open and* $\partial\Omega \in C^{0,1}$. *Let* $u \in H^{1,2}(\Omega)$ *be a weak solution of*

$$Lu = -f_i^i, \; f^i \in L^p(\Omega), \; p > n.$$

Then there hold
(1) $\sup_{B_\rho} |u| \leq c(\|u\|_{2,B_{2\rho}}, \|f\|_p, n, p, L) \;\forall B_{2\rho} \Subset \Omega$ *and*
(2) Let $x_0 \in \partial\Omega$, $\Omega_\rho(x_0) = \Omega \cap B_\rho(x_0)$, $\Gamma_\rho = \partial\Omega \cap B_\rho(x_0)$ *and suppose* $\sup_{\Gamma_{2\rho}} |u| \leq \gamma < \infty$, *then there holds*

$$\sup_{\Omega_\rho} |u| \leq c(\gamma, \|f\|_p, \|u\|_{2,\Omega_{2\rho}}, L, p, n).$$

Proof. We only prove part (2), since the first part works identically. Let $0 < \rho_1 < \rho_2 < 2\rho < 1$ and

$$0 \leq \eta \in C_0^{0,1}(B_{\rho_2}(x_0)),$$

such that

$$\eta_{|B_{\rho_1}} = 1 \ \wedge \ |D\eta| \leq \frac{1}{\rho_2 - \rho_1}.$$

Furthermore let $k_0 \geq \max(\gamma, 1)$, $v = \log u$ on $\{u > 0\}$ and

$$v_k = \max(v - k, 0), k \geq k_0.$$

Thus, if $v_k > 0$, it follows $u > 1$. Multiply the equation by

$$v_k \eta^2 \in H_0^{1,2}.$$

Then, using the ϵ-trick,

$$\begin{aligned}\int_\Omega |Dv_k|^2 \eta^2 u \leq c \int_\Omega |b||Dv_k| v_k u \eta^2 + c \int_\Omega v_k \eta^2 u \\ + c \int_\Omega |f||Dv_k|\eta^2 + c \int_\Omega |f| v_k |D\eta|\eta \\ + c \int_\Omega |v_k|^2 |D\eta|^2 u.\end{aligned}$$

Define

$$A(k, \eta) = \{v_k \eta^2 > 0\}, A(k, \rho) = \{v > k\} \cap B_\rho(x_0)$$

as well as the measure

$$|A(k, \eta)| = \int_{A(k,\eta)} u.$$

$$\begin{aligned}\int_\Omega |Dv_k|^2 \eta^2 u \leq c \int_\Omega v_k^2 (\eta^2 + |D\eta|^2) u \\ + c \int_{A(k,\eta)} |f|^2 u^{-1} |\eta|^2 + c \int_\Omega v_k \eta^2 u.\end{aligned}$$

$$\begin{aligned}\left(\int_\Omega |Dv_k|^2 \eta^2 u\right)^{\frac{1}{2}} \leq \frac{c}{\rho_2 - \rho_1} \Bigg(\left(\int_{A(k,\eta)} v_k^p u\right)^{\frac{1}{p}} |A(k, \eta)|^{\frac{1}{2} - \frac{1}{p}} \\ + \left(\int_{A(k,\eta)} |f|^p\right)^{\frac{1}{p}} |A(k, \eta)|^{\frac{1}{2} - \frac{1}{p}} \\ + \left(\int_\Omega v_k^r u\right)^{\frac{1}{2r}} |A(k, \eta)|^{\frac{1}{2} - \frac{1}{2r}} \Bigg)\end{aligned}$$

Setting $\kappa = \frac{n}{n-1}$ and applying the Sobolev embeddings we obtain

$$\begin{aligned}\left(\int_\Omega (v_k\eta^2 u)^\kappa\right)^{\frac{1}{\kappa}} &\le c\int_\Omega (|Dv_k|\eta^2 u + v_k|D\eta|\eta u + uv_k\eta^2|Dv_k|)\\ &\le c\left(\int_\Omega |Dv_k|^2\eta^2 u\right)^{\frac{1}{2}}|A(k,\eta)|^{\frac{1}{2}}\\ &+ c\frac{1}{\rho_2-\rho_1}\left(\int_\Omega v_k^r u\right)^{\frac{1}{r}}|A(k,\eta)|^{1-\frac{1}{r}}\\ &+ c\left(\int_\Omega |Dv_k|^2\eta^2 u\right)^{\frac{1}{2}}\left(\int_\Omega v_k^2\eta^2 u\right)^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\begin{aligned}\left(\int_\Omega (v_k\eta^2 u)^\kappa\right)^{\frac{1}{\kappa}} \le{}& \frac{c}{\rho_2-\rho_1}\left(\|f\|_p + \left(\int_{A(k,\eta)} v_k^p u\right)^{\frac{1}{p}}\right)|A(k,\eta)|^{1-\frac{1}{p}}\\ &+ \frac{c}{\rho_2-\rho_1}\left(\int_{A(k,\eta)} v_k^r u\right)^{\frac{1}{r}}|A(k,\eta)|^{1-\frac{1}{r}}\\ &+ \frac{c}{\rho_2-\rho_1}\left(\|f\|_p + \left(\int_{A(k,\eta)} v_k^p u\right)^{\frac{1}{p}}\right)\left(\int_\Omega v_k^r u\eta^2\right)^{\frac{1}{r}}|A(k,\eta)|^{1-\frac{1}{r}-\frac{1}{p}}.\end{aligned} \tag{2.7}$$

Since $\forall 1 < r < \infty$ we have

$$\left(\int_{A(k,\eta)} v_k^r u\right)^{\frac{1}{r}} \le c_r\left(\int_{A(k,\eta)} |u|^2\right)^{\frac{1}{2}},$$

it follows

$$\left(\int_\Omega (v_k\eta^2 u)^\kappa\right)^{\frac{1}{\kappa}} \le \frac{\tilde{c}}{\rho_2-\rho_1}|A(k,\eta)|^{1-\frac{1}{r}-\frac{1}{p}}.$$

$$\Rightarrow (h-k)|A(k,\eta)| \le \int_\Omega v_k\eta^2 u \le \frac{\tilde{c}}{\rho_2-\rho_1}|A(k,\eta)|^{1+\frac{1}{n}-\frac{1}{r}-\frac{1}{p}}.$$

Choose r such that $\frac{1}{r} < \frac{1}{n} - \frac{1}{p}$ and set $\beta = 1 + \frac{1}{n} - \frac{1}{r} - \frac{1}{p} > 1$. Then for $h > k > k_0$ we find

$$|A(h,\rho_1)| \le \frac{\tilde{c}}{\rho_2-\rho_1}\frac{1}{h-k}|A(k,\rho_2)|^\beta \quad \forall 0 < \rho_1 < \rho_2 < 1.$$

Then by 2.2.6 we obtain

$$|A(k_0+d,\rho)| = 0$$

with

$$d = 4^{\frac{\beta}{\beta-1}}\tilde{c}\frac{|A(k_0, 2\rho)|^{\beta-1}}{\rho}$$

$$\Rightarrow \sup_{B_\rho} u \le k_0 + d.$$

The same for $-u$ implies the claim. $\square$

2.3 Hoelder estimates near the boundary

2.3.1 Assumptions of this section. Let $\Omega \Subset \mathbb{R}^n$ be open, $n \ge 2$. We consider solutions $u \in H^{1,2}(\Omega)$ of

$$Lu = -(a^{ij}u_j)_i + b^i u_i + cu = -(f^i)_i,$$

$$u_{|\partial\Omega} = \phi,$$

where

$$a^{ij}, b^i, c \in L^\infty(\Omega),$$

$$(f^i) \in L^p(\Omega, \mathbb{R}^n),\ n < p < \infty,$$

$\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \le 1$ and a^{ij} is uniformly elliptic. Furthermore we define the operator

$$\tilde{L} = L - c.$$

2.3.2 Definition. We say, $\partial\Omega$ satisfies an *exterior cone condition*, $\partial\Omega \in (K)$, if for each $x_0 \in \partial\Omega$ there is a cone with uniform angle starting in x_0, such that for a uniform $\rho > 0$ we have

$$K_\rho(x_0) = K \cap B_\rho(x_0) \subset \Omega^c.$$

2.3.3 Example. $\partial\Omega \in C^{0,1} \Rightarrow \partial\Omega \in (K)$.

2.3.4 Remark. $\partial\Omega \in (K) \Rightarrow \exists \epsilon_0 > 0\ \forall x_0 \in \partial\Omega\colon \frac{|B_\rho(x_0)\backslash\Omega|}{\rho^n} \ge \epsilon_0$.

2.3.5 Theorem. *Let* $0 \le u$, $Lu \ge 0$, $x_0 \in \partial\Omega$ *and* $R > 0$. *Set*

$$m = \inf\{u(x)\colon x \in \partial\Omega \cap B_{4R}(x_0)\}$$

and

$$\bar{u} = \begin{cases} \min(u, m), & x \in \Omega \cap B_{4R} \\ m, & x \in B_{4R}\backslash\Omega. \end{cases}$$

Then there holds for all $p < 0$

$$\left(\frac{1}{R^n}\int_{B_{2R}} \bar{u}^p\right)^{\frac{1}{p}} \le c \inf_{B_R} \bar{u},$$

$c = c(n, L, p)$.

Proof. Let $p < 1$, $\eta \in C_0^{0,1}(B_{2R})$, $\delta > 0$, $\bar{u}_\delta = \bar{u} + \delta$ and $m_\delta = m + \delta$. Multiply $Lu \geq 0$ by the test function

$$(\bar{u}_\delta^{p-1} - m_\delta^{p-1})\eta^2 \in H_0^{1,2}(\Omega).$$

As in the proof of 1.7.3 we obtain

$$|p-1| \int_\Omega |D\bar{u}_\delta|^2 \bar{u}_\delta^{p-2} \eta^2 \leq c \int_\Omega \left(\frac{|D\eta|^2}{|p-1|} + \eta^2 \right) \bar{u}_\delta^p.$$

We also may integrate outside Ω in the full ball. Using the ϵ-trick and Sobolevs embedding, $\kappa = \frac{n}{n-1}$, $R < 1$, we obtain

$$\left(\int_{B_{2R}} \bar{u}_\delta^\kappa \eta^{2\kappa} \right)^{\frac{1}{\kappa}} \leq c \left(\frac{p^2}{|p-1|} + 1 \right) \int_{B_{2R}} (R|D\eta|^2 + \eta^2 + \frac{1}{R}\eta^2) \bar{u}_\delta^p. \tag{2.8}$$

Let $q < 0$ and $p = q\kappa^r$, $r \in \mathbb{N}$. Then by iteration we obtain

$$\left(\frac{1}{R^n} \int_{B_{2R}} \bar{u}_\delta^q \right)^{\frac{1}{q}} \leq c \inf_{B_R} \bar{u}_\delta.$$

$\delta \to 0$ implies the claim. □

2.3.6 Lemma. *Under the assumptions of the preceding theorem let* $0 < q < 1$. *Then there is* $p > 1$, *such that*

$$\left(\frac{1}{R^n} \int_{B_{2R}} \bar{u}^p \right)^{\frac{1}{p}} \leq c \left(\frac{1}{R^n} \int_{B_{4R}} \bar{u}^q \right)^{\frac{1}{q}},$$

$c = c(n, L, p, q)$.

Proof. It suffices to prove the claim for almost every $0 < q < 1$. So let $0 < q < 1$ such that $q\kappa^r \neq 1$ for all $r \in \mathbb{N}$, $\kappa = \frac{n}{n-1}$. Choose r_0 minimally such that $p = q\kappa^{r_0} > 1$ then by (2.8) we obtain using iteration

$$\left(\frac{1}{R^n} \int_{B_{2R}} \bar{u}^p \right)^{\frac{1}{p}} \leq c(\frac{1}{R^n} \int_{B_{4R}} \bar{u}^q)^{\frac{1}{q}}.$$

□

2.3.7 Lemma. *Under the assumptions of the preceding lemma let* $\Omega_{4R} = \Omega \cap B_{4R}(x_0)$ *and* $v = \log(\bar{u})$. *Then*

$$\forall B_{2\rho}(y) \subset B_{4R}(x_0) \colon \int_{B_\rho} |Dv| \leq A\rho^{n-1}.$$

Proof. Let $0 \le \eta \in C_0^{0,1}(B_{2\rho})$, $\eta_{|B_\rho} = 1$, $|D\eta| \le \frac{1}{\rho}$ and $\epsilon > 0$. Let furthermore $\bar{u}_\epsilon = \bar{u} + \epsilon$ and $v_\epsilon = \log \bar{u}_\epsilon$. Using the test function

$$((\bar{u}+\epsilon)^{-1} - (m+\epsilon)^{-1})\eta^2$$

we obtain

$$\begin{aligned}\int_\Omega |D\bar{u}_\epsilon|^2 \bar{u}_\epsilon^{-2}\eta^2 &\le c\int_\Omega |D\bar{u}_\epsilon|\bar{u}_\epsilon^{-1}|D\eta|\eta \\ &\quad + c\int_\Omega |D\bar{u}_\epsilon|\bar{u}_\epsilon^{-1}\eta^2 \\ &\quad + c\int_\Omega \eta^2\end{aligned}$$

and thus

$$\int_\Omega |Dv_\epsilon|^2\eta^2 \le c\rho^{-2}\int_\Omega \eta^2 \le c\rho^{n-2}.$$

$$\Rightarrow \int_{B_\rho} |Dv_\epsilon| \le A\rho^{n-1}.$$

□

2.3.8 Lemma. *Under the assumptions of the preceding lemma there exist $\alpha > 0$ and $c > 0$ such that*

$$\left(\frac{1}{R^n}\int_{B_R} \bar{u}^\alpha\right)^{\frac{1}{\alpha}} \le c\left(\frac{1}{R^n}\int_{B_R} \bar{u}^{-\alpha}\right)^{-\frac{1}{\alpha}}.$$

Proof. As 1.7.8. □

2.3.9 Theorem. *(Weak Harnack inequality)*
Let $0 \le u$, $Lu \ge 0$ and $m, \bar{u}$ as in the preceding theorem, then there exist $p > 1$ and $c = c(n,p,L)$, such that

$$\left(\frac{1}{R^n}\int_{B_R} \bar{u}^p\right)^{\frac{1}{p}} \le c\inf_{B_R} \bar{u}.$$

Proof. (i) $\exists q > 0$ such that

$$\begin{aligned}\left(\frac{1}{R^n}\int_{B_{2R}} \bar{u}^q\right)^{\frac{1}{q}} &\le \left(\frac{1}{R^n}\int_{B_{2R}} \bar{u}^{-q}\right)^{-\frac{1}{q}} \\ &\le c\inf_{B_R} \bar{u}.\end{aligned}$$

(ii) Furthermore there is $p > 1$, such that

$$\left(\frac{1}{R^n}\int_{B_R} \bar{u}^p\right)^{\frac{1}{p}} \le c\left(\frac{1}{R^n}\int_{B_{2R}} \bar{u}^q\right)^{\frac{1}{q}}$$

□

2.3.10 Corollary. *Under the assumptions of the preceding theorem there holds*

$$\frac{1}{R^n}\int_{B_R}\bar{u} \le c\inf_{B_R}\bar{u}.$$

2.3.11 Theorem. *Let $u \in H^{1,2}(\Omega)$ be a solution of the equation*

$$\tilde{L}u = 0 \text{ in } \Omega_{R_0} = \Omega \cap B_{R_0}(x_0),$$

$x_0 \in \partial\Omega$. *Let $\Gamma = B_{R_0} \cap \partial\Omega \in C^{0,1}$ and $\phi = u_{|\Gamma} \in C^{0,\alpha}$, $0 < \alpha < 1$. Then there exists $0 < a < 1$, such that for $0 < \rho < \frac{R_0}{4}$ and for*

$$\omega(\rho) = \sup_{x,y\in\Omega_\rho(z_0)} |u(x) - u(y)|$$

and

$$\tilde{\omega}(\rho) = \sup_{\partial\Omega\cap B_\rho(z_0)} |u(x) - u(y)|$$

we have

$$\omega(\rho) \le a\omega(4\rho) + \tilde{\omega}(4\rho).$$

Proof. Let

$$M(\rho) = \sup_{\Omega_\rho} u,\ m(\rho) = \inf_{\Omega_\rho} u,$$

$$\tilde{M}(\rho) = \sup_{\partial\Omega\cap B_\rho} u,\ \tilde{m}(\rho) = \inf_{\partial\Omega\cap B_\rho} u.$$

(i) Consider $v = M(4\rho) - u \ge 0$ in $\Omega_{4\rho}$, then we have

$$\tilde{L}v = 0.$$

Thus by the preceding corollary we have

$$\frac{1}{\rho^n}\int_{B_\rho}\bar{v} \le c\inf_{B_\rho}\bar{v}$$

$$\Rightarrow \rho^{-n}\bar{v}|B_\rho\backslash\Omega| \le c\inf_{\Omega_\rho}\bar{v} \le c\inf_{\Omega_\rho} v \le c(M(4\rho) - M(\rho)).$$

Since $\partial\Omega \in (K)$ we have

$$M(4\rho) - \tilde{M}(4\rho) \le c(M(4\rho) - M(\rho)).$$

(ii) Set $v = u - m(4\rho) \ge 0$ in $\Omega_{4\rho}$. Then $\tilde{L}v = 0$. Thus we again have

$$\frac{1}{\rho^n}\int_{B_\rho}\bar{v} \le c\inf_{B_\rho}\bar{v} \le c\inf_{\Omega_\rho} v \le c(m(\rho) - m(4\rho)).$$

$$\Rightarrow \tilde{m}(4\rho) - m(4\rho) \le c(m(\rho) - m(4\rho)).$$

(iii) Add the two inequalities to obtain

$$\omega(4\rho) - \tilde{\omega}(4\rho) \le c(\omega(4\rho) - \omega(\rho)),\ \ c > 1$$
$$\Rightarrow \omega(\rho) \le \frac{c-1}{c}\omega(4\rho) + \frac{1}{c}\tilde{\omega}(4\rho).$$

□

2.3.12 Theorem. *Let $\partial\Omega \in C^{0,1}$ and $u \in H^{1,2}(\Omega)$ be a solution of the Dirichlet problem*

$$Lu = -f^i_i$$
$$u_{|\partial\Omega} = \phi,$$

where $f^i \in L^p(\Omega)$, $p > n$.
Let $x_0 \in \partial\Omega$, $\Gamma_{4R} = \partial\Omega \cap B_{4R}(x_0)$ and $\phi \in C^{0,\alpha}(\Gamma_{4R})$, then there holds

$$u \in C^{0,\lambda}(\Omega \cup \Gamma_R),$$

$0 < \lambda \le \min(\alpha, 1 - \frac{n}{p})$.

Proof. (i) By 2.2.7 we have $u \in L^\infty(\Omega_{2R})$. Solving

$$-\Delta w = -cu$$

we obtain

$$\tilde{L}u = -(f^i + D^i w)_i \equiv -f^i_i.$$

(ii) Having extended the data to $B_{8\rho}$, solve

$$\tilde{L}w = -f^i_i \text{ in } B_{8\rho}$$
$$w_{|\partial B_{8\rho}} = 0,$$

for such small $\rho < 1$, that $\tilde{L}$ is coercitive.

$$\Rightarrow \sup|w| \le c\rho^{1-\frac{n}{p}}\|f\|_p.$$

Setting

$$v = u - w,$$

we have

$$\tilde{L}v = 0$$

in $\Omega_{8\rho}$. Thus by the preceding theorem we have

$$\omega_v(\rho) \le a\omega_v(4\rho) + \tilde{\omega}_v(4\rho).$$

$$\Rightarrow \omega_v(\rho) \le a\omega_u(4\rho) + a\omega_w(4\rho) + \tilde{\omega}_u(4\rho) + \tilde{\omega}_w(4\rho)$$
$$\le a\omega_u(4\rho) + c\|f\|_p\rho^{1-\frac{n}{p}} + c[\phi]_\alpha\rho^\alpha$$

By the De Giorgi lemma we obtain

$$\omega_u(\rho) \le c\rho^\lambda.$$

□

2.4 Application to nonlinear equations

Consider a general elliptic PDE of second order

$$F(\cdot, u, Du, D^2u) = 0,$$

$$a^{ij} = \frac{\partial F}{\partial u_{ij}} > 0,$$

where F is uniformly elliptic in compact subsets of the domain of definition of F. If the regularity of the equations admits, we may differentiate for x_k to obtain

$$0 = a^{ij}u_{kij} + \frac{\partial F}{\partial p_j}u_{kj} + \frac{\partial F}{\partial u}u_k + \frac{\partial F}{\partial x_k},$$

which is a linear equation for $v = u_k$. If it is a priori possible to obtain C^3 estimates, we thus obtain $v \in C^{2,\alpha}$ by Schauder theory. Obtaining C^3 estimates is quite difficult in general. The results of Evans, Krylov for the elliptic case and Krylov, Safonov for the parabolic case ensure $C^{2,\alpha}$ estimates only knowing C^2 bounds and the concavity of $F(\cdot, u, Du, \cdot)$. We now turn our attention to quasilinear equations.

2.4.1 Assumptions of this section. Let $\Omega \Subset \mathbb{R}^n$ be open. We consider the quasilinear equation

$$\begin{aligned} Au = -(a^i(\cdot, u, Du))_i &= f \\ u_{|\partial\Omega} &= \phi, \end{aligned} \tag{2.9}$$

where

$$a^{ij} = \frac{\partial a^i}{\partial p_j}$$

is locally uniformly elliptic, $f \in L^p(\Omega)$, $p > n \geq 2$, $\partial\Omega \in C^2$ and $a^i \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$.

2.4.2 Theorem. *(i) Let $u \in C^{0,1}(\bar{\Omega})$ be a solution of (2.9), $\phi \in H^{2,p}(\Omega)$, $f \in L^p(\Omega)$. Then we have*

$$u \in H^{2,p}(\Omega).$$

(ii) Suppose furthermore $f \in C^{0,\alpha}(\bar{\Omega})$, $\phi \in C^{2,\alpha}(\bar{\Omega})$ and $\partial\Omega \in C^{2,\alpha}$, then we have

$$u \in C^{2,\alpha}(\bar{\Omega}).$$

For the proof we first need several things.

2.4.3 Theorem. *Under the assumptions of the preceding theorem, (i), we have*

$$u \in C^{1,\alpha}(\bar{\Omega}),$$

for some $0 < \alpha \leq 1 - \frac{n}{p}$.

Proof. (i) $u \in C^{0,1}(\bar{\Omega}) \Rightarrow a^i(\cdot, u, p_j) \equiv a^i(\cdot, p_j)$ and

$$\Lambda|\xi|^2 \geq a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \ \lambda > 0.$$

The L^2 estimates imply $u \in H^{2,2}(\Omega)$ and

$$\|u\|_{2,2} \leq c(\|f\|_2 + \|u\|_2).$$

(ii) Let $1 \leq k \leq n$, $v = u_k \in H^{1,2}(\Omega)$. Use ζ_k as test function to obtain

$$-(a^{ij}v_j)_i - \left(\frac{\partial a^i}{\partial u}v\right)_i - \left(\frac{\partial a^i}{\partial x^k}\right)_i = f_k = (\delta^i_k f)_i$$

$$\Rightarrow -(a^{ij}v_j)_i \equiv -f^i_i,$$

$f^i \in L^p(\Omega)$. By the De Giorgi-Nash results we obtain $v \in C^{0,\alpha}(\Omega)$ with corresponding a priori esimates.

(iii) Boundary estimates. By local flattening we may assume the equation reads

$$\begin{aligned} Au &= f \text{ in } \Omega = B^+_1(0) \\ u_{|\Gamma} &= \phi, \end{aligned}$$

where $\Gamma = \partial\Omega \cap \{x^n = 0\}$.

Let $1 \leq k \leq n-1$ and $v = u_k$. Then v solves the Dirichlet problem

$$\begin{aligned} -(a^{ij}v_j)_i &= -f^i_i \text{ in } \Omega \\ v_{|\Gamma} &= \phi_k \in C^{0,\beta}, \end{aligned} \tag{2.10}$$

$\beta = 1 - \frac{n}{p}$. De Giorgi-Nash implies

$$v \in C^{0,\alpha}(\bar{B^+_R}(0)),$$

$0 < R < 1$.

(iv) In order to prove

$$u_n \in C^{0,\alpha}(B^+_{\frac{1}{2}}(0)),$$

we have to prove a so-called *Morrey condition* for Du. Let v be defined as in (iii). Let $0 < R < 1$ and $\xi \in B^+_R(0)$. Choose $0 < \rho < 1$, such that $B_{2\rho}(\xi) \subset B_R(0)$ and let $\eta \in C^{0,1}_0(B_{2\rho}(\xi))$, such that $\eta_{|B_\rho} = 1$ and $|D\eta| \leq \frac{1}{\rho}$.

Distinguish two cases:

(1) $B_{2\rho}(\xi) \cap \Gamma \neq \emptyset$. Then we choose $\xi_0 \in B_{2\rho}(\xi) \cap \Gamma$ and multiply (2.10) by

$$(v - \phi_k(\xi_0) - (\phi_k - \phi_k(\xi_0)))\eta^2 = (v - \phi_k)\eta^2 \in H^{1,2}_0(\Omega), \ \Omega = B^+_1(0).$$

(2) $B_{2\rho(\xi)} \subset B^+_R(0)$, then we multiply (2.10) by

$$(v - v(\xi))\eta^2.$$

In both cases integrate by parts. We only consider case (1).

$$\Rightarrow |v - \phi_k(\xi_0)| = |v - v(\xi_0)| \le c|x - \xi_0|^\alpha \le c\rho^\alpha$$

and

$$|\phi_k - \phi_k(\xi_0)| \le c|x - \xi_0|^\beta \le c\rho^\beta \le c\rho^\alpha.$$

$$\begin{aligned}\int_\Omega a^{ij} v_i v_j \eta^2 \le \int_\Omega &a^{ij} v_j \phi_{ki} \eta^2 \\ &- 2\int_\Omega a^{ij} v_j (v - \phi_k(\xi_0) - (\phi_k - \phi_k(\xi_0)))\eta_i \eta \\ &+ \int_\Omega f^i (v_i - \phi_{ki})\eta^2 \\ &+ 2\int_\Omega f^i (v - \phi_k(\xi_0) - (\phi_k - \phi_k(\xi_0)))\eta_i \eta\end{aligned}$$

By the standard ϵ-trick we obtain

$$\begin{aligned}\int_{B_\rho(\xi)\cap\Omega} |Dv|^2 &\le c\int_{B_{2\rho(\xi)}\cap\Omega} |D^2\phi|^2 + c\rho^{-2}\int_{B_{2\rho}\cap\Omega} (|v - v(\xi_0)|^2 + |\phi_k - \phi_k(\xi_0)|^2) \\ &+ c\int_{B_{2\rho}\cap\Omega} (|f|^2 + |D^2\phi|^2) \\ &+ c\rho^{-1}\int_{B_{2\rho}\cap\Omega} |f|(|v - v(\xi_0)| + |\phi_k - \phi_k(\xi_0)|) \\ &\equiv I_1 + I_2 + I_3 + I_4\end{aligned}$$

We have

$$\begin{aligned}I_1 &\le c\|\phi\|_{2,p}^2 \rho^{n - \frac{2n}{p}}, \\ I_2 &\le c([v]_\alpha^2 + [D\phi]_\alpha^2)\rho^{n-2+2\alpha}, \\ I_3 &\le c(\|f\|_p^2 + \|D^2\phi\|_p^2)\rho^{n - \frac{2n}{p}}\end{aligned}$$

and

$$I_4 \le c\|f\|_p \rho^{n-1-\frac{n}{p}+\alpha}.$$

$$\Rightarrow \int_{B_\rho\cap\Omega} |Dv|^2 \le cL^2\rho^{n-2+2\lambda},$$

$\lambda = \min(\alpha, 1 - \frac{n}{p})$ and $L^2 = \|\phi\|_{2,p}^2 + \|f\|_p^2 + [v]_\alpha^2 + [D\phi]_\alpha^2 + \|f\|_p$. This is the Morrey condition.

Now we show, that $v = u_n$ satisfies a Morrey condition as well. We use the equation:

$$-a^{ij}u_{ij} - \frac{\partial a^i}{\partial x^i} - \frac{\partial a^i}{\partial u} u_i = f.$$

$$\Rightarrow \int_{B_\rho\cap\Omega} |u_{nn}|^2 \le cL^2\rho^{n-2+2\lambda}.$$

Using the following lemma we obtain

$$v \in C^{0,\lambda}(\bar{B}^+_{\frac{R}{4}}(0)).$$

□

2.4.4 Lemma. *(Morrey)*
Let $\Omega = B_R(0)$ or $\Omega = B^+_R(0)$ and suppose for $u \in H^{1,p}(\Omega)$ and $1 \leq p \leq n$ there holds

$$\int_{B_\rho(\xi)} |Du|^p \leq cL^p \rho^{n-p+p\lambda}, \ \lambda > 0 \tag{2.11}$$

for all $0 < \rho \leq \frac{R}{4}$ and for all $\xi \in B_{\frac{R}{4}}(0)$ or for all $\xi \in \bar{B}^+_{\frac{R}{4}}(0)$ respectively. Then

$$u \in C^{0,\lambda}(\bar{B}_{\frac{R}{4}}(0))$$

or

$$u \in C^{0,\lambda}(\bar{B}^+_{\frac{R}{4}}(0))$$

respectively and

$$[u]_\lambda \leq cL.$$

Proof. Prove only the case $\Omega = B_R(0)$. Let $u \in C^1(\Omega)$ and $x, \xi \in B_{\frac{R}{4}}(0)$. Set

$$\bar{x} = \frac{1}{2}(x+\xi), \ \rho = \frac{|x-\xi|}{2}.$$

For $y \in B_\rho(\bar{x})$ we then have

$$u(y) - u(\xi) = \int_0^1 \frac{d}{dt} u(ty + (1-t)\xi) = \int_0^1 u_i(y^i - \xi^i).$$

$$\Rightarrow |B_\rho(\bar{x})|^{-1} \int_{B_\rho(\bar{x})} |u(y) - u(\xi)| \leq 2\rho |B_\rho(\bar{x})|^{-1} \int_0^1 \int_{B_\rho(\bar{x})} |Du(ty + (1-t)\xi)|.$$

Transform

$$z = ty + (1-t)\xi, \ \bar{z} = t\bar{x} + (1-t)\xi$$

to obtain

$$\begin{aligned}
\int_0^1 \int_{B_\rho(\bar{x})} |Du(ty + (1-t)\xi)| dy dt &= \int_0^1 t^{-n} \int_{B_{t\rho}(\bar{z})} |Du(z)| dz \\
&\leq c \int_0^1 t^{-n} (t\rho)^{n\frac{p-1}{p}} (\int_{B_{t\rho}(\bar{z})} |Du|^p)^{\frac{1}{p}} \\
&\leq c \int_0^1 t^{-n} (t\rho)^{n\frac{p-1}{p}} L (t\rho)^{\frac{n}{p} - 1 + \lambda}
\end{aligned}$$

$$\Rightarrow |B_{\rho(\bar{x})}|^{-1} \int_{B_{\rho(\bar{x})}} |u(y) - u(\xi)| \leq cL\rho^\lambda$$

and analogously for $x = \xi$.

$$\Rightarrow |u(x) - u(\xi)| \le |B_\rho(\bar{x})|^{-1} \int_{B_{\rho(\bar{x})}} (|u(y) - u(x)| + |u(y) - u(\xi)|) \le cL\rho^\lambda.$$

□

Now let $\partial\Omega \in C^{2,\alpha}$, $\phi \in C^{2,\alpha}(\bar{\Omega})$, $f \in C^{0,\alpha}(\bar{\Omega})$ and $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution of the problem (2.9). If we are able to prove C^0 and C^1 estimates, then by De Giorgi-Nash we obtain $C^{0,\alpha}$ coefficients, bounded by $|u|_{1,\alpha}$. Schauder theory then yields $C^{2,\alpha}$ estimates.

We now prove that Lipschitz solutions are already classical solutions.

2.4.5 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open and let $\partial\Omega \in C^{2,\alpha}$, $a^i, a \in C^{1,\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $\phi \in C^{2,\alpha}(\bar{\Omega})$ and $u \in C^{0,1}(\bar{\Omega})$ a solution of*

$$\begin{aligned} Au + a(\cdot, u, Du) &= 0 \\ u_{|\partial\Omega} &= \phi, \end{aligned} \tag{2.12}$$

then we have

$$u \in C^{2,\beta}(\bar{\Omega}),$$

for some $0 < \beta \le \alpha$.

Proof. (i) Let $u_0 \in C^{0,1}(\bar{\Omega})$ be a solution and let

$$1 + |u_0| + |Du_0| \le M.$$

Let $\theta = \theta(t)$ be a real function

$$\theta(t) = \begin{cases} t, & |t| \le M \\ \pm(M+1), & |t| \ge M+1, \end{cases}$$

$\dot{\theta} \ge 0$.
(ii) Let w, g be real functions

$$w(t) = \begin{cases} 1, & 0 \le t \le 2M \\ 0, & t \ge 3M \end{cases}$$

and

$$g(t) = \begin{cases} 0, & 0 \le t \le M \\ ct - k, & t \ge 2M, \end{cases}$$

such that g is convex.
(iii) Set

$$\tilde{a}^i(x, t, p) = a^i(x, \theta(t), p)w(|p|^2) + kg'(|p|^2)p^i,$$

where k is large. Furthermore set

$$\tilde{a}(x,t,p) = a(x,\theta(t),p)w(|p|^2).$$

There holds

$$|\tilde{a}(x,t,p)| \le c(1+|p|)$$

and $\tilde{a}^{ij}$ is uniformly positive definite. Thus the corresponding operator

$$\tilde{A}u + \tilde{a}(\cdot,u,Du)$$

is a uniformly elliptic differential operator. If $\gamma > 0$ is chosen large enough, then

$$\Phi u := \tilde{A}u + \tilde{a}(\cdot,u,Du) + \gamma(u-u_0)$$

is coercitive i.e. for $u_1, u_2 \in H^{1,2}(\Omega)$ such that $u_1 = u_2$ on $\partial\Omega$ we have

$$\langle \Phi u_1 - \Phi u_2, u_1 - u_2\rangle \ge c\|u_1-u_2\|^2_{1,2},\ c>0.$$

Using the exercises we obtain $u \in H^{1,2}(\Omega)$, solving

$$\begin{aligned} \Phi u &= 0 \\ u_{|\partial\Omega} &= \phi. \end{aligned} \tag{2.13}$$

By L^2 estimates and De Giorgi-Nash we obtain

$$u \in C^{1,\alpha}(\bar{\Omega}) \cap H^{2,2}(\Omega).$$

There holds

$$\Phi u_0 = Au_0 + a(\cdot,u_0,Du_0).$$

Thus, if (2.13) has a $C^{2,\alpha}$ solution u, then we must have $u = u_0$.
(iv) (2.13) has a $C^{2,\beta}(\bar{\Omega})$ solution. The linearization reads

$$\begin{aligned} -\tilde{a}^{ij}u_{ij} + \hat{a}(\cdot,u,Du) + \gamma(u-u_0) &= 0 \\ u_{|\partial\Omega} &= \phi \end{aligned} \tag{2.14}$$

with $\frac{\partial\hat{a}}{\partial t} + \gamma > 0$.
First, we need an a priori estimate:
(1) By the maximum principle we obtain a C^0 estimate.
(2) Using Thm 15.2 in Gilbarg-Trudinger we obtain a C^1 estimate, also cf. Chapter 3.2.
(3) L^2 estimates yield $u \in H^{2,2}(\Omega)$ and by De Giorgi-Nash we obtain $u \in C^{1,\lambda}$, $0<\lambda<1$.
(4) Schauder theory then yields $C^{2,\alpha}$ estimates.

(v) We now employ the Leray-Schauder fix point theorems, cf. next chapter, to obtain a solution. Let $0 < \sigma < 1$ and consider

$$\begin{aligned} \tilde{A}u + \sigma \tilde{a}(\cdot, u, Du) + (1-\sigma)\frac{\partial \tilde{a}^i}{\partial x^i}(\cdot, u, Du) + \gamma(u - \sigma u_0) = 0 \\ u_{|\partial\Omega} = \sigma\phi. \end{aligned} \tag{2.15}$$

For this equation we also have to prove $C^{2,\alpha}$ bounds. Choosing γ large enough, then (2.15) is also coercitive and we obtain estimates independently of σ. Leray-Schauder then implies , that there is a solution for $\sigma = 1$. □

2.4.6 Proposition. *Let $a^i, a \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and let $u_0 \in C^{0,1}(\bar{\Omega})$ be a weak solution of*

$$\begin{aligned} Au + a(\cdot, u, Du) = 0 \\ u_{|\partial\Omega} = \phi \in H^{2,p}(\Omega),\ p > n. \end{aligned}$$

Then we have

$$u_0 \in H^{2,p}(\Omega).$$

Proof. The same proof as the one of the preceding theorem is applicable. However, we have to use the L^p-theory of Calderon-Zygmund instead of Schauder theory. □

Chapter 3

Quasilinear operators and Leray-Schauder theory

3.1 Fixed point theorems, Leray-Schauder theorem and applications

3.1.1 Theorem. *(Schauder's fixed point theorem)*
Let V be a Banach space, $K \subset V$ compact and convex and $T\colon K \to K$ continuous. Then T has a fixed point.

Proof. We use Brouwer's fixed point theorem. Let $k \in \mathbb{N}$, then there exist $(u_i)_{1\le i\le N}$, $u_i \in K$, such that

$$K \subset \bigcup_{i=1}^{N} B_{\frac{1}{k}}(u_i).$$

Set

$$B_i := B_{\frac{1}{k}}(u_i).$$

Let

$$S_k := \operatorname{conv}(u_1, ..., u_N)$$

and define

$$\begin{aligned} J_k(u)\colon K &\to S_k \\ u &\mapsto \frac{\sum_{i=1}^N \operatorname{dist}(u, K\backslash B_i)u_i}{\sum_{i=1}^N \operatorname{dist}(u, K\backslash B_i)}. \end{aligned}$$

There holds

$$\|J_k u - u\| = \frac{\sum_{i=1}^N \operatorname{dist}(u, K\backslash B_i)(u_i - u)}{\sum_{i=1}^N \operatorname{dist}(u, K\backslash B_i)} < \frac{1}{k}$$

and since $J_k \circ T \colon S_k \to S_k$ is continuous, it has a fixed point v_k. By compactness there is a subsequence $v_k \to v \in K$. There holds

$$\|v_k - Tv_k\| = \|J_k T v_k - T v_k\| < \frac{1}{k}.$$

$$\Rightarrow v = Tv.$$

□

3.1.2 Corollary. *Let V be a Banach space, $K \subset V$ closed and convex and let $T \colon K \to K$ be continuous and $T(K)$ precompact. Then T has a fixed point.*

Proof. (i) Let A be a precompact set, then $\mathrm{conv}(A)$ is also precompact, because:
Let $\epsilon > 0$, then

$$\exists x_i \in A, 1 \leq i \leq N \colon A \subset \bigcup_{i=1}^{N} B_\epsilon(x_i).$$

Now let $y \in \mathrm{conv}(A)$,

$$y = \sum_k \lambda_k y_k.$$

Then there exist $x_{i_k} : y_k \in B_\epsilon(x_{i_k})$ and thus

$$\|y - \sum_k \lambda_k x_{i_k}\| \leq \sum_k \lambda_k \|y_k - x_{i_k}\| < \epsilon.$$

$$\Rightarrow \;\; \forall y \in \mathrm{conv}(A)\ \exists \bar{x} \in \mathrm{conv}(x_i) \colon \|y - \bar{x}\| < \epsilon.$$

$$\Rightarrow y \in \bigcup_{i=1}^{N} B_{2\epsilon}(x_i),$$

since $\mathrm{conv}(x_i)$ is precompact.
(ii) Let

$$C = \mathrm{conv}(\bar{T}(K)) \subset K.$$

Then $T \colon C \to C$ has a fixed point. □

3.1.3 Theorem. *(Schaefer)*
Let V be a Banach space, $T \colon V \to V$ continuous and compact. Suppose there is an $M > 0$, such that for all solutions of

$$u = \sigma T u,\ 0 < \sigma < 1,$$

so-called quasi fixed points, we have $\|u\| < M$, then T has a fixed point.

Proof. Without loss of generality we may assume $M = 1$, for otherwise consider $M^{-1}TM$. Define

$$T^*u = \begin{cases} Tu, & \|Tu\| < 1 \\ \frac{Tu}{\|Tu\|}, & \|Tu\| \geq 1. \end{cases}$$

Then

$$T^* \colon \bar{B}_1 \to \bar{B}_1$$

is continuous and $T^*(\bar{B}_1)$ is precompact. Thus T^* has a fixed point

$$u = T^*(u).$$

If $\|Tu\| > 1$ we obtain

$$u = \frac{1}{\|Tu\|} Tu,$$

which contradicts the a priori estimate. Thus $\|Tu\| \leq 1$ and so

$$u = Tu.$$

□

3.1.4 Lemma. *Let V be a Banach space and $B = B_1(0)$. Let $T \colon \bar{B} \to V$ be continuous, $T(\bar{B})$ be precompact and $T(\partial B) \subset \bar{B}$. Then T has a fixed point in $\bar{B}$. If $T(\partial B) \subset B$, then the fixed point lies in B.*

Proof. Define

$$T^*u = \begin{cases} Tu, & \|Tu\| \leq 1 \\ \frac{Tu}{\|Tu\|}, & \|Tu\| > 1. \end{cases}$$

Then $T^* \colon \bar{B}_1 \to \bar{B}_1$ is continuous and $T^*(\bar{B}_1)$ precompact. Thus

$$\exists u \in \bar{B}_1 \colon T^*u = u.$$

$$\Rightarrow Tu = u,$$

for otherwise we had $\|Tu\| > 1$. □

3.1.5 Theorem. *(Leray-Schauder)*
Let V be a Banach space and $T \colon V \times [0,1] \to V$ continuous and compact. Suppose

$$\forall u \in V \colon T(u, 0) = 0$$

and suppose

$$\exists M > 0 \ \forall 0 < \sigma < 1 \colon u = T(u, \sigma) \Rightarrow \|u\| < M.$$

Then

$$\exists u \in V \colon u = T(u, 1).$$

Proof. Without loss of generality let $M = 1$, i.e.

$$u = T(u, \sigma) \Rightarrow \|u\| < 1. \tag{3.1}$$

Let $0 < \epsilon \leq 1$ and let $T^*\colon \bar{B_1}(0) \to V$ be defined by

$$T^* u = T^*_\epsilon u = \begin{cases} T\left(\frac{u}{\|u\|}, \frac{1-\|u\|}{\epsilon}\right), & 1-\epsilon \leq \|u\| \leq 1 \\ T(\frac{u}{1-\epsilon}, 1), & \|u\| \leq 1-\epsilon. \end{cases}$$

Thus T^* is continuous, $T^*(\bar{B}_1)$ is precompact and $T^*(\partial B) = \{0\}$.
Thus there exists u_ϵ such that

$$u_\epsilon = T^* u_\epsilon.$$

Defining

$$\sigma_\epsilon = \begin{cases} \epsilon^{-1}(1 - \|u_\epsilon\|), & 1-\epsilon \leq \|u_\epsilon\| \leq 1 \\ 1, & \|u_\epsilon\| < 1-\epsilon, \end{cases}$$

we obtain

$$u_\epsilon = \begin{cases} T\left(\frac{u_\epsilon}{\|u_\epsilon\|}, \sigma_\epsilon\right), & 1-\epsilon \leq \|u_\epsilon\| \leq 1 \\ T(\frac{u_\epsilon}{1-\epsilon}, \sigma_\epsilon), & \|u_\epsilon\| < 1-\epsilon. \end{cases}$$

$\epsilon \to 0$ implies that for a subsequence we have

$$(u_\epsilon, \sigma_e) \to (u, \sigma),\ 0 \leq \sigma \leq 1.$$

There clearly holds

$$u = T(u, \sigma).$$

Furthermore $\sigma = 1$, for otherwise we would find

$$\|u_\epsilon\| \to 1$$

and thus

$$\|u\| = 1,$$

which is a contradiction. □

3.1.6 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ be open with $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$. Let $a^i \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $a \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, a^i elliptic and $\phi \in C^{2,\alpha}(\bar{\Omega})$. Suppose that for all $0 < \sigma < 1$ and for all solutions of the boundary value problem*

$$\begin{aligned} Au + \sigma a(\cdot, u, Du) + (1-\sigma)\frac{\partial a^i}{\partial x^i}(\cdot, u, Du) &= 0 \\ u_{|\partial\Omega} &= \sigma\phi \end{aligned} \tag{3.2}$$

there holds

$$|u| + |Du| \leq M.$$

Then the Dirichlet problem

$$
\begin{aligned}
Au + a(\cdot, u, Du) &= 0 \\
u_{|\partial\Omega} &= \phi
\end{aligned}
\tag{3.3}
$$

has a solution $u \in C^{2,\alpha}(\bar{\Omega})$.

Proof. Let $v \in C^{1,\alpha}(\bar{\Omega})$. Consider the equation

$$
-a^{ij}(\cdot, v, Dv)u_{ij} - \frac{\partial a^i}{\partial u}(\cdot, v, Dv)u_i - \frac{\partial a^i}{\partial x^i}(\cdot, v, Dv) + a(\cdot, v, Dv) = 0.
$$

Write

$$
Lu = -a^{ij}(\cdot, v, Dv)u_{ij} - \frac{\partial a^i}{\partial u}(\cdot, v, Dv)u_i.
$$

Then L is a uniformly elliptic differential operator with hoelder continuous coefficients. From Schauder theory we conclude, that the boundary value problem

$$
\begin{aligned}
Lu + a(\cdot, v, Dv) + \frac{\partial a^i}{\partial x^i}(\cdot, v, Dv) &= 0 \\
u_{|\partial\Omega} &= \phi
\end{aligned}
\tag{3.4}
$$

has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ and

$$
|u|_{2,\alpha,\Omega} \le c(|\phi|_{2,\alpha} + |v|_{1,\alpha,\Omega}),
$$

where $c = c(\lambda, |a|_{1,\bar{\Omega}\times[-|v|_0,|v|_0]\times[-|Dv|_0,|Dv|_0]}, |a^i|_{2,\bar{\Omega}\times[-|v|_0,|v|_0]\times[-|Dv|_0,|Dv|_0]})$.
Define

$$
\begin{aligned}
T\colon C^{1,\alpha}(\bar{\Omega}) &\to C^{2,\alpha}(\bar{\Omega}) \\
v &\mapsto u = Tv,
\end{aligned}
$$

where u is a solution of (3.4).
T is compact: Let (v^k) be bounded, then $u^k = Tv^k$ is bounded. Thus we obtain subsequences, such that

$$
v^k \to v \text{ in } C^1
$$

and

$$
u^k \to u \text{ in } C^2.
$$

$$
\Rightarrow u = Tv
$$

and by uniqueness the whole sequences must converge.
T is continuous: Write (3.4) in the form

$$
\begin{aligned}
Lu^k &= f^k \\
u_{|\partial\Omega} &= \phi.
\end{aligned}
$$

Let $v^k \to v$ and denote the u^k to be the corresponding solutions. Then $u^k - u^l$ solves

$$\begin{aligned}
&a^{ij}(\cdot, v^k, Dv^k)(u^k - u^l)_{ij} - \frac{\partial a^i}{\partial u}(\cdot, v^k, Dv^k)(u^k - u^l)_i \\
+&(a^{ij}(\cdot, v^l, Dv^l) - a^{ij}(\cdot, v^k, Dv^k))u^l_{ij} \\
+&(\frac{\partial a^i}{\partial u}(\cdot, v^l Dv^l) - \frac{\partial a^i}{\partial u}(\cdot, v^k, Dv^k))u^l_i \\
\equiv& f^k - f^l + F^{kl}
\end{aligned}$$

and thus

$$|u^k - u^l|_{2,\alpha} \le c(|f^k - f^l|_{0,\alpha} + |F^{kl}|_{0,\alpha}) \to 0.$$

We have to show that all quasi fixed points are a priori bounded. So let $u = \sigma T u$, $0 < \sigma < 1$. This means

$$-a^{ij}(\cdot, u, Du)u_{ij} + \sigma a(\cdot, u, Du) + \sigma\frac{\partial a^i}{\partial x^i}(\cdot, u, Du) + \frac{\partial a^i}{\partial u}(\cdot, u, Du)u_i = 0$$

$$u_{|\partial\Omega} = \sigma\phi.$$

By assumption we have $|u| + |Du| \le M_1$. Thus by the L^2 estimates and DeGiorgi-Nash we find

$$|u|_{1,\lambda} \le M_2.$$

Schauder implies

$$|u|_{2,\lambda} \le M_3$$

and repeating those arguments we find

$$|u|_{1,\alpha} \le M_4.$$

Setting

$$M = M_4 + 1$$

implies the claim.

□

3.2 Gradient bounds

3.2.1 Theorem. *Let a^i, a be the coefficients of the modified operator in the proof of Theorem 2.4.5.*

$$Au + a + \gamma(u - u_0) \equiv -(a^i(\cdot, u, Du))_i + a(\cdot, u, Du) + \gamma(u - u_0),$$

$a^i \in C^1$, $a \in C^0$, a^i uniformly elliptic, $a = 0$ for $|Du| > M$ and $(\frac{\partial a^i}{\partial x^i}, \frac{\partial a^i}{\partial u}) = 0$ for $|Du| > 1$. Let $\partial\Omega \in C^2$, $\phi \in C^2(\bar\Omega)$ and for $u \in H^{1,2}(\Omega)$

$$Au + a + \gamma(u - u_0) = 0$$

$$u_{|\partial\Omega} = \phi,$$

then $|Du| \le c$.

Proof. The L^2 estimates imply $u \in H^{2,2}(\Omega)$.
Let $|Du|_{\partial\Omega} \leq k_0$, then

$$|Du| \leq c(k_0, ...).$$

Let $1 \leq k \leq n$ and $v = u_k$. Differentiate the equation for x_k to obtain

$$-(a^{ij}u_j)_i + \frac{\partial a^i}{\partial u}v + \frac{\partial a^i}{\partial x^i} + D_k a + \gamma(v - v_0) = 0.$$

Multiply this equation by

$$v_k := \max(v - k, 0),$$

where $k > k_0$.

$$\Rightarrow \int_\Omega a^{ij} D_i v D_j v_k + \gamma \int_\Omega v v_k \equiv \int_\Omega f v_k, \ f \in L^\infty.$$

By the Stampacchia method we obtain

$$v \leq k_0 + d$$

and analogueously from below.
Bounds up to the boundary: Choose a tubular neighborhood Ω_ϵ with $0 \leq d \in C^2(\bar{\Omega}_\epsilon)$. Define an upper barrier $w \equiv w^+$ by

$$w = \phi + \Lambda h(d), \ 0 \leq d \leq \epsilon.$$

$$\Rightarrow w_{ij} = \phi_{ij} + \Lambda h' d_{ij} + \Lambda h'' d_i d_j.$$

$$-a^{ij}u_{ij} = f \in L^\infty.$$

Choose $h(d) = \log(1 + \alpha d)$, where α is large. Choose $\epsilon = \frac{1}{\alpha}$. Then h'' is the dominant term and thus

$$-a^{ij}w_{ij} > f.$$

$$w_{|\{d=\epsilon\}} = \phi + \Lambda \log 2 > u$$

$$\Rightarrow u \leq w.$$

Bound it from below by using $w^- = \phi - \Lambda h(d)$. $\square$