Partial differential equations 1

Based on lectures by Claus Gerhardt

Contents

Chapter 1

General remarks

1.1 Introduction

1.1.1 Definition. Let $\Omega \subset \mathbb{R}^n$ be open. A *partial differential equation* (PDE) of p-th order is an equation of the form

$$F(x, (D^\alpha u(x))_{|\alpha|\leq p}) = 0, \quad x \in \Omega,\ \alpha \in \mathbb{N}^n,\ D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

1.1.2 Definition. A PDE is called *quasilinear*, if

$$F(\cdot, u, Du, ..., D^p u) = 0$$

is linear in the highest derivative, i.e.

$$\sum_{|\alpha|=p} a_\alpha(\cdot, u, ..., D^{p-1}u) D^\alpha u + a(\cdot, u, ..., D^{p-1}u) = 0.$$

1.1.3 Definition. Let $\Omega \subset \mathbb{R}^n$ be open. A *linear differential operator of second order* is a map $L : C^2(\Omega) \to C^0(\Omega)$ of the form

$$\begin{aligned} Lu &= -a^{ij}(\cdot)u_{ij} + b^i(\cdot)u_i + c(\cdot)u \\ &\equiv Au + Bu + Cu. \end{aligned}$$

A is called *main term* of L. The *symbol* of L in $x \in \Omega$ in direction $\xi \in \mathbb{R}^n$ is defined by

$$\sigma(L; x, \xi) := a^{ij}(x)\xi_i\xi_j.$$

1.1.4 Remark. For $u \in C^2(\Omega)$ there holds

$$\begin{aligned} a^{ij}u_{ij} &= \frac{1}{2}(a^{ij} + a^{ji})u_{ij} + \frac{1}{2}(a^{ij} - a^{ji})u_{ij} \\ &= \frac{1}{2}(a^{ij} + a^{ji})u_{ij} + \frac{1}{2}(a^{ij}u_{ij} - a^{ji}u_{ji}) \\ &= \frac{1}{2}(a^{ij} + a^{ji})u_{ij} \end{aligned}$$

Thus we may suppose that a^{ij} is symmetric.

1.2 Examples in $\mathbb{R}^2$

1.2.1 $u_x = 0$.

$$\begin{aligned} & u_x(x,y) = 0 \\ \Rightarrow \ & 0 = \int_{x_0}^{x} u_t(t,y)dt = u(x,y) - u(x_0,y) \\ \Rightarrow \ & u(x,y) = \phi(y). \end{aligned}$$

Thus the general solution of $u_x = 0$ is given by the set of functions being independent of x.

1.2.2 Polar coordinates

In polar coordinates

$$x = r\cos\omega$$

$$y = r\sin\omega$$

for $u_\omega = 0$ one obtains the so-called radially symmetric functions $\phi = \phi(r)$.

1.2.3 $u_{xy} = 0$.

If $u \in C^2(\Omega)$ and Ω is convex, we get $u_x(x,y) = \tilde{\phi}(x)$

$$\begin{aligned} \Rightarrow \quad & \int_{x_0}^{x} u_t(t,y)dt = \int_{x_0}^{x} \tilde{\phi}(t)dt \\ \Rightarrow \ & u(x,y) = \phi(x) + \psi(y). \end{aligned}$$

1.2.4 $u_{xy} = f$.

As above we find

$$u(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} f + \phi(x) + \psi(y).$$

Notice: The general solution of an inhomogeneous linear PDE is given by the sum of a special solution and the general solution of the homogeneous equation.

1.2.5 The wave equation

$$u_{xx} - u_{yy} = 0, \quad u \in C^2(\Omega).$$

Using the global linear coordinate transformation $\Phi = \Phi(x, y)$,

$$\begin{aligned} \xi &= x + y \\ \eta &= x - y \\ \tilde{u} &:= u \circ \Phi^{-1} \end{aligned}$$

we obtain

$$\begin{aligned} & u_x = \tilde{u}_\xi \circ \Phi + \tilde{u}_\eta \circ \Phi \\ \Rightarrow \quad & u_{xx} = (\tilde{u}_{\xi\xi} + 2\tilde{u}_{\eta\xi} + \tilde{u}_{\eta\eta}) \circ \Phi \end{aligned}$$

as well as

$$\begin{aligned} & u_y = (\tilde{u}_\xi - \tilde{u}_\eta) \circ \Phi \\ \Rightarrow \quad & u_{yy} = (\tilde{u}_{\xi\xi} - 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}) \circ \Phi. \end{aligned}$$

Thus

$$0 = u_{xx} - u_{yy} = 4\tilde{u}_{\xi\eta} \circ \Phi$$

implying

$$\tilde{u}_{\xi\eta} \circ \Phi = 0.$$

From the previous examples it follows

$$\tilde{u}(\xi, \eta) = \phi(\xi) + \psi(\eta)$$

and

$$u(x, y) = \phi(x + y) + \psi(x - y).$$

1.2.6 $u_{xx} - c^{-2}u_{yy} = 0, c \neq 0.$

As above we obtain

$$u(x, y) = \phi(x + cy) + \psi(x - cy).$$

1.2.7 Laplace equation

The so-called *Laplace operator* in $\mathbb{R}^n$ is defined by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Functions satisfying $\Delta u = 0$ in Ω are called *harmonic functions*.

1.3 The Euler-Lagrange equations of the calculus of variations

1.3.1 Lemma (Fundamental lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in L^1_{loc}(\Omega)$ and let*

$$\forall \eta \in C^\infty_c(\Omega)\colon \int_\Omega f\eta = 0.$$

Then there holds

$$f = 0 \quad a.e.$$

Proof. Wlog let $\Omega \Subset \mathbb{R}^n$ and $f \in L^1(\Omega)$. Let

$$g(x) = \begin{cases} \frac{f(x)}{|f(x)|}, & f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\forall 1 \leq p < \infty\colon g \in L^p(\Omega).$$

Choose $\eta_\epsilon \to g$ in $L^1(\Omega)$, $\eta_\epsilon \in C^\infty_c(\Omega)$ and wlog $\eta_\epsilon \to g$ a.e. Let

$$\tilde{\theta}(t) = \begin{cases} t, & |t| \leq 2 \\ -2, & t < -2 \\ 2, & t > 2. \end{cases}$$

Then

$$\tilde{\eta}_\epsilon := \tilde{\theta} \circ \eta_\epsilon \to g \ \text{ a.e.}$$

Let $\tilde{\theta}_\alpha$ be the mollification of $\tilde{\theta}$, the for $x \in \Omega$ and $\tilde{\eta}^\alpha_\epsilon := \tilde{\theta}_\alpha \circ \eta_\epsilon$ we have

$$|\tilde{\eta}^\alpha_\epsilon(x) - g(x)| \leq |\tilde{\eta}^\alpha_\epsilon(x) - \tilde{\theta} \circ \eta_\epsilon(x)| + |\tilde{\theta} \circ \eta_\epsilon(x) - g(x)|.$$

Using $\|\tilde{\eta}^\alpha_\epsilon\|_\infty \leq \|\tilde{\theta}\|_\infty \leq 2$ we find

$$0 = \int_\Omega f\tilde{\eta}^\alpha_\epsilon \to \int_\Omega fg = \int_\Omega |f|.$$

□

1.3.2 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ and $K \subset C^1(\bar{\Omega})$, $F \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, as well as $u \in K$ be a solution of the variational problem*

$$J(v) = \int_\Omega F(\cdot, v, Dv) \to \min,\ v \in K. \tag{1.1}$$

Then there hold

(i) $\forall \eta \in C^\infty_c(\Omega)\ \exists \epsilon_0 > 0\ \forall |\epsilon| < \epsilon_0\colon u + \epsilon\eta \in K$

$$\Rightarrow 0 = \int_\Omega F_u\eta + F_{p_i}D_i\eta.$$

(ii) If furthermore $u \in C^2(\Omega), F \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and $\Omega \in C^1$, we obtain

$$-D_i(F_{p_i}) + F_u = 0. \tag{1.2}$$

Proof. (i) $\phi(\epsilon) := J(u + \epsilon\eta)$

$$\Rightarrow \phi \in C^1(-\epsilon_0, \epsilon_0)$$

and

$$\phi'(\epsilon) = \textstyle\int_\Omega F_u(\cdot, u + \epsilon\eta, Du + \epsilon D\eta)\eta + F_{p_i}(\cdot, u + \epsilon\eta, Du + \epsilon D\eta)D_i\eta$$

$$\Rightarrow 0 = \phi'(0) = \int_\Omega F_u(\cdot, u, Du)\eta + F_{p_i}(\cdot, u, Du)D_i\eta. \tag{1.3}$$

(ii) Partial integration. □

1.3.3 Remark.
(i) The expression $\phi'(0)$ in (1.3) is called *1. variation of* J at 0 in direction η. We also write $\delta J(u; \eta)$.
(ii) The equation (1.2) is called *Euler-Lagrange equation* of the problem (1.1).

1.3.4 Example. Plateau's problem, minimal surface equation

Let $\Omega \Subset \mathbb{R}^n$, $\Omega \in C^1$ and $\Gamma := \{(x, \psi(x)) : x \in \partial\Omega\}$. Let $u \in C^1(\bar{\Omega}), u_{|\partial\Omega} = \psi$. We define

$$J(u) := |\text{graph u}| = \int_\Omega \sqrt{1 + |Du|^2}$$

and consider the *variational problem*

$$J(v) \to \min, \quad v \in K = \{v \in C^1(\bar{\Omega}) : v_{|\partial\Omega} = \psi\}. \tag{1.4}$$

Let $u \in K$ be a solution of (1.4). Then by the previous theorem

$$\delta J(u; \eta) = \int_\Omega \frac{Du \cdot D\eta}{\sqrt{1 + |Du|^2}}.$$

The corresponding Euler-Lagrange equation is

$$-\text{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0.$$

This differential operator is called *minimal surface operator*.
Call this operator A. We have shown, that the variational problem leads to a so-called *Dirichlet problem*, a boundary value problem with prescribed boundary values,

$$\begin{aligned} Au &= 0 \text{ in } \Omega \\ u &= \psi \text{ on } \partial\Omega. \end{aligned}$$

1.4 Natural boundary conditions

1.4.1 Example. The capillarity problem

Let $\Omega \Subset \mathbb{R}^n$, $\Omega \in C^1$. On the set $K = C^1(\bar{\Omega})$ we consider the functional

$$J(v) = \int_\Omega \sqrt{1 + |Dv|^2} + \frac{\kappa}{2}\int_\Omega v^2 + \int_{\partial\Omega} \beta v \to \text{ min}, \tag{1.5}$$

$\kappa \in \mathbb{R}$, $\beta \in C^0(\partial\Omega)$. Let $u \in K$ be a solution, so

$$\forall \eta \in K\colon \delta J(u;\eta) = 0.$$

An easy calculation shows

$$\delta J(u;\eta) = \textstyle\int_\Omega \frac{D^i u}{\sqrt{1+|Du|^2}} D_i\eta + \int_\Omega \kappa u \eta + \int_{\partial\Omega} \beta\eta = 0$$

Now let $u \in C^2(\bar{\Omega}), \eta \in C^1_c(\Omega)$, then we have

$$Au + \kappa u = 0 \text{ in } \Omega,$$

where A is the minimal surface operator. This equation is called *capillarity equation.*
Now we also admit $\eta \in C^1(\bar{\Omega})$ and find after partial integration

$$\forall \eta \in C^1(\overline{\Omega})\colon 0 = \textstyle\int_\Omega (Au + \kappa u)\eta + \int_{\partial\Omega} \left(\frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} + \beta \right) \eta.$$

We want to show that

$$\forall \eta \in C^0(\partial\Omega)\colon 0 = \int_{\partial\Omega} \left(\frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} + \beta \right) \eta \tag{1.6}$$

and need the

Theorem. *Let $\eta \in C^0(\partial\Omega)$, then there exist $\eta_\epsilon \in C^1_c(\mathbb{R}^n)$, such that*

$$\eta_\epsilon \to \eta \textit{ uniformly on } \partial\Omega.$$

Proof. Tietze-Urysohn implies the existence of $\tilde{\eta} \in C^0_c(\mathbb{R}^n)$, such that $\tilde{\eta}_{|\partial\Omega} = \eta$. The convolutional sequence $\tilde{\eta}_\epsilon$ satisfies the desired properties. □

Thus for we have (1.6)

$$0 = \textstyle\int_{\partial\Omega} \left(\frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} + \beta \right) \eta_\epsilon \to \int_{\partial\Omega} \left(\frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} + \beta \right) \eta$$

Choose $\eta := \frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} + \beta$, then

$$-\frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} = \beta.$$

As a necessary condition we obtain $|\beta| < 1$.
We have thus solved a *Neumann boundary value problem*, asking for certain boundary derivatives.

$$\begin{aligned} Au + \kappa u &= 0 \text{ in } \Omega \\ -\frac{D_i u \nu^i}{\sqrt{1+|Du|^2}} + \beta &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Note, that this is not the normal derivative $\frac{\partial u}{\partial \nu}$, but the so-called *conormal derivative*, arising naturally from the variational problem.
Generally let $Au = -D_i(a^i(\cdot, u, Du))$ be an operator, then we call $-a_i \nu^i$ the *conormal* of A. Dirichlet boundary conditions are sometimes also called *boundary conditions of first kind*, Neumann boundary conditions *boundary conditions of second kind*.

1.5 Variational problems under side conditions

Consider

$$J(v) = \int_\Omega F(\cdot, v, Dv) \to \min \tag{1.7}$$

over the set

$$K = \{v \in C^1(\bar{\Omega}) : H(v) = \int_\Omega h(\cdot, v, Dv) = 0, v_{|\partial\Omega} = b\}.$$

Here we have $F, h \in C^p(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $p \geq 1$. Such a side condition is called *isoperimetric side condition*. Let u be a solution and $\eta, \zeta \in C_c^\infty(\Omega), \epsilon_1, \epsilon_2 \in \mathbb{R}$. Set

$$\begin{aligned} \phi(\epsilon_1, \epsilon_2) &= J(u + \epsilon_1 \eta + \epsilon_2 \zeta) \\ \psi(\epsilon_1, \epsilon_2) &= H(u + \epsilon_1 \eta + \epsilon_2 \zeta) \end{aligned}$$

If $D\psi(0,0) \neq 0$, then, using Analysis II, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$D\phi(0,0) + \lambda D\psi(0,0) = 0.$$

We have

$$D\phi(0,0) = (\delta J(u;\eta), \delta J(u;\zeta))$$

and

$$D\psi(0,0) = (\delta H(u;\eta), \delta H(u;\zeta)).$$

It follows

$$\delta J(u;\eta) + \lambda \delta H(u;\eta) = 0$$
$$\text{and } \delta J(u;\zeta) + \lambda \delta H(u;\zeta) = 0.$$

1.5.1 Theorem. *Let u be a solution of the variational problem (1.7) and suppose*

$$\exists \eta \in C_c^\infty(\Omega) : \delta H(u;\eta) \neq 0.$$

Then

$$\exists! \lambda \in \mathbb{R}\ \forall \eta \in C_c^\infty(\Omega) : \delta J(u;\eta) + \lambda \delta H(u;\eta) = 0.$$

Proof. Let $\zeta \in C_c^\infty(\Omega) : \delta H(u;\zeta) \neq 0$. Furthermore let $\eta, \tilde{\eta} \in C_c^\infty(\Omega)$.
Then there exist $\lambda, \mu \in \mathbb{R}$:

$$\delta J(u;\eta) + \lambda \delta H(u;\eta) = 0$$
$$\delta J(u;\zeta) + \lambda \delta H(u;\zeta) = 0 \text{ and}$$

$$\delta J(u;\tilde{\eta}) + \mu \delta H(u;\tilde{\eta}) = 0$$
$$\delta J(u;\zeta) + \mu \delta H(u;\zeta) = 0$$

$$\delta H(u;\zeta) \neq 0 \Rightarrow \lambda = \mu.$$

□

1.5.2 Remark. Let $\Omega \Subset \mathbb{R}^n$, $\Omega \in C^1$ and $u, h \in C^2(\bar{\Omega})$, then there holds:

$$\exists \zeta \in C_c^\infty(\Omega) : \delta H(u;\zeta) \neq 0 \Leftrightarrow -D_i(h_{p_i}(\cdot, u, Du)) + h_u(\cdot, u, Du) \neq 0.$$

1.5.3 Example. Let $\Omega \Subset \mathbb{R}^n$, $\Omega \in C^1$. Consider the variational problem

$$J(v) = \int_\Omega \sqrt{1 + |Dv|^2} \to \min \tag{1.8}$$

over the set $K = \{v \in C^1(\bar{\Omega}) : v_{|\partial\Omega} = \phi \wedge \int_\Omega v = V\}$.
This means, that we minimize the surface area at prescribed volume and fixed boundary values.

$$h(v) := v - \frac{1}{|\Omega|} V \Rightarrow \forall v \in K\colon \int_\Omega h(v) = 0$$

and

$$h_v = 1 \neq 0.$$

Let $u \in C^2(\bar{\Omega})$ be a solution of (1.8).

$$\Rightarrow \exists \lambda \in \mathbb{R} : \delta J(u; \eta) + \lambda H(u; \eta) = 0$$

and thus

$$\begin{aligned} Au + \lambda &= 0 \text{ in } \Omega \\ u_{|\partial\Omega} &= \phi \end{aligned}$$

1.5.4 Example. Let $\Omega \Subset \mathbb{R}^n$ and $\Omega \in C^1$.

$$J(v) = \frac{1}{2} \int_\Omega |Dv|^2 \to \min$$

over $K = \{v \in C^2(\bar{\Omega}) : v_{|\partial\Omega} = 0 \wedge \frac{1}{2} \int_\Omega v^2 = 1\}$.

$$h(v) = \frac{1}{2} v^2 \Rightarrow h_v(v) = v.$$

Let u be a solution of

$$J(v) \to \min$$

$$\Rightarrow u \neq 0$$

$$\Rightarrow h_v(u) = u \neq 0$$

$$\Rightarrow \exists \lambda \in \mathbb{R} : -\Delta u = \lambda u.$$

Chapter 2

The maximum principle

2.1 Linear elliptic operators of second order

2.1.1 Definition. A linear differential operator of second order L is called *elliptic in* $x \in \Omega$, if

$$\exists \lambda = \lambda(x) > 0 \; \forall \xi \in \mathbb{R}^n \colon \sigma(L; x, \xi) \geq \lambda |\xi|^2.$$

L is called *elliptic in* Ω, if L is elliptic in every $x \in \Omega$.
L is called *uniformly elliptic*, if $a^{ij} \in L^\infty(\Omega)$ and

$$\exists \lambda > 0 \; \forall x \in \Omega \; \forall \xi \in \mathbb{R}^n \colon \sigma(L; x, \xi) \geq \lambda |\xi|^2.$$

2.1.2 Remark. (Operators in *divergence form*)
Operators of the form

$$Lu = -D_i(a^i(x, u, Du)) + a(x, u, Du)$$

are named in correspondence to 2.1.1, if the corresponding properties are fulfilled by $a^{ij} = \frac{\partial a^i}{\partial u_j}$.

2.1.3 Proposition. *(Coordinate transformation)*
Let L be a linear differential operator of second order in Ω and $\tilde{x} \in \mathrm{Diff}^2(\Omega, \tilde{\Omega})$, then in the new coordinates L has the form

$$\tilde{L} = -\tilde{a}^{ij} \tilde{D}_i \tilde{D}_j + \tilde{b}^i \tilde{D}_i + \tilde{c},$$

where

$$\tilde{a}^{ij} = (a^{kl} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l}) \circ \tilde{x}^{-1},$$

$$\tilde{b}^i = (b^k \frac{\partial \tilde{x}^i}{\partial x^k} + a^{kl} \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^l}) \circ \tilde{x}^{-1},$$

$$\tilde{c} = c \circ \tilde{x}^{-1}.$$

Proof. Let $\tilde{u} : \tilde{\Omega} \to \mathbb{R}$ be defined by $\tilde{u}(\tilde{x}) = u \circ \tilde{x}^{-1}$, such that $u(x) = \tilde{u}(\tilde{x}(x))$.

$$\begin{aligned}
u_i &= \tilde{u}_k \frac{\partial \tilde{x}^k}{\partial x^i} \\
\Rightarrow u_{ij} &= \tilde{u}_{kl} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} + \tilde{u}_k \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j} \\
\Rightarrow Lu &= a^{ij} u_{ij} + b^i u_i + cu \\
&= a^{ij} \tilde{u}_{kl} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} + a^{ij} \tilde{u}_k \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j} \\
&+ b^i \tilde{u}_k \frac{\partial \tilde{x}^k}{\partial x^i} + c\tilde{u},
\end{aligned}$$

where a^{ij}, b^i and c are evaluated in Ω and $\tilde{u}$ in $\tilde{\Omega}$.
Thus we have

$$\tilde{L}\tilde{u}(\tilde{x}) = Lu \circ \tilde{x}^{-1} = \tilde{a}^{kl} \tilde{u}_{kl} + \tilde{b}^k \tilde{u}_k + \tilde{c}\tilde{u}.$$

$\square$

2.1.4 Remark. A differential operator in divergence form

$$Lu = -D_i(a^{ij} u_j) = -\mathrm{div}(A \cdot Du),$$

where (x^i) are Euclidian coordinates, transforms like

$$\tilde{L}\tilde{u} = -\frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^i} (\sqrt{\tilde{g}} \tilde{a}^{ij} \tilde{u}_j),$$

where $\tilde{g}_{ij} = \delta_{kl} \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j}$, $\tilde{g} = \det(\tilde{g}_{ij})$ and $\tilde{a}^{ij} = a^{kl} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l}$.

2.1.5 Example. (Straightening the boundary)
Let $\Omega \in C^2$. Write $\partial\Omega$ locally as a C^2-function, i.e. for every $y_0 \in \partial\Omega$ there is a neighbourhood $U = U(y_0)$, a coordinate system $x \in C^2(U, x(U))$, which arises from Euclidian coordinates by a permutation of $\{1, \ldots, n\}$, as well as a $\phi \in C^2(\hat{x}(U))$, such that

$$x(\partial\Omega \cap U) = \{(\hat{x}, \phi(\hat{x})) \colon \hat{x} \in \hat{x}(U)\},$$

$\hat{x} = (x^1, ..., x^{n-1})$. Let $V := \hat{x}(U)$ and define a transformation

$$\tilde{x} \colon V \times \mathbb{R} \to \tilde{x}(V \times \mathbb{R})$$

$$\forall 1 \leq i \leq n-1 \colon \tilde{x}^i = x^i,$$

$$\tilde{x}^n = \phi(\hat{x}) - x^n.$$

Then one obtains

$$\tilde{x}(\Gamma) \subset \{\tilde{x}^n = 0\},$$

where $\Gamma = \phi(V)$. Then $\tilde{x}$ is a C^2-diffeomorphism with $\det(\frac{\partial \tilde{x}}{\partial x}) = -1$. Thus a divergence form equation transforms like

$$-D_i(a^i) = -\frac{\partial}{\partial \tilde{x}^i}(a^k \frac{\partial \tilde{x}^i}{\partial x^k} \circ \tilde{x}^{-1}) = -\frac{\partial}{\partial \tilde{x}^i}(\tilde{a}^i).$$

There holds $\tilde{x}(x(\partial\Omega \cap U)) = \{\tilde{x}^n = 0\}$. Thus

$$\tilde{x}^n(x) \colon V \times \mathbb{R} \to \tilde{x}^n(V \times \mathbb{R})$$

has $x(\partial\Omega \cap U)$ as a hypersurface, which is why the normal has, in x-coordinates, the form

$$\nu_i = \pm \frac{(\frac{\partial \tilde{x}^n}{\partial x^i})}{\sqrt{1 + |D\phi|^2}}.$$

It follows:

2.1.6 Theorem. *Let Ω be a domain with $\partial\Omega \in C^{m,\alpha}$, $m \geq 2$, $0 \leq \alpha \leq 1$. Then*

$$\forall x_0 \in \partial\Omega \ \exists U \in \mathcal{U}(x_0) \ \exists \tilde{x} \in \operatorname{Diff}^{m,\alpha}(U, B_1(0)) :$$
$$\tilde{x}(U \cap \Omega) = B_1^+(0),$$
$$\tilde{x}(U \cap \partial\Omega) = B_1(0) \cap \{\tilde{x}^n = 0\}.$$

A PDE of the form

$$-a^{ij}u_{ij} + b^i u_i + cu = 0,$$
$$-D_i(a^i(x, u, Du)) + a(x, u, Du) = 0$$

respectively, transforms into one of the same structure in $B_1^+(0)$.
This also holds, if the coefficients only depend on (x, Du).

2.1.7 Remark. Let $\Omega \subset \mathbb{R}^n$ and $\tilde{x} \in \operatorname{Diff}^2(\bar{\Omega}, \tilde{x}(\bar{\Omega}))$. Then there holds:
If L is elliptic in any sense, this also holds for $\tilde{L}$.

Proof. The proof is valid for both kinds of operators. The main term transforms like

$$\tilde{a}^{ij} = a^{kl} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l}$$

$$\Rightarrow \tilde{a}^{ij}\xi_i\xi_j = a^{kl}(\frac{\partial \tilde{x}^i}{\partial x^k}\xi_i)(\frac{\partial \tilde{x}^j}{\partial x^l}\xi_j) \equiv a^{kl}\eta_k\eta_l \geq \lambda|\eta|^2.$$

Since $\frac{\partial \tilde{x}}{\partial x}$ is uniformly invertible due to $\tilde{x} \in \operatorname{Diff}^2(\bar{\Omega}, \tilde{x}(\bar{\Omega}))$, we obtain

$$|\eta| \geq c|\xi|.$$

□

2.1.8 Proposition. *Let L be elliptic in $x_0 \in \Omega$. Then there is an orthogonal transformation $\mathcal{O}$, such that the main term of $\tilde{L}$ has, with respect to $\mathcal{O}x$, in $\mathcal{O}x_0$ the form*

$$-\sum_i \lambda^i u_{ii},$$

where $\lambda^i > 0$.

Proof. Diagonalize the bilinear form $a^{ij}(x_0)$ using an orthogonal transformation $\mathcal{O} = (\mathcal{O}^i_j)$, i.e.

$$\mathcal{O}^i_k a^{kl} \mathcal{O}^j_l = \operatorname{diag}(\lambda^1, \ldots, \lambda^n),$$

where the λ^i are the eigenvalues of the positive definite matrix $a^{ij}(x_0)$. Then the global coordinate transformation

$$\tilde{x}(x) = \mathcal{O}x$$

will yield the desired representation. □

2.2 The maximum principle and applications

2.2.1 Lemma. *Let $L = -a^{ij}D_iD_j + b^iD_i$ be elliptic in $\Omega \subset \mathbb{R}^n$. Let $u \in C^2(\Omega)$ and suppose u attains a relative maximum in $x_0 \in \Omega$. Then*

$$Lu(x_0) \geq 0.$$

Proof. There holds $Lu(x_0) = -a^{ij}u_{ij}(x_0)$.

$$B := (u_{ij}(x_0)) \leq 0, \;\; A := (-a^{ij}(x_0)) < 0.$$

$$\Rightarrow Lu(x_0) = \operatorname{tr}(AB) = \operatorname{tr}(\mathcal{O}^*AB\mathcal{O}) = \operatorname{tr}(\mathcal{O}^*A\mathcal{O}\mathcal{O}^*B\mathcal{O}).$$

Choose $\mathcal{O}$ such that $\mathcal{O}^*B\mathcal{O} = \operatorname{diag}(\mu_i),\;\; \mu_i \leq 0$. Furthermore there holds $(\mathcal{O}^*A\mathcal{O})^{ii} < 0$.

$$\Rightarrow Lu(x_0) = \sum_i (\mathcal{O}^*A\mathcal{O})^{ii}\mu_i \geq 0.$$

□

2.2.2 Corollary. *Let $u \in C^2(\Omega)$ and $Lu < 0$ in Ω, where $Lu = -a^{ij}u_{ij} + b^iu_i$ is elliptic. Then u does not attain a relative maximum in Ω.*

2.2.3 Lemma. *(E.Hopf)*
Let $B_0 \subset \mathbb{R}^n$ be a ball with radius r_0 and $x_0 \in \partial B_0$.
Let $L = -a^{ij}D_iD_j + b^iD_i$ be uniformly elliptic in B_0 with bounded coefficients a^{ii}, b^i. Let $u \in C^2(B_0) \cap C^0(B_0 \cup \{x_0\})$ satisfy

$$Lu \leq 0 \text{ in } B_0 \;\wedge\; \forall x \in B_0 \colon u(x) < u(x_0).$$

Then there holds

$$\frac{\partial u}{\partial \nu}(x_0) := \liminf_{t \nearrow 0} \frac{u(x_0 + t\nu) - u(x_0)}{t} > 0,$$

where ν is the outer normal to B_0 in x_0.

Proof. Let wlog $B_1 = B_{r_1}(0)$ be an inner ball touching x_0 and $B_2 = B_{r_2}(0)$ be a concentric ball. $B' := B_1 \backslash \overline{B_2}$. We aim to find a function h in B' such that

$$Lh < 0,$$

$$\frac{\partial h}{\partial \nu}(x_0) < 0$$

and

$$h(x_0) = 0.$$

For $v = u + h$ we want

$$\sup_{B'} v = u(x_0)$$

to hold.
Then we had $\frac{\partial v}{\partial \nu}(x_0) \geq 0$ and thus $\frac{\partial u}{\partial \nu}(x_0) > 0$. We define

$$\delta(x) := e^{-\alpha|x|^2} - e^{-\alpha r_1^2},\ x \in B',\ \alpha > 1.$$

$$\Rightarrow \delta > 0 \text{ in } B_1 \text{ and } \delta_{|\partial B_1} \equiv 0.$$

There holds

$$D_i\delta(x) = -2\alpha e^{-\alpha|x|^2} x_i$$

and

$$D_i D_j \delta = (4\alpha^2 x_i x_j - 2\alpha\delta_{ij}) e^{-\alpha|x|^2}.$$

$$\begin{aligned} L\delta(x) &= -a^{ij} D_i D_j \delta(x) + b^i D_i \delta(x) \\ &= (-(4\alpha^2 a^{ij} x_i x_j - 2\alpha a_i^i) - 2b^i x_i \alpha) e^{-\alpha|x|^2} \\ &\leq -(4\alpha^2 \lambda |x|^2 - 2\alpha a_i^i - 2|b^i| r_1 \alpha) e^{-\alpha|x|^2} \\ &\leq -(4\alpha^2 \lambda r_2^2 - 2\alpha a_i^i - 2|b^i| r_1 \alpha) e^{-\alpha|x|^2} \\ &< 0, \end{aligned}$$

if α is large enough.

$$\frac{\partial \delta}{\partial \nu}(x_0) = D_i \delta(x_0) \frac{x_0^i}{|x_0|} < 0.$$

Set $h := \epsilon\delta$ for an ϵ yet to be determined.

$$v := u + h \ \Rightarrow Lv < 0 \text{ in } B'$$

and h fulfills the first three conditions.
We now show, that for small ϵ there holds

$$\sup_{B'} v = v(x_0).$$

We know that $\sup_{B'} v = \sup_{\partial B'} v$. On ∂B_1 we have $v = u$, and thus

$$\sup_{\partial B_1} v = u(x_0) = v(x_0).$$

Furthermore there holds

$$\sup_{\partial B_2} u < u(x_0) - \gamma,$$

γ small. Choose

$$\epsilon < \gamma,$$

to obtain the claim. □

2.2.4 Theorem. *(Strong maximum principle)*
Let $\Omega \subset \mathbb{R}^n$ be a domain and

$$L = -a^{ij} D_i D_j + b^i D_i + c,\ c \geq 0$$

be locally uniformly elliptic with locally bounded coefficients a^{ii}, b^i. Let $u \in C^2(\Omega)$ and $Lu \leq 0$, then u does not attain a positive maximum in Ω, if u is not constant.

Proof. Suppose, $x_0 \in \Omega$ and $\forall x \in \Omega\colon u(x) \leq u(x_0)$, as well as $\gamma := u(x_0) > 0$.

$$M := \{u = \gamma\} \subset \Omega.$$

$$\Rightarrow M \neq \emptyset,\ M \text{ closed in } \Omega.$$

If M was not open, then

$$\exists x_1 \in M\ \exists r_0 > 0\colon B_{r_0}(x_1) \cap \Omega \backslash M \neq \emptyset \wedge u_{|B_{3r_0}} \geq \frac{\gamma}{2}.$$

Let $x_2 \in B_{r_0}(x_1) \cap \Omega \backslash M$.

$$\Rightarrow d(x_2, M) =: r_1 > 0 \wedge r_1 = |x_2 - \bar{x}_0|,\ \bar{x}_0 \in \partial M.$$

There holds $r_1 < r_0$ and $B_{r_1}(x_2) \subset \Omega \backslash M$, as well as $B_{r_1}(x_2) \subset B_{3r_0}(x_1)$.

$$\Rightarrow u_{|B_{r_1}} \geq 0.$$

$L'u := Lu - cu \leq 0$ in $B_{r_1}(x_2)$. Thus $L', u, \bar{x}_0, B_{r_1}(x_2)$ satisfy the conditions of the Hopf lemma.

$$\Rightarrow \frac{\partial u}{\partial \nu}(\bar{x}_0) > 0 = Du(\bar{x}_0),$$

a contradiction. □

2.2.5 Theorem.
Under the same conditions as in 2.2.4,
(i) a function $u \in C^2(\Omega)$ satisfying $Lu \geq 0$ in Ω does not attain a negative minimum, unless it is constant and
(ii) if $u \in C^2(\Omega)$ is a solution of $-a^{ij}u_{ij} + b^i u_i = 0$, then

$$\inf_{\partial\Omega} u \leq u \leq \sup_{\partial\Omega} u$$

and equality holds in an $x \in \Omega$, if and only if u is constant.

Proof. (i) follows using $u \to -u$ from 2.2.4.
(ii) In case $c = 0$ you may conclude as in the proof of 2.2.4, where $L' = L$ and one does not need the positivity of γ. □

2.2.6 Lemma. *(Comparison lemma)*
Let L satisfy the conditions of 2.2.4 and let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy the inequality

$$Lu \geq Lv,$$

then we have

$$\min(0, \inf_{\partial\Omega} u - v) \leq u - v \text{ in } \Omega$$

and

$$v - u \leq \max(0, \sup_{\partial\Omega} v - u) \text{ in } \Omega.$$

Proof. Set $\phi := u - v \Rightarrow L\phi \geq 0$. Apply the preceeding theorem to ϕ. □

2.2.7 Definition. (Interior sphere conditions, ISC)
Let $\Omega \subset \mathbb{R}^n$ be open. We say, Ω satisfies an interior sphere condition, *ISC*, with radius R, if

$$\exists R > 0 \ \forall x \in \partial\Omega \ \exists x_0 \in \Omega \colon B_R(x_0) \subset \Omega \wedge B_R\bar{(}x_0) \cap \partial\Omega = \{x\}.$$

2.2.8 Example.
$\Omega \Subset \mathbb{R}^n, \partial\Omega \in C^2 \Rightarrow \Omega$ satisfies an ISC.

Proof. Exercise. □

2.2.9 Proposition. *Let L as in 2.2.4. Let $\Omega \Subset \mathbb{R}^n$ be connected and satisfy an ISC, then the solutions $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of*

$$Lu = -a^{ij}u_{ij} + b^i u_i = f \text{ in } \Omega$$

$$-\frac{\partial u}{\partial \nu} = \beta \text{ on } \partial\Omega$$

are unique up to an additive constant.

Proof. Since this is a linear boundary value problem, all you have to show is, that solutions of the homogeneous problem are constant.
If $u \neq \text{const}$, then we had

$$\sup_{\Omega} u = u(x_0) > u_{|\Omega}, \; x_0 \in \partial\Omega,$$

using the maximum principle. Choose an inner ball touching x_0, then by the Hopf lemma we obtain $-\frac{\partial u}{\partial \nu}(x_0) < 0$. $\square$

2.2.10 Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F = F(x,u,p,w) \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2})$. F is called *elliptic* in (x,u,p,w), if

$$F^{ij} := \frac{\partial F}{\partial w_{ij}}$$

is positive definite in (x,u,p,w).

2.2.11 Example. (Monge-Ampére-operator)
Let $M = M^n$ be a Riemannian manifold with metric (g_{ij}), $u \in C^2(M)$ and

$$F = \frac{\det(ug_{ij} + u_{ij})}{\det(g_{ij})} > 0.$$

$$\Rightarrow F^{ij} = F\tilde{g}^{kl}\frac{\partial \tilde{g}_{kl}}{\partial u_{ij}} = F\tilde{g}^{ij}, \; \tilde{g}_{ij} := ug_{ij} + u_{ij}.$$

Thus, F is elliptic if and only if $\tilde{g}^{ij} > 0$.

2.2.12 Lemma. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $F = F(x,u,p,w)$ be a uniformly elliptic differential operator of second order, $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2})$ with $F_u \geq 0$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality*

$$F(\cdot, v, Dv, -D^2v) \geq F(\cdot, u, Du, -D^2u),$$

then

$$v - u \geq \min(0, \inf_{\partial\Omega}(v-u)).$$

Proof. Apply the main theorem of calculus.

$$\begin{aligned}
0 &\leq F(\cdot, v, Dv, -D^2v) - F(\cdot, u, Du, -D^2u) \\
&= \int_0^1 \frac{d}{dt} F(\cdot, z_t, Dz_t, -D^2z_t), \quad z_t = tv + (1-t)u \\
&= \int_0^1 \frac{\partial F}{\partial u}(v-u) + \frac{\partial F}{\partial p_i} D_i(v-u) - \frac{\partial F}{\partial w_{ij}} D_{ij}(v-u)dt \\
&= (\int_0^1 \frac{\partial F}{\partial u} dt)(v-u) + (\int_0^1 \frac{\partial F}{\partial p_i} dt) D_i(v-u) - (\int_0^1 \frac{\partial F}{\partial w_{ij}} dt) D_{ij}(v-u)
\end{aligned}$$

Using 2.2.4 and its corollaries we obtain the claim. $\square$

2.2.13 Corollary. *Let $\Omega \subset \mathbb{R}^n$ be open. Let L be the quasilinear operator*

$$Lu = -a^{ij}(x, Du)u_{ij} + a(x, u, Du)$$

with coefficients $a^{ij} \in C^0(\Omega \times \mathbb{R}^n)$, $\frac{\partial a^{ij}}{\partial p_k} \in C^0(\Omega \times \mathbb{R}^n)$, $a \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $\frac{\partial a}{\partial u}, \frac{\partial a}{\partial p_i} \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Let L be uniformly elliptic and let $\frac{\partial a}{\partial u} \geq 0$. Then for u, v with $Lu = Lv$ we have

$$|u - v| \leq \sup_{\partial\Omega} |u - v|.$$

The boundary value problem

$$\begin{aligned} Lu &= f \\ u_{|\partial\Omega} &= \phi \end{aligned} \tag{2.1}$$

thus has at most one solution.

2.3 C^0-estimates for quasilinear PDE

2.3.1 Theorem. *Let $\Omega \Subset \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of*

$$\begin{aligned} -a^{ij}(x, Du)u_{ij} + a(x, u, Du) &= f \\ u_{|\partial\Omega} &= \phi, \end{aligned} \tag{2.2}$$

$f \in C^0(\bar{\Omega})$, $\phi \in C^0(\partial\Omega)$. Let

$$a^{ij} \in C^0(\Omega \times \mathbb{R}^n), \ \frac{\partial a^{ij}}{\partial p_k} \in C^0(\Omega \times \mathbb{R}^n)$$

and

$$\exists \lambda, \mu > 0 \ \forall x \in \Omega \ \forall \xi \in \mathbb{R}^n : \ \lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2.$$

Let

$$a, \ \frac{\partial a}{\partial u}, \ \frac{\partial a}{\partial p_i} \in C^0(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n), \ \frac{\partial a}{\partial u} \geq 0,$$

and

$$|a(x, u, p)| \leq c(1 + |p|).$$

Then there hold

$$|u|_{0,\Omega} \leq |\phi|_{0,\partial\Omega} + c|f|_0,$$

if

$$|a(x, 0, p)| \leq |p|,$$

and

$$|u|_{0,\Omega} \leq |\phi|_{0,\partial\Omega} + c(1 + |f|_0), \ \textit{otherwise}.$$

Proof. Let $\forall x \in \bar{\Omega}\colon -d \leq x_1 \leq d$. We construct functions δ^-, $\delta^+ \in C^2(\Omega) \cap C^0(\overline{\Omega})$, such that

$$L\delta^+ \geq f \text{ and } \delta^+_{|\partial\Omega} \geq \phi,$$

and analogueously with reversed inequalities for δ^-. Using the comparison principles then it follows

$$\delta^- \leq u \leq \delta^+. \tag{2.3}$$

Case 1: $|a(x, 0, p)| \leq c|p|$.
Let $\alpha > 0$, define

$$\delta^+(x) := |\phi|_{0,\partial\Omega} + (e^{\alpha d} - e^{\alpha x_1})|f|_{0,\Omega}.$$

$$\Rightarrow \delta^+_1(x) = -\alpha e^{\alpha x_1}|f|_{0,\Omega} \ \wedge \ \ \delta^+_{11}(x) = -\alpha^2 e^{\alpha x_1}|f|_{0,\Omega}.$$

$$\begin{aligned} L\delta^+(x) &= \alpha^2 a^{11}(x) e^{\alpha x_1}|f|_{0,\Omega} + a(x, \delta^+(x), D\delta^+(x)) \\ &\geq \alpha^2 a^{11}(x) e^{\alpha x_1}|f|_{0,\Omega} + a(x, 0, D\delta^+(x)) \\ &\geq \alpha^2 a^{11}(x) e^{\alpha x_1}|f|_{0,\Omega} - c|D\delta^+(x)| \\ &\geq \alpha e^{\alpha x_1}|f|_{0,\Omega}(\alpha a^{11}(x) - c) \geq |f|_{0,\Omega}, \end{aligned}$$

for large α. Set $\delta^- := -\delta^+$, thus it follows (2.3) and with the special choice of δ^+ we obtain the claim.
Case 2: Define $\delta^+(x) := |\phi|_{0,\partial\Omega} + (e^{\alpha d} - e^{\alpha x_1})(1 + |f|_{0,\Omega})$ and δ^- as above.
$\square$

Chapter 3

Schauder estimates

3.1 Potentials

3.1.1 Definition. (i) The functions

$$\gamma(r) := \begin{cases} \frac{1}{r^{n-2}}, & n \geq 3 \\ \log(r), & n = 2 \end{cases}$$

are called *Newton potentials* in $\mathbb{R}^n$.
(ii) Let $\Omega \subset \mathbb{R}^n$ and $\rho \in L^1(\Omega)$, then its so called *volume potential* is defined by

$$u(x) := \int_\Omega \gamma(|x-y|)\rho(y)dy.$$

3.1.2 Remark. An easy calculation shows, that the Newton potential is for $r = |x-y|$ radially symmetric around y and harmonic in $\mathbb{R}^n \backslash \{y\}$ with respect to x.

3.1.3 Lemma. *Let $E \subset \mathbb{R}^n$ be measurable and $|E| < \infty$. Then*

$$\forall x \in \mathbb{R}^n\colon \int_E \frac{1}{|x-y|^{n-\alpha}} dy \leq \frac{\omega_n}{\alpha} n \left(\frac{|E|}{\omega_n} \right)^{\frac{\alpha}{n}},$$

where $n \geq 2$, $0 < \alpha < n$, $\omega_n = |B_n|$.

Proof. Let $R = \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}$, then $|B_R(x)| = |E|$.

Let $\tilde{B} = E \cap B_R$ and $r = |x-y|$. Then

$$\begin{aligned}
\int_E \frac{1}{r^{n-\alpha}} dy &= \int_{\tilde{B}} \frac{1}{r^{n-\alpha}} dy + \int_{E \setminus \tilde{B}} \frac{1}{r^{n-\alpha}} dy \\
&\leq \int_{\tilde{B}} \frac{1}{r^{n-\alpha}} dy + \frac{1}{R^{n-\alpha}} (|E| - |\tilde{B}|) \\
&= \int_{\tilde{B}} \frac{1}{r^{n-\alpha}} dy + \frac{1}{R^{n-\alpha}} (|B_R| - |\tilde{B}|) \\
&\leq \int_{\tilde{B}} \frac{1}{r^{n-\alpha}} dy + \int_{B_R \setminus \tilde{B}} \frac{1}{r^{n-\alpha}} dy = \int_{B_R} \frac{1}{r^{n-\alpha}} \\
&= \int_{\mathbb{S}^{n-1}} \int_0^R \frac{1}{r^{n-\alpha}} r^{n-1} = \frac{1}{\alpha} R^\alpha |\mathbb{S}^{n-1}| = \frac{1}{\alpha} R^\alpha n \omega_n.
\end{aligned}$$

□

3.1.4 Corollary.

$$(i) \quad \forall x, x_0 \in \mathbb{R}^n, \ 0 < \alpha < n \colon \int_{B_\delta(x_0)} \frac{1}{|x-y|^{n-\alpha}} dy \leq \frac{n\omega_n}{\alpha} \delta^\alpha. \tag{3.1}$$

$$(ii) \int_{B_\delta(x_0)} |\log(r)| = O(\delta), \ \text{if } n \geq 2. \tag{3.2}$$

Proof.
(i) follows from the preceeding lemma.
(ii) $|\log(r)| r \leq \text{const in } B_\delta(x_0), \ r = |y - x_0|$,

$$\Rightarrow \int_0^\delta |log(r)| r^{n-1} dr \leq c \int_0^\delta r^{n-2} dr = c\delta^{n-1}.$$

□

3.1.5 Theorem. *(Gauß)*
Let $E \subset \mathbb{R}^n$ be measurable and bounded, $n \geq 2$ and $f \in L^\infty(E)$. Then the integrals

$$u(x) = \int_E \gamma(r) f(y) dy, \ x \in \mathbb{R}^n$$

and

$$u_i(x) = \int_E \frac{\partial}{\partial x^i} \gamma(r) f(y) dy, \ x \in \mathbb{R}^n,$$

converge absolutely and there holds

$$u \in C^1(\mathbb{R}^n),$$

$$\forall 1 \leq i \leq n \colon D_i u = u_i$$

as well as

$$|u|_{1,0,\mathbb{R}^n} \leq C(E) \|f\|_\infty.$$

Proof. The absolute convergence follows from 3.1.4.
Let $h > 0,\ r_h := (r^2 + h)^{\frac{1}{2}},\ u_h(x) := \int_E \gamma(r_h) f(y) dy \in C^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
|u(x) - u_h(x)| &\leq \int_E |\gamma(r) - \gamma(r_h)||f(y)|dy \leq \|f\|_\infty \int_E |\gamma(r) - \gamma(r_h)|dy \\
&\leq \|f\|_\infty \left(\int_{E \setminus B_\delta(x)} |\gamma(r) - \gamma(r_h)|dy + \int_{B_\delta(x)} |\gamma(r) - \gamma(r_h)|dy \right) \\
&\leq \|f\|_\infty \left(\int_{E \setminus B_\delta(x)} \left| \int_0^1 \frac{\partial}{\partial t} \gamma(r_{th}) dt \right| dy + \int_{B_\delta(x)} |2\gamma(r)|dy \right) \\
&\leq \|f\|_\infty \left(\int_{E \setminus B_\delta(x)} \int_0^1 |\dot{\gamma}(r_{th})| \frac{h}{(r^2 + th)^{\frac{1}{2}}} dt dy + O(\delta) \right) \\
&\leq \|f\|_\infty (C(E, \delta)h + O(\delta)) < \epsilon, \text{ if } h < h_0 = h_0(\delta).
\end{aligned}$$

The derivatives are treated likewise and the estimates follow from the preceding results.

□

3.1.6 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $f \in C^{0,\alpha}(\bar{\Omega}),\ 0 < \alpha \leq 1$. Then the volume potential $\omega(x) = \int_\Omega \gamma(r) f(y) dy$ satisfies*
(i) $\omega \in C^2(\Omega)$ and

$$-\Delta\omega = \begin{cases} -2\pi f, & \text{if } n = 2 \\ n(n-2)\omega_n f, & \text{if } n \geq 3. \end{cases}$$

(ii) Let $\Omega \subset \Omega_0 \Subset \mathbb{R}^n,\ \partial\Omega_0 \in C^1$ and set $f \equiv 0$ in $\Omega_0 \setminus \bar{\Omega}$, then the so called Dini formula holds:

$$D_i D_j \omega(x) = \int_{\Omega_0} D_i D_j \gamma(r)(f(y) - f(x)) dy - f(x) \int_{\partial\Omega_0} D_i \gamma(r) \nu_j, \quad (3.3)$$

where the derivatives are to be taken with respect to x.
(iii) $\forall \Omega' \Subset \Omega$: $|D^2\omega|_{\Omega'} \leq C(\Omega', \alpha)|f|_{0,\alpha,\Omega}$.

Proof. (ii) Set

$$u(x) := \int_{\Omega_0} D_i D_j \gamma(r)(f(y) - f(x)) dy - f(x) \int_{\partial\Omega_0} D_i \gamma(r) \nu_j,\ x \in \Omega.$$

u is well defined, since f is Hoelder continuous. Set $v := D_i\omega$ and choose $\eta \in C^1([0, \infty))$, such that $\eta_{|[0,1]} \equiv 0,\ \eta_{|[2,\infty)} \equiv 1$ and $|\dot{\eta}| \leq 2$. Then set $\eta_\epsilon(t) := \eta(\frac{t}{\epsilon})$. Now let

$$v_\epsilon(x) := \int_{\Omega_0} D_i \gamma(r) \eta_\epsilon(r) f(y) dy$$

$$\Rightarrow v_\epsilon \in C^\infty(\Omega)$$

and

$$\begin{aligned}
\Rightarrow D_j v_\epsilon(x) &= \int_\Omega D_j(D_i\gamma(r)\eta_\epsilon(r))f(y)dy \\
&= \int_{\Omega_0} D_j(D_i\gamma(r)\eta_\epsilon(r))(f(y)-f(x))dy \\
&+ f(x)\int_{\Omega_0} D_j(D_i\gamma(r)\eta_\epsilon(r))dy \\
&= \int_{\Omega_0} D_j(D_i\gamma(r)\eta_\epsilon(r))(f(y)-f(x))dy \\
&- f(x)\int_{\partial\Omega_0} D_i\gamma(r)\eta_\epsilon(r)\nu_j
\end{aligned}$$

Thus we have for ϵ sufficiently small

$$\begin{aligned}
|u(x) - D_j v_\epsilon(x)| &\le \left|\int_{\Omega_0} D_j((1-\eta_\epsilon)D_i\gamma)(f(y)-f(x))\right| \\
&\le [f]_\alpha \int_{|x-y|<2\epsilon} (D_iD_j\gamma|x-y|^\alpha + \frac{2}{\epsilon}|D_i\gamma||x-y|^\alpha) \\
&\le [f]_\alpha c\epsilon^\alpha
\end{aligned}$$

$$\Rightarrow \forall \Omega' \Subset \Omega\colon D_j v_\epsilon \rightrightarrows u \text{ in } \Omega'$$

$$\Rightarrow u \in C^0(\Omega)$$

and $v_\epsilon \rightrightarrows v = D_i\omega$.

$$\Rightarrow \omega \in C^2(\Omega),\ D_jD_i\omega = u.$$

(i) Let $x \in \Omega$, choose a ball $\Omega_0 = B_R(x),\ \Omega \Subset B_r(x)$ and apply (3.3)

$$\Rightarrow -\Delta\omega(x) = -f(x)\int_{\partial B_R}(2-n)R^{1-n} = n(n-2)\omega_n f(x),\ n \ge 3.$$

Analogueously for $n = 2$.
(iii) Apply (3.3) to $B_R(x)$.

□

3.1.7 Definition. Let $\Omega \subset \mathbb{R}^n$ be bounded, $d = \operatorname{diam}(\Omega)$, then we define in $C^k(\bar\Omega)$ resp. $C^{k,\alpha}(\bar\Omega)$ *dimension invariant norms,*

$$|u|'_{k,\Omega} = \sum_{j=0}^k d^j |u|_{j,\Omega}$$

$$|u|'_{k,\alpha,\Omega} = |u|'_{k,\Omega} + d^{k+\alpha}[D^k u]_{\alpha,\Omega}.$$

3.1.8 Remark. Let $u \in C^{0,\alpha}(\bar{\Omega}), v \in C^{0,\beta}(\bar{\Omega}), w \in C^1(\bar{\Omega})$ and $\gamma = \min(\alpha, \beta)$

$$\Rightarrow uv \in C^{0,\gamma}(\overline{\Omega}),$$

$$|uv|'_{0,\gamma,\Omega} \leq |u|'_{0,\alpha,\Omega}|v|'_{0,\beta,\Omega}$$

as well as

$$|uv|_{0,\gamma} \leq [u]_{\alpha,\Omega}|v|_{0,\beta} + [v]_{\beta,\Omega}|u|_{0,\alpha}.$$

Furthermore there holds

$$u \circ w \in C^{0,\alpha}(\overline{\Omega})$$

and

$$|u \circ w|_{0,\alpha} \leq [u]_{\alpha,\Omega}|w|^{\alpha}_{1,\Omega}.$$

Proof. Exercise. □

3.1.9 Example.

$$-\Delta u = f \text{ in } B_{2R}(0)$$

$$\Rightarrow |Du|_{B_R} \leq c|f|_{B_{2R}}.$$

In this case, however, the constant can not depend on n only. This becomes visible via a *scaling argument*.
Let $\epsilon > 0,\ u_\epsilon = u(\epsilon x)$

$$\Rightarrow Du_\epsilon = \epsilon Du(\epsilon x),\ -\Delta u_\epsilon = -\epsilon^2 \Delta u(\epsilon x)$$

$$\Rightarrow -\Delta u_\epsilon = \epsilon^2 f(\epsilon x) \equiv \tilde{f}$$

If the estimate holds, we have

$$|Du_\epsilon|_{B_R} \leq c|\tilde{f}|_{B_{2R}}$$

$$\Rightarrow \epsilon|Du|_{B_{\epsilon R}} \leq c\epsilon^2|f|_{B_{2\epsilon R}}.$$

Set $\epsilon = R^{-1}$, then $R \to \infty$ leads to a contradiction. Using the new norms, this problem does not arise, since the radius scales.

3.1.10 Theorem. *Let* $B_1 = B_R(x_0),\ B_2 = B_{3R}(x_0),\ f \in C^{0,\alpha}(\bar{B}_2)$, $0 < \alpha < 1$ *and*

$$\omega(x) = \int_{B_2} \gamma(r) f(y) dy.$$

Then $\omega \in C^{2,\alpha}(\bar{B}_1)$ *and*

$$|D^2\omega|'_{0,\alpha,B_1} \leq C|f|'_{0,\alpha,B_2},\ C = C(n, \alpha).$$

Proof. Let $x \in B_1$, then by (3.3) we have

$$D_iD_j\omega(x) = \int_{B_2} D_iD_j\gamma(r)(f(y) - f(x))dy - f(x)\int_{\partial B_2} D_i\gamma(r)\nu_j(y)$$

$$\Rightarrow |D_iD_j\omega(x)| \le C(n)\int_{B_2} \frac{1}{r^{n-\alpha}}[f]_{\alpha,B_2} + C|f|_{0,B_2} \le (CR^\alpha + C)|f|_{0,\alpha,B_2}.$$

Let $\overline{x} \in B_1$ be another point and $\overline{r} = |y - \overline{x}|$. Set

$$\delta = |x - \overline{x}|,\ \xi = \frac{1}{2}(x + \overline{x}).$$

Then

$$\begin{aligned} D_iD_j\omega(\overline{x}) - D_iD_j\omega(x) = f(x)I_1 + (f(x) - f(\overline{x}))I_2 + I_3 \\ + I_4 + (f(x) - f(\overline{x}))I_5 + I_6, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} I_1 &= \int_{\partial B_2} (D_i\gamma(r) - D_i\gamma(\overline{r}))\nu_j(y) \\ I_2 &= \int_{\partial B_2} D_i\gamma(\overline{r})\nu_j(y) \\ I_3 &= \int_{B_\delta(\xi)} D_iD_j\gamma(r)(f(x) - f(y)) \\ I_4 &= \int_{B_\delta(\xi)} D_iD_j\gamma(\overline{r})(f(y) - f(\overline{x})) \\ I_5 &= \int_{B_2\setminus B_\delta(\xi)} D_iD_j\gamma(r) \\ I_6 &= \int_{B_2\setminus B_\delta(\xi)} (D_iD_j\gamma(r) - D_iD_j\gamma(\overline{r}))(f(\overline{x}) - f(y)) \end{aligned}$$

Let $r_t = |tx + (1-t)\overline{x} - y|$. We derive the estimates

$$\begin{aligned} |I_1| &\le C\int_0^1 \frac{1}{r_t^n}\delta \le CR^{-1}\delta \le 2C(\frac{\delta}{2R})^\alpha = CR^{-\alpha}\delta^\alpha, \\ |I_2| &\le C, \\ |I_3| &\le C\int_{B_\delta(\xi)} \frac{1}{r^{n-\alpha}}[f]_\alpha = C\delta^\alpha[f]_\alpha, \\ |I_4| &\le C\delta^\alpha[f]_\alpha, \\ |I_5| &= \left|\int_{\partial(B_2\setminus B_\delta(\xi))} D_i\gamma(r)\nu_j\right| \le \int_{\partial B_2} |D_i\gamma(r)| + \int_{\partial B_\delta(\xi)} |D_i\gamma(r)| \le C, \\ |I_6| &\le \delta\int_0^1\int_{B_2\setminus B_\delta(\xi)} |DD_iD_j\gamma(r_t)||f(\overline{x}) - f(y)| \\ &\le \delta\int_0^1\int_{|y-\xi|\ge\delta} \frac{|\overline{x} - y|^\alpha}{|x_t - y|^{n+1}}[f]_\alpha. \end{aligned}$$

Now for $|y-\xi| \geq \delta$, we have

$$|\overline{x}-y| \leq |\overline{x}-\xi| + |\xi - y| < 2|\xi - y|$$

and

$$|x_t - y| \geq |y-\xi| - |\xi - x_t| \geq |y-\xi| - \frac{\delta}{2} \geq \frac{|y-\xi|}{2}.$$

$\rho := |y-\xi|$

$$\Rightarrow |I_6| \leq \delta C[f]_\alpha \int_\delta^\infty \frac{\rho^\alpha}{\rho^{n+1}}\rho^{n-1}d\rho = C\delta\frac{1}{1-\alpha}\delta^{\alpha-1}[f]_\alpha = C\frac{1}{1-\alpha}\delta^\alpha[f]_\alpha.$$

Combining the single estimates implies the claim.

□

3.1.11 Remark. Let $f \in C_c^{0,\alpha}(\mathbb{R}^n),\ 0<\alpha<1,\ \operatorname{supp} f \subset B_R(0)$

$$\omega(x) = \int_{\mathbb{R}^n} \gamma(r)f(y)dy.$$

Then

$$\omega \in C^{2,\alpha}(\mathbb{R}^n)$$

and

$$\begin{aligned}
(i)\ & [D^2\omega]_{\alpha,\mathbb{R}^n} \leq C[f]_{\alpha,\mathbb{R}^n},\ C = C(n,\alpha)\\
(ii)\ & |D^2\omega|'_{0,\alpha,B_R} \leq C|f|'_{0,\alpha,B_R},\ C = C(n,\alpha)\\
(iii)\ & |\omega|'_{1,B_R} \leq CR^2|f|_{0,B_R},\ C = C(n),\ n \geq 3.
\end{aligned}$$

Proof. (ii) follows from the previous theorem, since $f \in C_c^{0,\alpha}(B_R(0))$.
(iii) From 3.1.4 we obtain $|D\omega|_{0,B_R} \leq CR|f|_{0,B_R}$ and $|\omega| \leq CR^2|f|_{0,B_R}$

$$\Rightarrow |\omega|'_{1,B_R} = |\omega|_{0,B_R} + R|D\omega|_{0,B_R} \leq CR^2|f|_{0,B_R}.$$

(i) From (ii) we deduce

$$R^\alpha[D^2\omega]_{\alpha,B_R} \leq C(|f|_{0,B_R} + R^\alpha[f]_{\alpha,B_R}).$$

Dividing by R^α and $R\to\infty$ imply the claim. □

3.1.12 Theorem. *Let $f \in C_c^{0,\alpha}(\mathbb{R}^n)$, $u \in C_c^{2,\alpha}(\mathbb{R}^n)$ and $-\Delta u = f$.*

$$\Rightarrow u = c_n \int_{\mathbb{R}^n} \gamma(r)f(y),$$

where

$$c_n = \begin{cases} \frac{1}{n(n-2)\omega_n}, & n \geq 3 \\ -\frac{1}{2\pi}, & n = 2 \end{cases}.$$

Proof. Let $\omega = c_n \int_{\mathbb{R}^n} \gamma(r) f(y)$, then $-\Delta\omega = f$. Set $v := u - \omega$ to obtain $\Delta v = 0$. By the maximum principle we have $\sup v = \limsup_{|x|\to\infty} v(x) = \limsup_{|x|\to\infty}(-\omega)$. But

$$\omega = -c_n \int_{\mathbb{R}^n} \gamma(r)\Delta u \to 0.$$

$\square$

3.2 Boundary estimates for potentials

3.2.1 Theorem. *Let* $\mathbb{R}^n_+ = \{x^n > 0\}$, $x_0 \in \partial\mathbb{R}^n_+$, $B_1^+(x_0) = B_R^+(x_0) = B_R(x_0) \cap \mathbb{R}^n_+$, $B_2^+(x_0) = B_{3R}^+(x_0)$, $f \in C^{0,\alpha}(\bar{B}_2^+)$, $0 < \alpha < 1$. *Then for* $\omega(x) = \int_{B_2^+} \gamma(r) f(y) dy$ *we have*

$$\omega \in C^{2,\alpha}(\bar{B}_1^+)$$

and

$$|D^2\omega|_{0,B_1^+} + R^\alpha [D^2\omega]_{\alpha,B_1^+} \le c(|f|_{0,B_2^+} + R^\alpha [f]_{\alpha,B_2^+}). \tag{3.5}$$

Proof. Let $x \in B_1^+$ and apply Dini's formula to B_2^+, where the boundary integral does not vanish over $\partial B_2^+(x_0) \cap \mathbb{R}^n_+$ only, since $\nu_j = 0 \ \forall 1 \le j < n$. Then the proof of 3.1.10 carries over literally for either $i \ne n$ or $j \ne n$. For $i = j = n$ we use the equation and the estimates for $\omega_{kk}, 1 \le k < n$. $\square$

3.2.2 Corollary. *Let* $f \in C_c^{0,\alpha}(\overline{\mathbb{R}^n_+})$, $0 < \alpha < 1$, $\omega(x) = \int_{\mathbb{R}^n_+} \gamma(r) f(y)$, *then*

$$[D^2\omega]_{\alpha,\mathbb{R}^n_+} \le c(n,\alpha)[f]_{\alpha,\mathbb{R}^n_+}.$$

Proof. (3.5) implies

$$R^\alpha [D^2\omega]_{\alpha,B_R^+} \le c(|f|_{0,B_{3R}^+} + R^\alpha [f]_{\alpha,B_{3R}^+}).$$

Divide by R^α and send $R \to \infty$. $\square$

3.3 Harmonic functions and Green's function

3.3.1 Definition. Let $\Omega \subset \mathbb{R}^n$ be open. A function $u \in C^2(\Omega)$ is called *harmonic, subharmonic* or *superharmonic*, if $-\Delta u = 0, -\Delta u \le 0$ or $-\Delta u \ge 0$.

3.3.2 Theorem. *Let* $u \in C^2(\Omega)$, $-\Delta u = 0$ $(\le 0, \ge 0)$. *Then* $\forall B_R(y) \Subset \Omega$:

$$u(y) = (\le, \ge) \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u \tag{3.6}$$

and

$$u(y) = (\leq, \geq) \frac{1}{\omega_n R^n} \int_{B_R} u. \tag{3.7}$$

Proof. We only show this for subharmonic functions, the other cases follow by considering $u \to -u$. Let $0 < \rho < R$, $B_\rho = B_\rho(y)$.

$$\Rightarrow \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} = \int_{B_\rho} \Delta u \geq 0.$$

Let (r, ξ^i) be polar coordinates centered at y, $x = y + r\xi$. Then we have

$$\begin{aligned} 0 \leq \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} &= \rho^{n-1} \int_{\mathbb{S}^{n-1}} D_i u(y + \rho\xi)\xi^i \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\mathbb{S}^{n-1}} u(y + \rho\xi) \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} (\rho^{1-n} \int_{\partial B_\rho} u). \end{aligned}$$

$$\Rightarrow \rho^{1-n} \int_{\partial B_\rho} u \leq R^{1-n} \int_{\partial B_R} u.$$

$\rho \to 0$ implies (3.6). For all $\rho \leq R$ there holds (3.6). Integrating on both sides from 0 to R yields (3.7). □

3.3.3 Theorem. *(Harnack)*
Let $0 \leq u \in C^2(\Omega)$ be harmonic and Ω connected. Then there holds

$$\forall \Omega' \Subset \Omega\colon \sup_{\Omega'} u \leq c(\Omega') \inf_{\Omega'} u.$$

Proof. Let $y \in \Omega, B_{4R}(y) \subset \Omega$. $x_1, x_2 \in B_R(y)$

$$\Rightarrow u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} u \leq \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} u$$

and

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int_{B_{3R}(x_1)} u \geq \frac{1}{\omega_n 3^n R^n} \int_{B_{2R}(y)} u.$$

$$\Rightarrow u(x_1) \leq 3^n u(x_2) \ \forall x_i \in B_R(y),$$

$$\Rightarrow \sup_{B_R(y)} u \leq 3^n \inf_{B_R(y)} u.$$

Let $\Omega' \Subset \Omega$ wlog connected. Then finitely many balls $B_R(x_i), 1 \leq i \leq N$, cover $\overline{\Omega'}$ and satisfy $B_{4R}(x_i) \subset \Omega$. Let $x, y \in \Omega'$. We claim, that there is a continuous path γ with the following properties:

$$\Gamma := \{\gamma(t) : 0 \le t \le 1\} \subset \Omega', \ \gamma(0) = x, \ \gamma(1) = y,$$
$$\Gamma \subset \bigcup_{k=1}^{l} B_R(x_{i_k}), \ B_R(x_{i_k}) \cap B_R(x_{i_{k+1}}) \neq \emptyset,$$
$$B_R(x_{i_k}) \neq B_R(x_{i_m}), \ m \neq k, \ \Gamma \cap B_R(x_{i_k}) \neq \emptyset \text{ and}$$
$$\gamma(1) \in B_R(x_{i_l}).$$

Proof of existence: Let γ be a continuous path in Ω' from x to y. Let

$$\Lambda := \{t \in [0,1] : \gamma(0) \text{ and } \gamma(t) \text{ can be connected this way}\}.$$

$$\Rightarrow \Lambda \neq \emptyset, \ \Lambda \text{ open.}$$

Let $t_n \in \Lambda, \ t_n \to t_0, \ \gamma(t_0) \in B_R(x_i)$

$$\Rightarrow \gamma(t_n) \in B_R(x_i), n \text{ large.}$$

Let γ_n be such a path, connecting $\gamma(0)$ and $\gamma(t_n)$. Then there are two cases:
(a) $B_R(x_i) \neq B_R(x_{i_k})$ $\forall k$. Then set $B_R(x_{i_{l+1}}) = B_R(x_i)$.
(b) $B_R(x_i) = B_R(x_{i_k})$ for some k. Then $\gamma(t_0) \in B_R(x_{i_k})$ and $\gamma(t_n) \in B_R(x_{i_k})$. Thus you may connected inside the ball and obtain a new path of this kind.
Now let $y_1, y_2 \in \Omega'$ be connected by such a chain and $y \in B_R(x_{i_1}) \cap B_R(x_{i_2})$

$$\Rightarrow u(y_1) \le \sup_{B_R(x_{i_1})} u \le 3^n \inf_{B_R(x_{i_1})} u \le 3^n u(y) \le 3^n \sup_{B_R(x_{i_2})} u \le \dots \le 3^{nN} u(y_2).$$

Taking the supremum and infimum implies the claim. □

3.3.4 Proposition. *(Greensche Identitäten)*
Let $\partial\Omega \in C^{0,1}, \ u, v \in C^2(\Omega) \cap C^1(\bar{\Omega}), \ \Delta u \in L^1(\Omega)$. Then there hold

$$\int_\Omega v\Delta u + \int_\Omega Du \cdot Dv = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \tag{3.8}$$

and

$$\int_\Omega (v\Delta u - u\Delta v) = \int_{\partial\Omega} (v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}) \tag{3.9}$$

Proof. (3.8) follows by applying the divergence theorem to $v\nabla u$. (3.9) follows by replacing u by v in (3.8) and then subtracting the equations. □

Now let $\Omega \Subset \mathbb{R}^n, \ y \in \Omega, \ B_\delta(y) \Subset \Omega, \ \Omega_\delta := \Omega \backslash B_\delta(y)$. Apply (3.9) in Ω_δ with $v = -c_n\gamma(r)$.

$$\Rightarrow \int_{\Omega_\delta} (-c_n)\gamma\Delta u = \int_{\partial\Omega} \left((-c_n)\gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \gamma}{\partial \nu}\right) + \int_{\partial B_\delta} (-c_n)\left(\gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \gamma}{\partial \nu}\right).$$

$\delta \to 0$

$$\Rightarrow u(y) = \int_\Omega (-c_n)\gamma\Delta u + \int_{\partial\Omega} \left(u\frac{\partial(-c_n)\gamma}{\partial\nu} - (-c_n)\gamma\frac{\partial u}{\partial\nu} \right) \tag{3.10}$$

This formula is called *Green's representation theorem.* If u is harmonic, we deduce that u uniquely determined by its boundary values. Since γ is real analytic away from the singularity, harmonic functions are also analytic.
Now let $h \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be harmonic, then (3.9) implies

$$0 = \int_{\partial\Omega} \left(u\frac{\partial h}{\partial\nu} - h\frac{\partial u}{\partial\nu} \right) + \int_\Omega h\Delta u.$$

Add this formula to the Green's representation, you obtain for $G = -c_n\gamma + h$

$$u(y) = \int_{\partial\Omega} (u\frac{\partial G}{\partial\nu} - G\frac{\partial u}{\partial\nu}) + \int_\Omega G\Delta u. \tag{3.11}$$

If h can be chosen to satisfy

$$\forall y \in \Omega \colon G(\cdot, y)_{|\partial\Omega} = 0.$$

it follows that

$$u(y) = \int_{\partial\Omega} u\frac{\partial G}{\partial\nu} + \int_\Omega G\Delta u.$$

G is then called *Green's function* for the Laplacian.

We now determine Green's function for a ball.

3.3.5 Definition. Let $B_R = B_R(0)$. Define the *inversion*

$$T : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n},\ \overline{x} = Tx$$

by

$$\overline{x} = R^2\frac{x}{|x|^2},\ x \neq 0,\ \overline{0} := \infty,\ \overline{\infty} = 0.$$

Let $\tilde{\gamma} = -c_n\gamma$. Define Green's function for $\Omega = B_R(0)$ by

$$\begin{aligned} G(x,y) &= \begin{cases} \tilde{\gamma}(|x-y|) - \tilde{\gamma}\left(\frac{|y|}{R}|x-\overline{y}|\right), & \text{if } y \neq 0 \\ \tilde{\gamma}(|x|) - \tilde{\gamma}(R), & \text{if } y = 0 \end{cases} \\ &= \tilde{\gamma}\left(\sqrt{x^2+y^2-2\langle x,y\rangle}\right) - \tilde{\gamma}\left(\sqrt{\left(\frac{|x||y|}{R}\right)^2 + R^2 - 2\langle x,y\rangle}\right) \end{aligned}$$

G has the following properties:

- $G(x,y) = G(y,x)$.

- $\forall z \in \Omega\colon \Delta_x G(\cdot, y)_{|\{x \neq y\}} = 0$.
- $\forall y \in \Omega\colon G(\cdot, y)_{|\partial B_R} = 0$.
- $x \in \partial B_R \wedge y \in B_R \Rightarrow \frac{\partial G}{\partial \nu}(x, y) = \frac{R^2 - |y|^2}{n\omega_n R}|x - y|^{-n} > 0$.
- $G(x, y) \leq 0$ by and the maximum principle.

Plug G into the representation formula, in case $\Delta u = 0$ the *Poisson integral formula* follows:

$$u(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B_R} \frac{u(y)}{|x - y|^n} \equiv \int_{\partial B_R} K(x, y) u(y) dy \tag{3.12}$$

K is called *Poisson kernel.* Using approximation one obtains this formula for all harmonic $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. On the other hand let $\phi \in C^0(\partial B_R)$ and

$$u(x) = \int_{\partial B_R} K(x, y)\phi(y) dy,$$

then $u \in C^\infty(B_R),\ \Delta u = 0$. Furthermore we have

3.3.6 Theorem. *Let $\phi \in C^0(\partial B_R)$ and*

$$u(x) = \int_{\partial B_R} K(x, y)\phi(y) dy,\ x \in B_R.$$

Then

$$u \in C^\infty(B_R) \cap C^0(\bar{B}_R)$$

and

$$\Delta u = 0,\ u_{|\partial B_R} = \phi.$$

Proof. It suffices to show $u \in C^0(\bar{B}_R)$. Choosing $u \equiv 1$ we obtain

$$\int_{\partial B_R} K(x, y) = 1.$$

Let $x_0 \in \partial B_R,\ \epsilon, \delta > 0$ such that

$$\forall |x - x_0| < \delta\colon |\phi(x) - \phi(x_0)| < \epsilon.$$

$$\begin{aligned}
|u(x) - \phi(x_0)| &= \left| \int_{\partial B_R} K(\phi - \phi(x_0)) \right| \\
&\leq \Big| \int_{\partial B_R \cap \{|y - x_0| < \delta\}} K(\phi - \phi(x_0)) \\
&+ \int_{\partial B_R \cap \{|y - x_0| \geq \delta\}} K(\phi - \phi(x_0)) \Big| \\
&\leq \epsilon + \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B_R \cap \{|y - x_0| \geq \delta\}} \frac{1}{|x - y|^n} \cdot 2\|\phi\|_\infty \qquad (3.13) \\
&\to \epsilon,\ x \to x_0.
\end{aligned}$$

□

3.3.7 Remark. The continuity up to the boundary also holds locally, even if ϕ is not continuous everywhere.

3.3.8 Proposition. *Let Ω be a domain and let $u \in C^0(\bar{\Omega})$ satisfy the mean value equality (3.7), then u does not attain a maximum in Ω, unless u is constant.*

Proof. Let $m = \sup_\Omega u < \infty$ and suppose $m = u(x_0),\ x_0 \in \Omega$. Set

$$\Lambda := \{u = m\}.$$

Then $\Lambda \neq \emptyset$ is closed.

$$m = u(y_0) \leq \frac{1}{\omega_n R^n} \int_{B_\rho(y_0)} u \leq m$$

$$\Rightarrow u_{|B_\rho} \equiv m,$$

since u is continuous. Thus Λ is open. □

3.3.9 Theorem. *$u \in C^0(\Omega)$ is harmonic, i.e. $u \in C^2(\Omega)$ and $\Delta u = 0$, if and only if u has the mean value property for all $y \in \Omega$ and $B_R(y) \Subset \Omega$.*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u,$$
$$u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u.$$

Proof. Since the second equation follows from the first and the 'only if' part has already been proven, it is left to show the harmonicity from the second equation. So let $B_\delta(y) \Subset \Omega$. Let $h \in C^2(B_\delta) \cap C^0(\bar{B}_\delta)$, such that $\Delta h = 0,\ h = u$ on $\partial B_\delta(y)$. Set

$$w := u - h,$$

then w satisfies the second mean value equation. There holds

$$0 = \inf_{\partial B_\delta(y)} w \leq w \leq \sup_{\partial B_\delta(y)} w = 0,$$

implying $w = 0$. □

3.3.10 Theorem. *Let $u_n \in C^0(\bar{\Omega})$ be a sequence of harmonic functions with $u_n \rightrightarrows u$, then u is harmonic.*

Proof. Follows from the theorem above and the stability of the integral under uniform convergence. □

3.3.11 Theorem. *Let Ω be a domain, u_n a monotone sequence of harmonic functions, converging in $y \in \Omega$. Then the whole sequence converges locally uniformly to a harmonic function.*

Proof. W.l.o.g. let the sequence be increasing. Let $\Omega' \Subset \Omega$, $y \in \Omega'$. Let $\epsilon > 0$

$$\Rightarrow \exists n_0 \in \mathbb{N}\ \forall l > k > n_0 \colon 0 \leq u_l(y) - u_k(y) < \epsilon.$$

Apply Harnack's inequality to $u_l - u_k$.

$$\Rightarrow \sup_{\Omega'}(u_l - u_k) \leq C \inf_{\Omega'}(u_l - u_k) \leq C\epsilon.$$

□

3.3.12 Theorem. *Let u be harmonic in Ω, $\Omega' \Subset \Omega$, $d = \operatorname{dist}(\Omega', \partial\Omega)$. Then*

$$\forall \alpha \in \mathbb{N}^n \colon |D^\alpha u|_{0,\Omega'} \leq \left(\frac{n \cdot 2^{|\alpha|}}{d}\right)^{|\alpha|} |u|_{0,\Omega}.$$

Proof. Induction for $|\alpha|$. $|\alpha| = 1$:
Let $B_R(y) \Subset \Omega$. Du is also harmonic, thus

$$Du(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} Du = \frac{1}{\omega_n R^n} \int_{\partial B_R(y)} u\nu$$

$$\Rightarrow |Du(y)| \leq \frac{n}{R} \sup_{B_R(y)} |u|.$$

Let $y \in \Omega'$. Choose $R = \frac{d}{2}$

$$\Rightarrow |Du(y)| \leq \frac{2n}{d} |u|_{0,\Omega}.$$

Let the claim be proven for $|\alpha| \geq 1$.

$$\begin{aligned}
|DD^\alpha u(y)| &\leq \frac{2n}{d} |D^\alpha u|_{0,B_{\frac{\delta}{2}}(y)} \\
&\leq \frac{2n}{d} \left(\frac{n \cdot 2^{|\alpha|}}{\frac{d}{2}}\right)^{|\alpha|} |u|_{0,\Omega} \\
&= \left(\frac{n}{d}\right)^{|\alpha|+1} \cdot 2 \cdot 2^{|\alpha|(|\alpha|+1)} |u|_{0,\Omega} \\
&\leq \left(\frac{n}{d}\right)^{|\alpha|+1} \cdot 2^{(|\alpha|+1)^2} |u|_{0,\Omega}.
\end{aligned}$$

□

3.3.13 Theorem. *Every bounded sequence (u_k) of harmonic functions contains a subsequence, converging uniformly on compact subsets to a harmonic function u.*

Proof. Let $\Omega' \Subset \Omega$

$$\Rightarrow |u_n|_{3,0,\Omega'} \leq c(\Omega')|u_n|_{0,\Omega}.$$

Arzela-Ascoli implies $u_{n_k} \to u_{\Omega'}$ in $C^2(\bar{\Omega}')$. Choosing an exhaustion and applying the diagonal method implies the claim. □

3.3.14 Theorem. *(Liouville)*
Let $u \in C^2(\mathbb{R}^n)$ be harmonic and bounded, then $u \equiv const.$

Proof. Let $y \in \mathbb{R}^n$

$$\Rightarrow |Du(y)| \leq \frac{n}{R}|u|_{0,B_R} \to 0, \;\; R \to \infty.$$

□

3.4 Perron's method

3.4.1 Definition. Let $\Omega \subset \mathbb{R}^n$ be open. A function $u \in C^0(\Omega)$ is called
(i) *subharmonic*, if

$$\forall B_\rho \Subset \Omega \; \forall h \in C^2(B_\rho) \cap C^0(\bar{B}_\rho)[\Delta h = 0, \;\; h_{|\partial B_\rho} \geq u_{|\partial B_\rho}] \colon h \geq u,$$

(ii) *superharmonic*, if

$$\forall B_\rho \Subset \Omega \; \forall h \in C^2(B_\rho) \cap C^0(\bar{B}_\rho)[\Delta h = 0, \;\; h_{|\partial B_\rho} \leq u_{|\partial B_\rho}] \colon h \leq u.$$

3.4.2 Remark. Another possibility to define sub- and superharmonicity for continuous functions is to demand (3.6), (3.7) for all compactly contained balls. Those definitions would then be equivalent.

Proof. Exercise. □

3.4.3 Proposition. *Let the notions of sub-, super- and -harmonicity be defined by mean value properties and Ω be connected. Then*
(i) u subharmonic in $\Omega \Rightarrow u < \sup_\Omega u$, unless u is constant.
(ii) u subharmonic, v superharmonic, $u, v \in C^0(\bar{\Omega}) \wedge v \geq u$ on $\partial\Omega$

$$\Rightarrow v > u \text{ in } \Omega \vee u \equiv v.$$

Proof. (i) has already been shown.

(ii) $u - v$ is subharmonic $\Rightarrow (u - v) < \sup_\Omega (u - v) \vee (u - v) \equiv$ const. □

3.4.4 Lemma. *Let u be subharmonic in Ω, $B = B_\rho \Subset \Omega$, h harmonic in B_ρ with $h_{|\partial B_\rho} = u$. Set*

$$\tilde{u} = \begin{cases} h & \text{in } B_\rho \\ u & \text{in } B_\rho^c \end{cases}$$

Then $\tilde{u} \in C^0(\Omega)$ and $\tilde{u}$ is subharmonic. $\tilde{u}$ is called harmonic substitute of u in B_ρ.

Proof. $\tilde{u}$ is clearly continuous. Let $B_R \Subset \Omega$, $\Delta v = 0$ in B_R, $v_{|\partial B_R} \geq \tilde{u}_{|\partial B_R}$.

$$u \leq \tilde{u} \text{ in } \Omega \Rightarrow u \leq v \text{ in } B_R$$

$$\Rightarrow \tilde{u} \leq v \text{ in } B_R \backslash B_\rho.$$

In $B_R \cap B_\rho$ v and $\tilde{u} = h$ are harmonic

$$\Rightarrow \tilde{u} - v \leq \sup_{\partial(B_R \cap B_\rho)} (\tilde{u} - v) \leq 0.$$

□

3.4.5 Lemma. *Let $u_1, ..., u_N$ be subharmonic, then $\max\{u_i\}_{1 \leq i \leq N}$ is also subharmonic. The minimum of finitely many superharmonic functions is superharmonic.*

Proof. Follows at once from the mean value properties. □

3.4.6 Definition. Let $\Omega \Subset \mathbb{R}^n$, $\phi \in L^\infty(\partial\Omega)$.
(i) $u \in C^0(\bar{\Omega})$ is called *subfunction rel* ϕ, if u is subharmonic and $u_{|\partial\Omega} \leq \phi$.
(ii) $u \in C^0(\bar{\Omega})$ is called *superfunction rel* ϕ, if u is superharmonic and $u_{|\partial\Omega} \geq \phi$.
(iii) S_ϕ is labeling the set of all subfunctions.

Especially constant functions satisfying the inequality on the boundary are subfunctions or superfunctions respectively.

3.4.7 Theorem. *(Perron)*
Let $\Omega \Subset \mathbb{R}^n$ and $\phi \in L^\infty(\partial\Omega)$. Let $u = \sup\{v \in S_\phi\}$. Then u is harmonic.

Proof. (i) Let $v \in S_\phi \Rightarrow v \leq \sup_{\partial\Omega} v \leq \sup_{\partial\Omega} \phi$. Thus u is well defined.
(ii) Let $y \in \Omega \Rightarrow \; \exists v_n \in S_\phi : v_n(y) \to u(y)$. W.l.o.g. let $|v_n| \leq$ const, otherwise consider

$$\max(v_n, \min(\inf_{\partial\Omega} \phi, u(y))) \in S_\phi.$$

Let $B_R(y) \Subset \Omega$ and $\tilde{v}_n$ the harmonic substitute of v_n in B_R.

$$\Rightarrow v_n \leq \tilde{v}_n \leq u$$

$$\Rightarrow \exists \tilde{v}_{n_k} \to v$$

locally uniformly in $B_R(y)$, such that $\Delta v = 0$ in $B_R(y)$.
Suppose $v \neq u$ in $B_R(y)$.

$$\Rightarrow \exists z \in B_R(y) \colon v(z) < u(z)$$

$$\Rightarrow \exists u_0 \in S_\phi \colon v(z) < u_0(z) \leq u(z).$$

Define

$$w_k := \max(u_0, \tilde{v}_{n_k}) \in S_\phi$$

and let $\tilde{w}_k$ be the harmonic substitute in $B_R(y)$.

$$\Rightarrow \tilde{v}_{n_k} \le w_k \le \tilde{w}_k$$

and

$$\exists \tilde{w}_{k_l} \rightrightarrows w$$

locally in $B_R(y)$, such that $\Delta w = 0$.

$$\Rightarrow v \le w \le u$$

$$\Rightarrow u(y) = v(y) \le w(y) \le u(y)$$
$$\Rightarrow v(y) = w(y)$$
$$\Rightarrow v \equiv w \text{ in } B_R(y),$$

in contradiction with $v(z) < w(z)$. □

3.4.8 Theorem. *Let $\Omega \Subset \mathbb{R}^n$. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solution of the boundary value problem*

$$\begin{aligned} \Delta u &= 0 \textit{ in } \Omega \\ u_{|\partial\Omega} &= \phi. \end{aligned} \tag{3.14}$$

Then

$$u = \sup\{v \in S_\phi\}.$$

Proof. Let $w = \sup\{v \in S_\phi\}$

$$\Rightarrow u \le w,$$

since $u \in S_\phi$. Let furthermore $v \in S_\phi$, then we have $v \le u$, since $v - u \in C^0(\bar{\Omega})$ is subharmonic and thus $v - u \le \sup_{\partial\Omega}(v - u)$. Thus there holds $w \le u$. □

u is then called the *Perron solution* of the boundary value problem.

3.4.9 Definition. Let $\Omega \subset \mathbb{R}^n$ be open. (i) Let $\xi \in \partial\Omega$. A function $w \in C^0(\bar{\Omega})$ is called *upper barrier in $\xi \in \partial\Omega$*, if

$$\begin{aligned} &(a)\ w \text{ is superharmonic} \\ &(b)\ w(\xi) = 0,\ w > 0 \text{ in } \bar{\Omega}\backslash\{\xi\}. \end{aligned}$$

(ii) w is called *local upper barrier in ξ*, if for some $R > 0$ we have $w \in C^0(\overline{\Omega \cap B_R(\xi)})$ as well as (a), (b) in $\bar{\Omega} \cap B_R(\xi)$.

3.4.10 Proposition. *If there is a local barrier in $\xi \in \partial\Omega$, then there is also a global one.*

Proof. Let w be the barrier in $\Omega \cap B_R(\xi)$. Set

$$m := \inf\{w(x)\colon\ x \in \Omega \cap B_R(\xi) \backslash B_{\frac{R}{2}}(\xi)\} > 0.$$

Define

$$\overline{w}(x) := \begin{cases} \min(m, w(x)), & x \in \bar{\Omega} \cap B_{\frac{R}{2}}(\xi) \\ m, & \text{otherwise.} \end{cases}$$

$\overline{w} \in C^0(\bar{\Omega})$, since $\overline{w}_{|\Omega \cap (B_R \backslash B_{\frac{R}{2}})} = m$. $\overline{w}$ is superharmonic, $\overline{w} > 0$ in $\bar{\Omega}\backslash\{\xi\}$ and $\overline{w}(\xi) = 0$. □

3.4.11 Definition. A boundary point $\xi \in \partial\Omega$ is called *regular with respect to* Δ, if there is an upper barrier in ξ.

3.4.12 Lemma. *Let $\Omega \Subset \mathbb{R}^n$. Let u be the Perron solution of the boundary value problem*

$$\Delta u = 0 \ \textit{in}\ \Omega$$
$$u_{|\partial\Omega} = \phi.$$

If $\xi \in \partial\Omega$ is regular and ϕ continuous in ξ, then

$$\lim_{x \to \xi} u(x) = \phi(\xi).$$

Proof. Let $\epsilon > 0$, $m = \sup_{\partial\Omega} |\phi|$. Let w be a barrier in ξ, then

$\exists \delta > 0\ \exists k > 0\colon |x - \xi| < \delta \Rightarrow |\phi(x) - \phi(\xi)| < \epsilon \wedge |x - \xi| \geq \delta \Rightarrow kw(x) \geq 2m.$

The functions

$$w^+(x) = \phi(\xi) + \epsilon + kw(x)$$

and

$$w^-(x) = \phi(\xi) - \epsilon - kw(x)$$

are super- (sub-) solutions.

$$w^- \in S_\phi \Rightarrow w^- \leq u.$$

Furthermore let $v \in S_\phi$. Then

$$v - w^+ \leq \sup_{\partial\Omega}(v - w^+) \leq 0$$
$$\Rightarrow \forall v \in S_\phi\colon v \leq w^+.$$
$$\Rightarrow u \leq w^+.$$

Thus we have

$$w^- \leq u \leq w^+$$

and the claim follows. □

3.4.13 Theorem. *Let $\Omega \Subset \mathbb{R}^n$. Then the classical Dirichlet problem*

$$\Delta u = 0 \text{ in } \Omega, \; u_{|\partial\Omega} = \phi \tag{3.15}$$

is solvable for arbitrary $\phi \in C^0(\partial\Omega)$ in $C^2(\Omega) \cap C^0(\bar{\Omega})$ if and only if every boundary point is regular.

Proof. By the preceeding lemma the Perron solution solves (3.15). So let $\xi \in \partial\Omega$. Define $\psi(x) = |x - \xi|$. Let $w \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be the solution of $\Delta w = 0, \; w_{|\partial\Omega} = \psi$. Then w is a barrier. □

3.4.14 Proposition. *Let $\Omega \Subset \mathbb{R}^n$. Suppose $\partial\Omega$ satisfies an exterior sphere condition with radius R. Then every boundary point is regular.*

Proof.

$$w(x) := \begin{cases} R^{2-n} - |x-y|^{2-n}, & n \geq 3 \\ \log(\frac{|x-y|}{R}), & n = 2, \end{cases}$$

where y is the center of the outer ball. Then

$$w \in C^\infty(\bar{\Omega}), \; \Delta w = 0.$$

Thus w is a barrier. □

3.5 Schauder a priori bounds

3.5.1 Lemma. *(Compactness lemma)*
Let E_i, $i = 1, 2, 3$, be Banach spaces and suppose we have embeddings

$$E_1 \xrightarrow{compact} E_2 \xrightarrow{continuous} E_3,$$

then there holds

$$\forall \epsilon > 0 \; \exists c \in \mathbb{R} \; \forall u \in E_1 \colon \|u\|_2 \leq \epsilon \|u\|_1 + c\|u\|_3.$$

Proof. Suppose the claim not to be true. Then there exists $\epsilon > 0$ and a sequence $(u_n)_{n \in \mathbb{N}}$ with $\|u_n\|_2 = 1$, such that

$$\forall n \in \mathbb{N} \colon 1 > \epsilon \|u_n\|_1 + n\|u_n\|_3. \tag{3.16}$$

Thus (u_n) is bounded in E_1 and contains a subsequence (u_{n_k}), which converges in E_2 to a limit u. By (3.16) this subsequence has to converge to $0 \in E_3$. By injectivity of the second map u must be zero in E_2, which is a contradiction.

□

In particular we obtain

3.5.2 Corollary. *Let $\Omega \Subset \mathbb{R}^n$, then for all $u \in C^{2,\alpha}(\bar{\Omega})$ and $\epsilon > 0$*

$$|u|_{2,0,\Omega} \leq \epsilon |u|_{2,\alpha,\Omega} + c_\epsilon |u|_{0,\Omega}.$$

3.5.3 Theorem. *(Schwarz reflection principle)*
Let $u \in C^2(\overline{B_R^+(0)})$ be harmonic, $u_{|\{x^n=0\}} = 0$. Then the reflectively extended function

$$\tilde{u}(\hat{x}, x^n) := \begin{cases} u(x), & x^n \geq 0 \\ -u(\hat{x}, -x^n), & x^n < 0 \end{cases}$$

is harmonic in $B_R(0)$.

Proof. The function is clearly continuous. In each point away from the axis $\{x^n = 0\}$ we have the mean value property. For centers in $\{x^n = 0\}$ we also deduce the mean value property, since the function is anti symmetric. □

3.5.4 Lemma. *Let $u \in C_c^2(\overline{\mathbb{R}^n_+})$, $f \in C^{0,\alpha}(\overline{\mathbb{R}^n_+})$, $0 < \alpha < 1$ and suppose*

$$\Delta u = f \text{ in } \mathbb{R}^n_+$$
$$u(\hat{x}, 0) = 0,$$

where $\hat{x} = (x^1, \dots, x^{n-1})$.
Then

$$u \in C^{2,\alpha}(\overline{\mathbb{R}^n_+})$$

and

$$[D^2 u]_{\alpha,\mathbb{R}^n_+} \leq c[f]_{\alpha,\mathbb{R}^n_+}, \ c = c(n, \alpha).$$

Proof. Let $\tilde{f}$ be the even reflection of f to $\mathbb{R}^n$.Then $\tilde{f} \in C_c^{0,\alpha}(\mathbb{R}^n)$ and there holds

$$[\tilde{f}]_{\alpha,\mathbb{R}^n} \leq 2[f]_{\alpha,\mathbb{R}^n_+}.$$

For $x = (\hat{x}, x^n)$ let $\tilde{x} = (\hat{x}, -x^n)$. In $\mathbb{R}^n_+$ define

$$\begin{aligned} \omega(x) &= c_n \int_{\mathbb{R}^n_+} (\gamma(|x-y|) - \gamma(|\tilde{x}-y|)) f(y) dy \\ &= c_n \int_{\mathbb{R}^n_+} (\gamma(|x-y|) - \gamma(|x-\tilde{y}|)) f(y) dy. \end{aligned}$$

$$\Rightarrow \Delta\omega = f, \ \omega \in C^{2,\alpha}(\mathbb{R}^n_+) \text{ and } \omega(\hat{x}, 0) = 0.$$

Furthermore there holds

$$\int_{\mathbb{R}^n_+} \gamma(|x-\tilde{y}|) f(y) = \int_{\mathbb{R}^n_-} \gamma(|x-y|) \tilde{f}(y).$$

$$
\begin{aligned}
\Rightarrow \omega(x) &= c_n \int_{\mathbb{R}^n_+} \gamma(|x-y|) f(y) - c_n \int_{\mathbb{R}^n_-} \gamma(|x-y|) \tilde{f}(y) \\
&+ c_n \int_{\mathbb{R}^n_+} \gamma(|x-y|) f(y) - c_n \int_{\mathbb{R}^n_+} \gamma(|x-y|) \tilde{f}(y) \\
&= 2c_n \int_{\mathbb{R}^n_+} \gamma(|x-y|) f(y) - c_n \int_{\mathbb{R}^n} \gamma(|x-y|) \tilde{f}(y) \\
&\equiv \omega_1(x) + \omega_2(x)
\end{aligned}
$$

$$
\Rightarrow [D^2 \omega_1]_{\alpha, \mathbb{R}^n_+} \le c [f]_{\alpha, \mathbb{R}^n_+} \text{ by } (3.5)
$$

and

$$
[D^2 \omega_2]_{\alpha, \mathbb{R}^n_+} \le c [\tilde{f}]_{\alpha, \mathbb{R}^n} \text{ by } 3.1.10.
$$

We now show $\omega = u$. Set $v := u - \omega$.

$$
\Rightarrow \Delta v = 0 \ \wedge \ v_{|\{x^n = 0\}} = 0.
$$

There holds

$$
\lim_{|x| \to \infty} v(x) = 0,
$$

since $\lim_{|x|\to\infty} u = 0$ and $\lim_{|x|\to\infty} \omega = 0$, since supp $f \Subset \overline{\mathbb{R}^n_+}$ and

$$
\sup_{y \in \text{supp f}} \gamma(|x-y|) - \gamma(|x-\tilde{y}|) \to 0.
$$

□

3.5.5 Lemma. *Let $\Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^{2,\alpha}$, $u \in C^{2,\alpha}(\bar{\Omega})$, $\phi \in \text{Diff}^{2,\alpha}(\bar{\Omega}, \phi(\bar{\Omega}))$ and $\tilde{u} := u \circ \psi$, where $\psi = \phi^{-1}$. Then*

$$
\tilde{u} \in C^{2,\alpha}(\phi(\bar{\Omega}))
$$

and

$$
|\tilde{u}|_{2,\alpha,\phi(\bar{\Omega})} \le c |u|_{2,\alpha,\bar{\Omega}},
$$

$c = c(\partial\Omega, |\psi|_{2,\alpha})$.

Proof. This is a simple computation taking 3.1.8 into account. □

3.5.6 Theorem. *(Schauder estimate)*
Let $\Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$ *and* $u \in C^{2,\alpha}(\bar{\Omega})$ *be a solution of the boundary value problem*

$$\begin{aligned} Lu = -a^{ij}u_{ij} + b^i u_i + cu &= f \\ u_{|\partial\Omega} &= \phi, \end{aligned}$$

$f \in C^{0,\alpha}(\bar{\Omega})$, $\phi \in C^{2,\alpha}(\partial\Omega)$, $a^{ij}, b^i, c \in C^{0,\alpha}(\bar{\Omega})$, (a^{ij}) *elliptic,* $c \geq 0$. *Then there holds*

$$|u|_{2,\alpha} \leq c(|f|_{0,\alpha,\Omega} + |\phi|_{2,\alpha,\partial\Omega}).$$

Here we have $c = c(\alpha, n, \Omega, |\partial\Omega|_{2,\alpha}, (a^{ij}), |b^i|_{0,\alpha,\Omega}, |c|_{0,\alpha,\Omega})$.

Proof. Since ϕ is extendable as $C_c^{2,\alpha}(\bar{\Omega})-$function to $\overline{\Omega}$ with

$$|\phi|_{2,\alpha,\mathbb{R}^n} \leq c|\phi|_{2,\alpha,\partial\Omega},$$

let wlog $\phi = 0$. Let $(U_k)_{1\leq k\leq N}$ be a finite cover of $\overline{\Omega}$ with local charts, in which $\partial\Omega$ can be flattened. Furthermore let the U_k be so small, that

$$\omega_{U_k}(a^{ij}) \leq \frac{1}{2c},$$

where $c = c(n, \alpha, (a^{ij}))$ is specified later. Let $(\zeta_k)_{1\leq k\leq N}$ be a subordinate partition of unity, such that

$$u = \sum_{k=1}^{N} u_k, \;\; u_k = u\zeta_k.$$

Multiply $Lu = f$ with ζ_k, then

$$a^{ij}D_iD_ju_k = F_k = -f\zeta_k + a^{ij}(2D_iuD_j\zeta_k + uD_iD_j\zeta_k) + b^iD_iu\zeta_k + cu\zeta_k \quad (3.17)$$

By 3.1.8 we have $F_k \in C^{0,\alpha}(\bar{\Omega})$. There are two cases:
(i) supp $\zeta_k \subset \Omega \Rightarrow$ supp $F_k \subset \Omega :$ Extend u_k and F_k outside Ω by 0 to the whole $\mathbb{R}^n$. Then we have (3.17) in $\mathbb{R}^n$.
(ii) $\partial\Omega \cap U_k \neq 0 :$ Choose new coordinates $y = y(x)$ in U_k straightening the boundary, such that $y(\Omega \cap U_k) \subset \mathbb{R}^n_+$, $y(\partial\Omega \cap U_k) \subset \{y^n = 0\}$ and (3.17) transforms to

$$\tilde{a}^{ij}D_{y^i}D_{y^j}\tilde{u}_k = \tilde{F}_k = F_k \circ y^{-1} - a^{kl}\frac{\partial^2 y^i}{\partial x^k \partial x^l} \circ y^{-1} D_{y^i}\tilde{u}_k, \quad (3.18)$$

where $\tilde{F}_k$ is also of class $C^{0,\alpha}(\bar{\Omega})$. Furthermore there holds

$$\tilde{u}_k(\hat{y}, 0) = 0,$$

which is why $\tilde{u}_k$ and $\tilde{F}_k$ can be extended to $\mathbb{R}^n_+$ by 0.
Freezing coefficients: Consider $x_0 \in U_k,\ y_0 \in y(U_k)$.

$$\Rightarrow a^{ij}(x_0)D_iD_ju_k = F_k - (a^{ij} - a^{ij}(x_0))D_iD_ju_k \tag{3.19}$$

and

$$\tilde{a}^{ij}(y_0)D_{y^i}D_{y^j}\tilde{u}_k = \tilde{F}_k - (\tilde{a}^{ij} - \tilde{a}^{ij}(y_0))D_{y^i}D_{y^j}\tilde{u}_k \tag{3.20}$$

A further linear transformation $\hat{x}(x)$ and $\hat{y}(y)$ leads to

$$\Delta_{\hat{x}}\hat{u}_k = \hat{F}_k \text{ in } \mathbb{R}^n \tag{3.21}$$

and

$$\Delta_{\hat{y}}\hat{\tilde{u}}_k = \hat{\tilde{F}} \text{ in } \hat{y}(\mathbb{R}^n_+), \tag{3.22}$$

where

$$\hat{F}_k = (F_k - (a^{ij} - a^{ij}(x_0))D_iD_ju_k) \circ \hat{x}^{-1}$$

and $\hat{\tilde{F}}$ analogously. Furthermore suppose w.l.o.g., that the boundary condition

$$\hat{u}_k(\hat{y}, 0) = 0$$

holds in the new coordinates, otherwise consider a further orthogonal transformation. The inner potential estimate, 3.1.10, the preceding lemma, as well as 3.1.8 imply for (3.21)

$$[D^2\hat{u}_k]_\alpha \le c(n,\alpha)[\hat{F}_k]_{\alpha,\mathbb{R}^n} \le c(|F_k|_{0,\alpha} + [a^{ij}]_\alpha|u|_2 + \omega_{U_k}(a^{ij})[D^2\hat{u}_k]_\alpha),$$

where now $c = c(n,\alpha,(a^{ij}))$ and a similar inequality for (3.22). Because of the special choice of the covering we obtain

$$\begin{aligned}
[D^2\hat{u}_k]_{\alpha,\hat{x}(\mathbb{R}^n)} &\le c(|F_k|_{0,\alpha,\hat{x}(\mathbb{R}^n)} + [a^{ij}]_{\alpha,\hat{x}(\mathbb{R}^n)}|\hat{u}|_2) \\
&\le c(|f|_{0,\alpha}|\zeta_k|_{2,\alpha} \\
&\quad + \max(|a^{ij}|_{0,\alpha}, |b^i|_{0,\alpha}, |c|_{0,\alpha})|\hat{u}|_2|\zeta|_{2,\alpha} + [a^{ij}]_\alpha|\hat{u}|_2) \\
&\le c(|f|_{0,\alpha,\hat{x}(\Omega)} + |\hat{u}|_{2,\hat{x}(\Omega)}).
\end{aligned}$$

Now add $|\hat{u}_k|_2$, then applying 3.5.5 to the set $\bar{\Omega}\cap\operatorname{supp}(\zeta_k)$ and the previous coordinate transformations we obtain

$$|u_k|_{2,\alpha,\Omega} \le c|\hat{u}|_{2,\alpha},$$

where c also depends of the norm of the linear transformation.
Sum over k,

$$\Rightarrow |u|_{2,\alpha,\Omega} \le c(|f|_{0,\alpha} + |u|_{2,0}) \le c(|f|_{0,\alpha} + \epsilon|u|_{2,\alpha} + \tilde{c}|u|_0)$$

$$\Rightarrow |u|_{2,\alpha,\Omega} \le (|f|_{0,\alpha} + |u|_0).$$

If furthermore $c \ge 0$, then by the C^0-estimates we have $|u|_0 \le c|f|_0$. $\square$

3.5.7 Remark. The constant in the Schauder estimates only depends on the special choice of the covering and the partition of unity, as well as on the $C^{2,\alpha}-$norms of $\partial\Omega$, diam Ω, the $C^{0,\alpha}-$norms of the coefficients and the ellipticity constant.

3.5.8 Theorem. *Let $\partial\Omega \in C^{l,\alpha}$, $f \in C^{l-2,\alpha}(\bar{\Omega})$, $\phi \in C^{l,\alpha}(\partial\Omega)$, $a^{ij}, b^i, c \in C^{l-2,\alpha}(\bar{\Omega})$, $l \geq 2$, (a^{ij}) elliptic, $c \geq 0$, then for a solution $u \in C^{l,\alpha}(\bar{\Omega})$ of $Lu = f$, $u_{|\partial\Omega} = \phi$ we have*

$$|u|_{l,\alpha,\Omega} \leq c(|f|_{l-2,\alpha,\Omega} + |\phi|_{l,\alpha,\partial\Omega}).$$

Proof. Exercise. □

Chapter 4

Existence theorems

4.1 The method of continuity

4.1.1 Theorem. *(Method of continuity)*
Let L_τ, $0 \le \tau \le 1$, be a family of differential operators satisfying the conditions of 3.5.6,

$$L_\tau = -a_\tau^{ij} D_i D_j + b_\tau^i D_i + c_\tau,$$

with the following properties

$$\exists \lambda > 0 \ \forall 0 \le \tau \le 1\colon \lambda \le \lambda_\tau, \tag{4.1}$$

where $\forall \xi \in \mathbb{R}^n\colon a_\tau^{ij} \xi_i \xi_j \ge \lambda_\tau |\xi|^2$,

$$\sup_\tau (|a_\tau^{ij}|_{0,\alpha} + |b_\tau^i|_{0,\alpha} + |c_\tau|_{0,\alpha}) \le c \tag{4.2}$$

and

$$\forall v \in C^{2,\alpha}(\bar\Omega)\colon |(L_\tau - L_{\tau_0})v|_{0,\alpha} \le \epsilon(\tau, \tau_0)|v|_{2,\alpha} \tag{4.3}$$

$$\epsilon(\tau, \tau_0) \to 0, \ (\tau \to \tau_0).$$

Let the L_τ be defined on

$$C_0^{2,\alpha}(\bar\Omega) := \{u \in C^{2,\alpha}(\bar\Omega) : u_{|\partial\Omega} = 0\},$$

such that

$$L_\tau \in \mathcal{L}(C_0^{2,\alpha}(\bar\Omega), C^{0,\alpha}(\bar\Omega)).$$

Then L_τ is a homeomorphism for all $0 \le \tau \le 1$, if L_0 is one.

Proof. (i) The L_τ are injective by the maximum principle. The inverse functions L_τ^{-1} are continuous because of the Schauder estimates.
(ii) Set $V := C_0^{2,\alpha}(\bar\Omega)$, $W := C^{0,\alpha}(\bar\Omega)$. Then

$$\Lambda := \{\tau \in [0,1] : \mathcal{R}(L_\tau) = W\} \ne \emptyset, \text{ since } 0 \in \Lambda.$$

Furthermore we have

$$\Lambda \text{ open},$$

since for $\tau_0 \in \Lambda$ write

$$L_\tau = L_{\tau_0} + (L_\tau - L_{\tau_0}) = L_{\tau_0}(I + L_{\tau_0}^{-1}(L_\tau - L_{\tau_0})) \equiv L_{\tau_0}(I - A)$$

$$\Rightarrow \|A\| \le \|L_{\tau_0}^{-1}\| \|L_\tau - L_{\tau_0}\| < \frac{1}{2}, \text{ if } |\tau - \tau_0| \text{ is small.}$$

Thus L_τ is invertible. Furthermore

$$\Lambda \text{ is closed},$$

since for $\tau_i \in \Lambda$, $\tau_i \to \tau_0 \in [0,1]$ and $f \in W$

$$\exists u_i \in V \colon L_{\tau_i} u_i = f.$$

$$3.5.6 \Rightarrow |u_i|_{2,\alpha} \le c|f|_{0,\alpha}$$

$$\Rightarrow u_i \xrightarrow{C^2} u \in C_0^{2,\alpha}(\bar{\Omega}),$$

for a subsequence u_i.

$$L_{\tau_0} u - f = L_{\tau_0} u - L_{\tau_i} u_i = L_{\tau_0} u_i - L_{\tau_i} u_i + L_{\tau_0}(u - u_i)$$

$$\Rightarrow |L_{\tau_0} u - f|_0 \le \|L_{\tau_0} - L_{\tau_i}\| |u_i|_{2,\alpha} + |L_{\tau_0}(u - u_i)|_0 \to 0.$$

□

4.1.2 Corollary. *Let $\partial\Omega \in C^{2,\alpha}$, $f \in C^{0,\alpha}(\bar{\Omega})$, $\phi \in C^{2,\alpha}(\bar{\Omega})$ and L a linear elliptic differential operator of second order, satisfying the conditions of 3.5.6. Then the boundary value problem*

$$\begin{aligned} Lu &= f \ \text{ in } \Omega \\ u_{|\partial\Omega} &= \phi \end{aligned}$$

has exactly one solution $u \in C^{2,\alpha}(\bar{\Omega})$.

Proof. Consider $u - \phi$ to assume $\phi \equiv 0$ wlog.

$$L_\tau = (1-\tau)(-\Delta) + \tau L$$

satisfy the conditions of the method of continuity.

$$\Rightarrow \mathcal{R}(L) = C^{0,\alpha}(\bar{\Omega}),$$

if $\mathcal{R}(-\Delta) = C^{0,\alpha}(\bar{\Omega})$. The following theorem implies the claim. □

4.1.3 Theorem. *Let* $\partial\Omega \in C^{2,\alpha}$*,* $f \in C^{0,\alpha}(\bar\Omega)$*,* $\phi \in C^{2,\alpha}(\bar\Omega)$*.*

$$-\Delta u = f \text{ in } \Omega$$
$$u_{|\partial\Omega} = \phi$$

has a solution $u \in C^{2,\alpha}(\bar\Omega)$*.*

Proof. Wlog let $\phi = 0$. We even may suppose $f \in C_c^\infty(\mathbb{R}^n)$, for otherwise we consider an extension $\tilde f \in C_c^{0,\alpha}(\mathbb{R}^n)$ with

$$|\tilde f|_{0,\alpha} \le c|f|_{0,\alpha}.$$

Let $f_\epsilon \in C_c^\infty(\mathbb{R}^n)$ be a mollification with

$$f_\epsilon \xrightarrow{C^0} f$$

and

$$[f_\epsilon]_{\alpha,\mathbb{R}^n} \le c[f]_\alpha$$

and let

$$-\Delta u_\epsilon = f_\epsilon,\ u_{\epsilon|\partial\Omega} = 0$$
$$\Rightarrow |u_\epsilon|_{2,\alpha} \le c|f_\epsilon|_{0,\alpha} \le c|f|_{0,\alpha}.$$

Claim: $\Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^\infty$, $f \in C_c^\infty(\mathbb{R}^n)$

$$\Rightarrow \exists u \in C^\infty(\bar\Omega):\ -\Delta u = f,\ u_{|\partial\Omega} = 0.$$

We will prove this in PDE 2 with the help of L^2- estimates.
Approximate Ω by $\Omega_\epsilon \Subset \Omega$, $\partial\Omega_\epsilon \in C^\infty$, $\Omega_\epsilon \nearrow \Omega$, $\partial\Omega_\epsilon \to \partial\Omega$ such that

$$|\partial\Omega_\epsilon|_{2,\alpha} \le c|\partial\Omega|_{2,\alpha}.$$

Solve the problem in Ω_ϵ, such that

$$|u_\epsilon|_{2,\alpha,\Omega_\epsilon} \le c|f|_{0,\alpha,\mathbb{R}^n}.$$

Let

$$\tilde u_\epsilon(x) := \begin{cases} u_\epsilon(x), & x \in \Omega_\epsilon \\ 0, & x \ne \Omega_\epsilon \end{cases}$$
$$\Rightarrow \tilde u_\epsilon \in C_c^{0,1}(\mathbb{R}^n),\ |D\tilde u_\epsilon| \le c|Du_\epsilon| \le c.$$

Thus there is a subsequence with

$$\tilde u_\epsilon \xrightarrow{C^0} u \in C^{0,\alpha}(\bar\Omega),\ u_{|\partial\Omega} = 0$$

and

$$\forall \Omega' \Subset \Omega\colon u_\epsilon \xrightarrow{C^2(\Omega')} u\ \wedge\ |u|_{2,\alpha,\Omega'} \le c|f|_{0,\alpha}.$$
$$\Rightarrow u \in C^{2,\alpha}(\bar\Omega)$$

and

$$-\Delta u = f,\ u_{|\partial\Omega} = 0.$$

□

4.2 Fredholm alternative

4.2.1 Lemma. *(Riesz)*
Let V be a normed space, $M \subset V$ a closed subspace and $M \neq V$. Then there holds

$$\forall 0 < \epsilon < 1 \ \exists u_\epsilon \in V \colon \|u_\epsilon\| = 1 \ \wedge \ \operatorname{dist}(u_\epsilon, M) \geq \epsilon.$$

Proof. Let $u \in V \backslash M$

$$\Rightarrow d := \operatorname{dist}(u, M) > 0.$$

$$\Rightarrow \forall 1 > \epsilon > 0 \ \exists v_\epsilon \in M \colon d < \|u - v_\epsilon\| < \frac{d}{\epsilon}.$$

Set

$$u_\epsilon := \frac{u - v_\epsilon}{\|u - v_\epsilon\|}.$$

Let $v \in M$.

$$\Rightarrow \|u_\epsilon - v\| = \frac{1}{\|u - v_\epsilon\|}\|u - v_\epsilon - \|u - v_\epsilon\| v\| \geq \epsilon.$$

□

4.2.2 Remark. If V is a Hilbert space, choose $\epsilon = 1$. u_ϵ then is orthogonal on M. In general we can only prove the existence of an *almost orthogonal element.*

4.2.3 Theorem. *(Fredhom alternative)*
Let V be a Banach space and $T \in K(V)$. Then $I - T$ is injective if and only if $I - T$ is surjective. In this case $(I - T)^{-1}$ is continuous.

Proof. The proof contains four steps.
1. Let $S := I - T$, $N := \ker(S)$. Then

$$\exists c > 0 \ \forall x \in V \colon \operatorname{dist}(x, N) \leq c\|Sx\|, \tag{4.4}$$

since if (4.4) was wrong, then

$$\exists x_n \in V \colon d_n = \operatorname{dist}(x_n, N) > n\|Sx_n\|$$

wlog $\|Sx_n\| = 1$, such that $d_n > n$. Choose $y_n \in N$ such that

$$d_n \leq \|x_n - y_n\| \leq 2d_n. \tag{4.5}$$

$$z_n := \frac{x_n - y_n}{\|x_n - y_n\|}$$

$$\Rightarrow \|z_n\| = 1 \ \wedge \ \|Sz_n\| = \frac{\|Sx_n\|}{\|x_n - y_n\|} \leq \frac{1}{d_n} \to 0.$$

$Sz_n = z_n - Tz_n$ and T compact

$$\Rightarrow Tz_n \to y_0$$

for a subsequence.

$$\Rightarrow z_n \to y_0 \ \Rightarrow Sz_n \to Sy_0 = 0.$$

$$\Rightarrow y_0 \in N,$$

which is a contradiction, since

$$\begin{aligned} \operatorname{dist}(z_n, N) &= \inf_{y \in N} \left\| \frac{x_n - y_n}{\|x_n - y_n\|} - y \right\| \\ &= \inf_{y \in N} \frac{1}{\|x_n - y_n\|} \|x_n - y\| = \frac{d_n}{\|x_n - y_n\|} \geq \frac{1}{2}, \end{aligned} \tag{4.6}$$

by (4.5).
2. $R = R(S)$ is closed: Let

$$Sx_n \to y \in V.$$

Step 1 implies $d_n \leq c\|Sx_n\| \leq c$. Choose $y_n \in N$ as in (4.5).

$$\Rightarrow \|w_n\| \equiv \|x_n - y_n\| \leq c.$$

$$Sw_n = Sx_n \to y.$$

T compact implies $Tw_n \to w_0$

$$\Rightarrow w_n \to y + w_0 \ \Rightarrow S(y + w_0) = y.$$

3. $N = \{0\} \Rightarrow R = R(S) = V$:
Let the claim be wrong and $R_j := S^j(V)$. Then $R_j \subset R_{j-1}$. Consider

$$S\colon R_j \to R_j.$$

By step 2 it follows that R_j is closed and

$$... \subset R_3 \subset R_2 \subset R_1 = R \subset V.$$

Claim:

$$\exists k \in \mathbb{N}\ \forall j \geq k\colon R_j = R_k.$$

Otherwise choose, using the Riesz lemma, for $n \in \mathbb{N}$

$$x_n \in R_n\colon \|x_n\| = 1 \ \wedge \ \operatorname{dist}(x_n, R_{n+1}) \geq \frac{1}{2}.$$

Let $n > m$

$$\Rightarrow Tx_m - Tx_n = x_m + (-x_n - Sx_m + Sx_n)$$

$$\Rightarrow \|Tx_m - Tx_n\| \geq \frac{1}{2},$$

which is in contradiction to the compactness of T.
So let $y \in V$, then $S^k y \in R_k = R_{k+1}$

$$\Rightarrow S^k y = S^{k+1}x \ \Rightarrow S^k(y - Sx) = 0$$

$$\Rightarrow y = Sx.$$

Thus S is surjective.
4. $R = V \Rightarrow N = \{0\}$:
The sequence $N_j = S^{-j}(0) = (S^j)^{-1}(0)$ consists of closed subspaces

$$N_1 \subset N_2 \subset ...$$

Claim:

$$\exists k \in \mathbb{N}\ \forall j \geq k\colon N_j = N_k.$$

It is clear that $S(N_i) \subset N_{i-1}$. If the claim was wrong,

$$\exists\ \|x_m\| = 1 : \mathrm{dist}(x_m, N_{m-1}) \geq \frac{1}{2}.$$

Let $m > n$ then, analogously to step 3, we obtain a contradiction, since

$$Tx_m - Tx_n = x_m + (-x_n - Sx_m + Sx_n).$$

$R = V \Rightarrow\ \forall k\colon R(S^k) = V$

$$\Rightarrow \forall y \in N_k\colon y = S^k x \Rightarrow 0 = S^k y = S^{2k}x$$

$$\Rightarrow x \in N_{2k} = N_k$$

$$\Rightarrow y = 0$$

$$\Rightarrow N_k = \{0\} \Rightarrow N = \{0\}.$$

□

4.2.4 Theorem. *A compact linear operator T of a Banach space into itself has at most countably many eigenvalues clustering at no value except possibly at 0. Each eigenvalue $\lambda \neq 0$ has finite multiplicity.*

Proof. Let λ_n be a sequence of eigenvalues and x_n a corresponding sequence of linearly independent eigenvectors. Suppose $\lambda_n \to \lambda \neq 0$.

$$S_{\lambda_n} := \lambda_n - T = \lambda_n(I - \lambda_n^{-1}T)$$

and

$$M_n := \langle x_1, ..., x_n \rangle \subset M_{n+1}.$$

By the Riesz lemma

$$\exists y_n \in M_n \colon \|y_n\| = 1 \ \wedge \ \operatorname{dist}(y_n, M_{n-1}) \geq \frac{1}{2}.$$

Let $n > m$ and consider

$$\lambda_n^{-1} T y_n - \lambda_m^{-1} T y_m = y_n + (-y_m - \lambda_n^{-1} S_{\lambda_n} y_n + \lambda_m^{-1} S_{\lambda_m} y_m) \equiv y_n - y.$$

We want to show, that $y \in M_{n-1}$. Except for $S_{\lambda_n} y_n$ this is obvious.

$$y_n = \beta^i x_i$$

$$\Rightarrow S_{\lambda_n} y_n = \lambda_n \beta^i x_i - \beta^i \lambda_i x_i \in M_{n-1}.$$

$$\Rightarrow \|\lambda_n^{-1} T y_n - \lambda_m^{-1} T y_m\| \geq \frac{1}{2},$$

which is a contradiction with the compactness of T and $\lambda_n \to \lambda \neq 0$. □

4.2.5 Lemma. *Let $T \in \mathcal{L}(H)$, H be a Hilbert space. Then*

$$T \text{ compact } \Leftrightarrow T^* \text{ compact.}$$

Proof. Since $T^{**} = T$, it suffice to prove the 'only if' part. So let $\|x_n\| \leq c$.

$$\begin{aligned} \|T^* x_n - T^* x_m\|^2 &= \langle T^*(x_n - x_m), T^*(x_n - x_m)\rangle \\ &= \langle x_n - x_m, TT^*(x_n - x_m)\rangle \\ &\leq c\|TT^*(x_n - x_m)\| \to 0 \end{aligned}$$

for a subsequence, since TT^* is compact. □

4.2.6 Lemma. *Let E, F be Banach spaces and $T \in K(E,F)$. Then*

$$T^* : F^* \to E^*$$

is also compact.

Proof. Let $y_n^* \in F^*$ be bounded.

$$\begin{aligned} \|T^* y_n^*\| &= \sup_{\|x\| \leq 1} |\langle T^* y_n^*, x\rangle| = \sup_{\|x\| \leq 1} |\langle y_n^*, Tx\rangle| \\ &\leq \sup_{x \in \overline{T(B_1(0))}} |\langle y_n^*, x\rangle| = \|y_n^*\|_{\infty, \overline{T(B_1(0))}}. \end{aligned}$$

$$\|y_n^*\| \leq c$$

implies, that (y_n^*) is equicontinuous on $\overline{T(B_1(0))}$, as well as bounded. Arzela-Ascoli guarantees a subsequence, such that

$$\|y_n^* - y_m^*\|_{\infty, \overline{T(B_1(0))}} \to 0, \ n, m \to \infty.$$

By the inequality above we obtain compactness. □

4.2.7 Lemma. *Let $T \in \mathcal{L}(H)$, then $\overline{\mathcal{R}(T)} = N(T^*)^\perp$.*

Proof. (i) Since $N(T^*)^\perp$ is closed, it suffices to show

$$\mathcal{R}(T) \subset N(T^*)^\perp.$$

So let $y \in \mathcal{R}(T)$, $z \in N(T^*)$ and $y = Tx$.

$$\Rightarrow \langle y, z \rangle = \langle Tx, z \rangle = \langle x, T^* z \rangle = 0.$$

(ii) $N(T^*)^\perp \subset \overline{\mathcal{R}(T)} \Leftrightarrow \overline{\mathcal{R}(T)}^\perp \subset N(T^*)$. So let $y \perp \mathcal{R}(T)$.

$$\Rightarrow \ \forall x \in H \colon 0 = \langle y, Tx \rangle.$$

$$\Rightarrow \ \forall x \in H \colon 0 = \langle T^* y, x \rangle$$

$$\Rightarrow T^* y = 0 \Rightarrow y \in N(T^*).$$

□

4.2.8 Remark. Let $A \in \mathcal{L}(H)$ be compact. From the proof of the Fredholm alternative we obtain

$$\mathcal{R}(I - A) \text{ closed}.$$

$$\Rightarrow H = \mathcal{R}(I - A) \oplus_\perp N(I - A^*) = \mathcal{R}(I - A^*) \oplus_\perp N(I - A).$$

4.2.9 Theorem. *Let $A \in \mathcal{L}(H)$ be compact. then the equation*

$$y = (I - A)x, \ \ y \in H$$

is solvable if and only if $y \perp N(I - A^*)$.

Proof. Follows at once from the previous remark. □

4.2.10 Corollary. *Let $A \in \mathcal{L}(H)$ be compact. Then*

$$N(I - A) = \{0\} \Leftrightarrow N(I - A^*) = \{0\}.$$

Proof. Follows from the Fredholm alternative. □

4.2.11 Theorem. *Let $\Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$. Let*

$$L = -a^{ij} D_i D_j + b^i D_i + c$$

be elliptic with coefficients in $C^{0,\alpha}(\bar{\Omega})$, i.e.

$$L : C_0^{2,\alpha}(\bar{\Omega}) \to C^{0,\alpha}(\bar{\Omega}).$$

Then

$$L \textit{ injective} \ \Leftrightarrow \ L \textit{ surjective}.$$

Proof. Choose $\lambda > 0$, such that $c + \lambda > 0$ and define

$$L_\lambda := L + \lambda.$$

Then L_λ is a homeomorphism and

$$L_\lambda^{-1} : C^{0,\alpha}(\bar{\Omega}) \to C_0^{2,\alpha}(\bar{\Omega}) \xrightarrow{compact} C^{0,\alpha}(\bar{\Omega}).$$

λL_λ^{-1} is also compact. Thus

$$I - \lambda L_\lambda^{-1} : C^{0,\alpha}(\bar{\Omega}) \to C^{0,\alpha}(\bar{\Omega})$$

is surjective if and only if it is injective. There holds

$$L = L_\lambda(I - \lambda L_\lambda^{-1}) \Rightarrow N(L) = N(I - \lambda L_\lambda^{-1})$$

and L is surjective if and only if $I - \lambda L_\lambda^{-1}$ is. $\square$

4.2.12 Definition. (i) Let E, F be Banach spaces, $A \in \mathcal{L}(E, F)$. We define the *cokernel* of A, $\operatorname{coker}(A)$, as the algebraic complement of $\mathcal{R}(A)$, i.e.

$$F = \mathcal{R}(A) \oplus_a \operatorname{coker}(A).$$

(ii) $A \in \mathcal{L}(E, F)$ is called *Fredholm operator*, if

$$\mathcal{R}(A) \text{ is closed}$$

and

$$\dim(N(A)),\ \dim(\operatorname{coker}(A)) < \infty.$$

(iii) Let A be Fredholm, then the *index* of A is defined by

$$\operatorname{ind}(A) := \dim(N(A)) - \dim(\operatorname{coker}(A)).$$

4.2.13 Proposition. *Let $A : E \to F$ be Fredholm, $K \in \mathcal{L}(E, F)$ compact. Then*

$$A + K \textit{ is Fredholm } \wedge\ \operatorname{ind}(A + K) = \operatorname{ind}(A)$$

We do not prove this theorem here.

4.2.14 Theorem. *Let $\Omega \Subset \mathbb{R}^n$, $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$ and*

$$L = -a^{ij} D_i D_j + b^i D_i + c$$

be elliptic with coefficients in $C^{0,\alpha}(\bar{\Omega})$, then

$$L : C_0^{2,\alpha}(\bar{\Omega}) \to C^{0,\alpha}(\bar{\Omega})$$

is Fredholm with index 0.

Proof. Let $\lambda > 0$ with $c + \lambda > 0$ and

$$L_\lambda := L + \lambda j,$$

where

$$j\colon C_0^{2,\alpha}(\bar{\Omega}) \xrightarrow{compact} C^{0,\alpha}(\bar{\Omega}).$$

L_λ is a homeomorphism, which is why L_λ is Fredholm with index 0. Thus this also holds for L. $\square$