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Hypersurfaces of prescribed Weingarten curvature

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0. Introduction

In a complete $(n+1)$-dimensional manifold N we want to find closed hypersurfaces M of *prescribed curvature*, so-called *Weingarten* hypersurfaces. To be more precise, let Ω be a connected open subset of N, $f \in C^{2,\alpha}(\bar{\Omega})$, F a smooth, symmetric function defined in the positive cone $\Gamma_+ \subset \mathbf{R}^n$, then we look for a convex hypersurface $M \subset \Omega$ such that

$$F|_M = f(x) \quad \forall x \in M \,, \tag{0.1}$$

where $F|_M$ means that F is evaluated at the vector $(\kappa_i(x))$ the components of which are the principal curvatures of M.

This is in general a fully nonlinear partial differential equation problem, which is elliptic if we assume F to satisfy

$$\frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+ \,. \tag{0.2}$$

Classical examples of curvature functions F are the elementary symmetric polynomials of order k, H_k, defined by

$$H_k = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \quad 1 \leqq k \leqq n \,. \tag{0.3}$$

H_1 is the mean curvature H, H_2 is the scalar curvature – for hypersurfaces in Euclidean space –, and H_n is the Gaussian curvature K.

For technical reasons it is convenient to consider the homogeneous polynomials of degree 1

$$\sigma_k = H_k^{1/k} \tag{0.4}$$

instead of H_k. Then, the σ_k's are not only monotone increasing but also *concave*. Their *inverses* $\tilde{\sigma}_k$, defined through

$$\tilde{\sigma}_k(\kappa_i) = \frac{1}{\sigma_k(\kappa_i^{-1})} \tag{0.5}$$

share these properties; a proof of this non-trivial result can be found in [11]. $\tilde{\sigma}_1$ is the so-called *harmonic curvature* G, and, evidently, we have $\tilde{\sigma}_n = \sigma_n$.

The existence of closed Weingarten hypersurfaces in $\mathbf{R}^{n+1}$ has been studied extensively in previous papers: the case $F = H$ by Bakelman and Kantor [2], Treibergs and Wei [13], the case $F = K$ by Oliker [12], Delanoë [5], and for general curvature functions by Caffarelli, Nirenberg and Spruck [4].

In a recent paper [8], we considered the existence problem for a class of curvature functions that included the n-th root of the Gaussian curvature and the inverses of the complete symmetric functions $\gamma_k, 1 \leqq k \leqq n$, which are defined through

$$\gamma_k(\kappa) = \left(\sum_{|\alpha|=k} \kappa^\alpha \right)^{1/k}, \tag{0.6}$$

and we could solve the problem provided the sectional curvature of the ambient space N was non-positive and the boundary of Ω consisted of two components which acted as barriers for the problem.

In this paper we want to remove the restriction on the sectional curvature of the ambient space, and we also redefine the class of curvature functions slightly leading to a much larger class that includes the inverses of the symmetric polynomials $\sigma_k,\ 1 \leqq k \leqq n$.

The existence prove in [8] remains valid for the larger class of curvature functions – the only modification is that instead of the function $\Phi(t) = -t^{-1}$ one has to choose $\Phi(t) = -t^{-m}$, m large, in [8, Sect. 8].

However, the sign condition on the sectional curvature of N cannot be dropped without using an entirely different technique in the existence proof. In our former paper we obtained the desired hypersurface as the stationary limit of the solution to an evolution equation. The time-dependent solutions $M(t)$ did only satisfy

$$F|_{M(t)} \leqq f \tag{0.7}$$

due to the choice of the initial hypersurface, and a positive lower bound on F could only be obtained under the additional restriction on the sign of the sectional curvature of N.

On the other hand, for smooth solutions of the equation (0.1), a priori estimates up to any order can be proved in an arbitrary ambient space N under very mild assumptions that are *automatically* satisfied if N is a space form or if $K_N \leqq 0$.

Thus, the main difficulty is to replace the evolutionary approach by a method that gives the same a priori estimates that can be proved for smooth solutions of the equation (0.1).

We use the method of successive approximation to accomplish this task.

The main assumption in the existence proof is a barrier assumption.

Definition 0.1. Let M_1, M_2 be strictly convex, closed hypersurfaces in N, homeomorphic to S^n and of class $C^{4,\alpha}$ which bound a connected open subset Ω, such that the mean curvature vector of M_1 points outside of Ω and the mean curvature vector of M_2 points inside of Ω. M_1, M_2 are barriers for (F, f) if

$$F|_{M_1} \leqq f \; . \tag{0.8}$$

and

$$F|_{M_2} \geqq f \; . \tag{0.9}$$

Remark 0.2. In view of the Harnack inequality we deduce from the properties of the barriers that they do not touch, unless both coincide and are solutions of our problem. In this case Ω would be empty.

The curvature functions we have in mind are defined in detail in Sect. 1, we shall call those functions to be of class (K); special functions belonging to that class are the inverses of σ_k and γ_k, and also the inverses of *convex*, symmetric curvature functions that are strictly monotone increasing and homogeneous of degree 1.

We need one more definition.

Definition 0.3. A coordinate system (x^α) in N is called a normal Gaussian coordinate system, if the metric takes the form

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha dx^\beta = dx^{0^2} + \bar{g}_{ij}\, dx^i dx^j \; . \tag{0.10}$$

Here, Greek indices range from 0 to n, Latin indices from 1 to n, and the summation convention is always used.

Then, we can prove

Theorem 0.4. *Let F be of class (K), $0 < f \in C^{2,\alpha}(\overline{\Omega})$ and assume that M_1, M_2 are barriers for (F, f), then the problem*

$$F|_M = f \tag{0.11}$$

has a strictly convex solution $M \subset \overline{\Omega}$ of class $C^{4,\alpha}$ provided $\overline{\Omega}$ is covered by a normal Gaussian coordinate system (x^α), such that the level hypersurfaces $\{x^0 = \text{const}\}$ are homeomorphic to S^n and the barriers M_i can be written as graphs over some level hypersurface S_0

$$M_i = \operatorname{graph} u_i|_{S_0} \; . \tag{0.12}$$

Furthermore, we assume the existence of a strictly convex function $\psi \in C^2(\overline{\Omega})$.

The solution M can be written as the graph of a function $u \in C^{4,\alpha}(S_0)$ and is therefore homeomorphic to S^n.

Remark 0.5. In [8, Sect. 4] we have proved that in case $K_N \leqq 0$ or if N is a space form, $\bar{\Omega}$ can always be covered by a normal Gaussian coordinate system as required in Theorem 0.4. Moreover, the level hypersurfaces $\{x^0 = \text{const}\}$ are strictly convex. A strictly convex function ψ in $\bar{\Omega}$ is then given by

$$\psi = \frac{1}{2}\left|x^0\right|^2 , \tag{0.13}$$

where we assume without loss of generality that $x^0 > 0$, and orient $\frac{\partial}{\partial x^0}$ appropriately.

The paper is organized as follows: In Sect. 1 we define the curvature functions of class (K) and prove that the process of *elliptic regularization* maps (K) into itself, i.e. each curvature function $F \in (K)$ can be approximated by curvature functions $F_\varepsilon \in (K)$ the first derivatives of which are bounded from above.

In Sect. 2 we introduce the notations and common definitions we rely on and emphasize some admissible simplifications.

In Sect. 3 we prove a priori estimates in the C^2-norm for solutions to an auxiliary problem and give in Sect. 4 an existence proof for the auxiliary problem which is valid for a large class of fully nonlinear elliptic operators of second order.

Finally, in Sect. 5 we demonstrate that the auxiliary solutions converge to a solution of the original problem.

1. Curvature functions

Let $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ be a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 ; \tag{1.1}$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices $\mathscr{S}_+$, for, let $(h_{ij}) \in \mathscr{S}_+$ with eigenvalues κ_i, $1 \leqq i \leqq n$, then define F on $\mathscr{S}_+$ by

$$F(h_{ij}) = F(\kappa_i) . \tag{1.2}$$

If we define

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} , \tag{1.3}$$

and

$$F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}} , \tag{1.4}$$

then

$$F^{ij}\xi^i\xi^j = \frac{\partial F}{\partial \kappa_i}|\xi^i|^2 \quad \forall \xi \in \mathbf{R}^n , \tag{1.5}$$

$$F^{ij} \text{ is diagonal if } h_{ij} \text{ is diagonal} \tag{1.6}$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}\eta_{ii}\eta_{jj} + \sum_{i\neq j}\frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2 \,, \tag{1.7}$$

for any $(\eta_{ij}) \in \mathscr{S}$, where $\mathscr{S}$ is the space of all symmetric matrices. The second term on the right-hand side of (1.7) is non-positive if F is concave and non-negative if it is convex and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

Furthermore, let $(h_{ij}) \in \mathscr{S}_+$ and consider a coordinate system such that $h_{ij} = \kappa_i\delta_{ij}$, then for any pair $(\hat{\eta}_{ij}), (\bar{\eta}_{ij}) \in \mathscr{S}$ satisfying

$$\hat{\eta}_{ij} = 0 \quad \text{for } i \neq j \tag{1.8}$$

and

$$\bar{\eta}_{ij} = 0 \quad \text{for } i = j \tag{1.9}$$

we have

$$F^{ij,kl}\,\hat{\eta}_{ij}\,\bar{\eta}_{kl} = 0 \tag{1.10}$$

as can be deduced from the proof of [8, Lemma 1.1].

Since any $(\eta_{ij}) \in \mathscr{S}$ can be decomposed in a diagonal part $(\hat{\eta}_{ij})$ and diagonal zero part $(\bar{\eta}_{ij})$, we conclude

$$F^{ij,kl}\eta_{ij}\eta_{kl} = F^{ij,kl}\,\hat{\eta}_{ij}\,\hat{\eta}_{kl} + F^{ij,kl}\,\bar{\eta}_{ij}\,\bar{\eta}_{kl} \,. \tag{1.11}$$

We now can define the class (K)

Definition 1.1. A symmetric function $F \in C^0(\bar{\Gamma}_+) \cap C^{2,\alpha}(\Gamma_+)$ homogeneous of degree 1 is said to be of class (K) if

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+ \,, \tag{1.12}$$

$$F \text{ is concave} \,, \tag{1.13}$$

$$F|_{\partial\Gamma_+} = 0 \,, \tag{1.14}$$

and there exists a constant $c = c(F)$ such that

$$F^{ij,kl}\eta_{ij}\eta_{kl} \leqq cF^{-1}(F^{ij}\eta_{ij})^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall \eta \in \mathscr{S} \tag{1.15}$$

where F is evaluated at $(h_{ij}) \in \mathscr{S}_+$ and $(\tilde{h}^{ij}) = (h_{ij})^{-1}$.

In our previous paper we postulated the inequality (1.15) with $c = 2$, but this restriction is totally unnecessary and excludes important curvature functions as we shall see in the following.

We immediately deduce from (1.15)

Lemma 1.2. *Let F be of class (K), let κ_r be the largest eigenvalue of $(h_{ij}) \in \mathscr{S}_+$, then for any $(\eta_{ij}) \in \mathscr{S}$ we have*

$$F^{ij,kl}\eta_{ij}\eta_{kl} \leqq cF^{-1}(F^{ij}\eta_{ij})^2 - \kappa_r^{-1}F^{ij}\eta_{ir}\eta_{jr} \,, \tag{1.16}$$

where F is evaluated at (h_{ij}).

Before we show that the $\tilde{\sigma}_k$ are of class (K), let us deduce a necessary consequence of (1.15)

Lemma 1.3. *Suppose a symmetric curvature function of class C^2 satisfies* (1.15), *then*

$$\frac{F_i - F_j}{\kappa_i - \kappa_j} \leqq -\frac{1}{2}(F_i\kappa_j^{-1} + F_j\kappa_i^{-1}) \tag{1.17}$$

for any $i \neq j$. Moreover, for any symmetric curvature function F on Γ_+ of class C^1 the inequality (1.17) *is equivalent to*

$$F_i\kappa_i \leqq F_j\kappa_j \quad \text{if } \kappa_j \leqq \kappa_i \,. \tag{1.18}$$

Proof. To prove the first part of the lemma, let $(h_{ij}) \in \mathcal{S}_+$ with eigenvalues κ_i, and without loss of generality we may assume that all eigenvalues are simple. Choose a coordinate system such that $h_{ij} = \kappa_i\delta_{ij}$ and choose (η_{ij}) in (1.15) such that $\eta_{12} = \eta_{21} = 1$ and all other components are zero, then we conclude (1.17) for $i = 1$ and $j = 2$ in view of the relations (1.6) and (1.7); but this also yields the general case for arbitrary indices $i \neq j$.

To prove (1.18) we assume $\kappa_j < \kappa_i$, multiply (1.17) with $(\kappa_i - \kappa_j)$ and rearrange terms to conclude the equivalence.

Remark 1.4. If a symmetric curvature function F on Γ_+ satisfies the relation (1.18), then the estimate (1.15) is valid for any diagonal zero matrix $(\bar{\eta}_{ii}) \in \mathcal{S}$. In this case, the estimate (1.15) is completely verified if it is shown in addition that (1.15) is valid for any $(\hat{\eta}_{ij}) \in \mathcal{S}$ that can be diagonalized together with (h_{ij}), cf. (1.11), since the right-hand side of (1.15) splits accordingly into parts which contain only $(\hat{\eta}_{ij})$ resp. $(\bar{\eta}_{ij})$.

Let us now prove that the $\tilde{\sigma}_k$ are of class (K).

Lemma 1.5. *The $\tilde{\sigma}_k$ are of class (K).*

Proof. The conditions (1.12) and (1.14) are easily checked, while the concavity is proved in [11].

Thus, it remains to verify the estimate (1.15). First, we observe that the $\tilde{\sigma}_k$ satisfy the condition (1.18) since the σ_k satisfy the reverse inequality. Hence, we only have to verify (1.15) for those (η_{ij}) that can be diagonalized together with (h_{ij}). Let $F = \tilde{\sigma}_k$, then, we can write F in the form

$$F(\kappa) = \frac{1}{(\sum_{\alpha\in I} \kappa^{-\alpha})^{1/k}} \,, \tag{1.19}$$

where I is the set of those multiindices α that represent a combination

$$i_1 < \cdots < i_k \quad \text{of } \{1, \ldots, n\} \,. \tag{1.20}$$

For each i, $1 \leqq i \leqq n$, and multiindex α we define the multiindex α_i through

$$\alpha_i(j) = \begin{cases} 0, & j = i \\ \alpha(j), & j \neq i \end{cases} \tag{1.21}$$

and set $\alpha_{ij} = (\alpha_i)_j$ for $1 \leqq j \leqq n$.

Then, we have

$$F_i = \frac{1}{k}\left(\sum_{\alpha\in I}\kappa^{-\alpha}\right)^{-\frac{1}{k}-1}\sum_{\alpha\in I}\kappa^{-\alpha_i}\alpha(i)\kappa_i^{-2} \tag{1.22}$$

and

(1.23)

$$\begin{aligned}\frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j} &= (k+1)F^{-1}F_iF_j - \frac{1}{k}\left(\sum_{\alpha\in I}\kappa^{-\alpha}\right)^{-\frac{1}{k}-1}\sum_{\alpha\in I}\kappa^{-\alpha_{ij}}\alpha(i)\alpha_i(j)\kappa_i^{-2}\kappa_j^{-2}\\ &\quad -\frac{2}{k}\left(\sum_{\alpha\in I}\kappa^{-\alpha}\right)^{-\frac{1}{k}-1}\sum_{\alpha\in I}\kappa^{-\alpha_i}\alpha(i)\kappa_i^{-3}\delta_{ij}\\ &\leqq (k+1)F^{-1}F_iF_j - 2F_i\kappa_j^{-1}\delta_{ij}\ .\end{aligned}$$

Now, let us choose a coordinate system such that $h_{ij} = \kappa_i\delta_{ij}$ and let (η_{ij}) be diagonal, then we conclude from (1.7) and the preceding estimate

$$F^{ij,kl}\eta_{ij}\eta_{kl} = \frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}\eta_{ii}\eta_{jj} \leqq (k+1)F^{-1}(F^{ij}\eta_{ij})^2 - 2F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl}\ , \tag{1.24}$$

i.e. the $\tilde{\sigma}_k$ are of class (K).

By combining the results of [8, Lemma 1.4] and the preceding lemma we can now state

Lemma 1.6. *Let $F \in C^{2,\alpha}(\Gamma_+)\cap C^0(\bar{\Gamma}_+)$ be symmetric, homogeneous of degree* 1, *and strictly monotone increasing, then the inverses $\tilde{F}$ are class (K) provided $F=\sigma_k$, $1\leqq k\leqq n$, or F is convex; especially, the inverses of the γ_k are therefore of class (K), but also the inverse of the length of the second fundamental form.*

Let us emphasize that the σ_k, $1\leqq k\leqq n-1$, are not of class (K).

Next, we introduce the notion of *elliptic regularization*, which is a useful tool in the existence proof that is to follow, where we have to approximate $F\in(K)$ by curvature functions $F_\varepsilon\in(K)$ the first derivatives of which are uniformly bounded.

Definition 1.7. Let F be a symmetric curvature function on Γ_+, then we define the elliptic regularization of F, F_ε, through

$$F_\varepsilon(\kappa_i) = F([\kappa_i^{-1}+\varepsilon\sigma]^{-1})\ , \tag{1.25}$$

where $\varepsilon > 0$ and

$$\sigma = \sum_{i=1}^n \kappa_i^{-1}\ . \tag{1.26}$$

The definition becomes more obvious if F is the inverse of a function φ, then F_ε is the inverse of the function $\varphi(\kappa_i + \varepsilon H)$, where H is the mean curvature.

Lemma 1.8. *Let $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ be symmetric, monotone increasing, homogeneous of degree 1 and concave, then the F_ε share these properties and in addition there holds*

$$\frac{\partial F_\varepsilon}{\partial \kappa_i} \leqq F(1, \ldots, 1)\varepsilon^{-1} . \tag{1.27}$$

Furthermore, let $c > 0$ and $\Lambda_{c,\varepsilon}$ be defined through

$$\Lambda_{c,\varepsilon} = \{\kappa \in \Gamma_+ : F_\varepsilon \geqq c\} , \tag{1.28}$$

then there exists $\varepsilon_0 > 0$ such that for all $1 \leqq i \leqq n$

$$\varepsilon_0 \leqq \kappa_i \quad \forall \kappa \in \Lambda_{c,\varepsilon} \tag{1.29}$$

and

$$\frac{\partial F_\varepsilon}{\partial \kappa_i} \geqq \varepsilon_0 \kappa_i^{-2} \quad \forall \kappa \in \Lambda_{c,\varepsilon} . \tag{1.30}$$

Proof. The F_ε are obviously homogeneous, monotone increasing and as smooth as F, so let us consider inequality (1.27). In view of the homogeneity, we have

$$\sum_{i=1}^{n} \frac{\partial F_\varepsilon}{\partial \kappa_i} \kappa_i = F_\varepsilon , \tag{1.31}$$

and hence, for a fixed but arbitrary i

$$\frac{\partial F_\varepsilon}{\partial \kappa_i} \leqq F_\varepsilon \kappa_i^{-1} = F(\kappa_i^{-1}[\kappa_k^{-1} + \varepsilon\sigma]^{-1}) \leqq F(1, \ldots, 1)\varepsilon^{-1} \tag{1.32}$$

because of the monotonicity.

To prove the concavity, let us define

$$\sigma_i^k = \begin{cases} \varepsilon, & k \neq i \\ 1 + \varepsilon, & k = i \end{cases} \tag{1.33}$$

then

$$\frac{\partial F_\varepsilon}{\partial \kappa_i} = F_k \sigma_i^k [\kappa_k^{-1} + \varepsilon\sigma]^{-2} \kappa_i^{-2} \tag{1.34}$$

and

$$\begin{aligned} \frac{\partial^2 F_\varepsilon}{\partial \kappa_i \partial \kappa_j} &= F_{kl} \sigma_i^k \sigma_j^l [\kappa_k^{-1} + \varepsilon\sigma]^{-2} [\kappa_l^{-1} + \varepsilon\sigma]^{-2} \kappa_i^{-2} \kappa_j^{-2} \\ &\quad + 2F_k \sigma_i^k [\kappa_k^{-1} + \varepsilon\sigma]^{-3} \sigma_j^k \kappa_i^{-2} \kappa_j^{-2} - 2F_k \sigma_i^k [\kappa_k^{-1} + \varepsilon\sigma]^{-2} \kappa_i^{-3} \delta_{ij} \end{aligned} \tag{1.35}$$

where F_{kl} stands for the second derivatives of F. The first term on the right-hand side is negative-semidefinite because F is concave.

To estimate the remaining terms, let $(\xi^i) \in \mathbf{R}^n$, and consider for fixed k

$$(1.36)\quad \sigma_i^k \kappa_i^{-2} \xi^i \sigma_j^k \kappa_j^{-2} \xi^j [\kappa_k^{-1} + \varepsilon\sigma]^{-1} \leqq [\sigma_i^k \kappa_i^{-3} |\xi^i|^2]^{\frac{1}{2}} [\sigma_l^k \kappa_l^{-1}]^{\frac{1}{2}} [\sigma_j^k \kappa_j^{-3} |\xi^j|^2]^{\frac{1}{2}} \cdot [\sigma_m^k \kappa_m^{-1}]^{\frac{1}{2}} [\kappa_k^{-1} + \varepsilon\sigma]^{-1} = \sigma_i^k \kappa_i^{-3} |\xi^i|^2$$

since

$$(1.37)\qquad \sigma_i^k \kappa_i^{-1} = \kappa_k^{-1} + \varepsilon\sigma\ .$$

The concavity of F_ε is therefore proved.

The remaining claims of the lemma (1.29) and (1.30) we do not need in the following and we therefore leave the proof to the interested reader.

Lemma 1.9. *Let $F \in (K)$ then F_ε is also of class (K).*

Proof. We only have to show that inequality (1.15) is valid. Let $(\eta_{ij}) \in \mathscr{S}$, then we choose a coordinate system such that $h_{ij} = \kappa_i \delta_{ij}$ and decompose (η_{ij}) in a diagonal part $(\hat{\eta}_{ij})$ and a diagonal zero part $(\bar{\eta}_{ij})$ and prove the estimates for each part separately.

First, we consider

$$(1.38)\qquad F_\varepsilon^{ij,kl} \bar{\eta}_{ij} \bar{\eta}_{kl}\ .$$

From the Remark 1.4. we conclude that the estimate (1.15) is valid for $(\bar{\eta}_{ij})$ if F_ε satisfies the relation (1.18). Thus, let $\kappa_j < \kappa_i$; then (no summation over i)

$$(1.39)\qquad \begin{aligned} \frac{\partial F_\varepsilon}{\partial \kappa_i} \kappa_i &= F_k \sigma_i^k [\kappa_k^{-1} + \varepsilon\sigma]^{-2} \kappa_i^{-1} \\ &= \varepsilon F_k [\kappa_k^{-1} + \varepsilon\sigma]^{-2} \kappa_i^{-1} + F_i [\kappa_i^{-1} + \varepsilon\sigma]^{-2} \kappa_i^{-1} \end{aligned}$$

where σ_i^k is defined as before, and we obtain

$$(1.40)\qquad \begin{aligned} \frac{\partial F_\varepsilon}{\partial \kappa_i} \kappa_i - \frac{\partial F_\varepsilon}{\partial \kappa_j} \kappa_j &= \varepsilon F_k [\kappa_k^{-1} + \varepsilon\sigma]^{-2} [\kappa_i^{-1} - \kappa_j^{-1}] \\ &\quad + F_i [\kappa_i^{-1} + \varepsilon\sigma]^{-1} \frac{\kappa_i^{-1}}{\kappa_i^{-1} + \varepsilon\sigma} - F_j [\kappa_j^{-1} + \varepsilon\sigma]^{-1} \frac{\kappa_j^{-1}}{\kappa_j^{-1} + \varepsilon\sigma} \\ &\leqq F_j [\kappa_j^{-1} + \varepsilon\sigma]^{-1} \left[\frac{\kappa_i^{-1}}{\kappa_i^{-1} + \varepsilon\sigma} - \frac{\kappa_j^{-1}}{\kappa_j^{-1} + \varepsilon\sigma} \right] \leqq 0 \end{aligned}$$

where we used that

$$(1.41)\qquad F_i [\kappa_i^{-1} + \varepsilon\sigma]^{-1} \leqq F_j [\kappa_j^{-1} + \varepsilon\sigma]^{-1}$$

because F satisfies (1.18).

Next, let us demonstrate

$$F_\varepsilon^{ij,kl}\,\hat\eta_{ij}\,\hat\eta_{kl} \leqq cF_\varepsilon^{-1}(F_\varepsilon^{ij}\,\hat\eta_{ij})^2 - F_\varepsilon^{ik}\,\tilde h^{jl}\,\hat\eta_{ij}\,\hat\eta_{kl} \tag{1.42}$$

if this inequality is valid for F.

In view of the special choice of $(\hat\eta_{ij})$ this inequality looks like

$$\frac{\partial^2 F_\varepsilon}{\partial\kappa_i\partial\kappa_j}\,\hat\eta_{ii}\,\hat\eta_{jj} \leqq cF_\varepsilon^{-1}\left(\frac{\partial F_\varepsilon}{\partial\kappa_i}\,\hat\eta_{ii}\right)^2 - \frac{\partial F_\varepsilon}{\partial\kappa_i}\kappa_i^{-1}|\,\hat\eta_{ii}|^2\,. \tag{1.43}$$

From (1.35) and (1.36) we deduce that

$$\begin{aligned}\frac{\partial^2 F_\varepsilon}{\partial\kappa_i\partial\kappa_j}\,\hat\eta_{ii}\,\hat\eta_{jj} \leqq{}& F_{kl}[\kappa_k^{-1}+\varepsilon\sigma]^{-2}[\kappa_l^{-1}+\varepsilon\sigma]^{-2}\sigma_i^k\kappa_i^{-2}\,\hat\eta_{ii}\sigma_j^l\kappa_j^{-2}\,\hat\eta_{jj}\\ &+F_k[\kappa_k^{-1}+\varepsilon\sigma]^{-3}[\sigma_i^k\kappa_i^{-2}\,\hat\eta_{ii}]^2 - F_k[\kappa_k^{-1}+\varepsilon\sigma]^{-2}\sigma_i^k\kappa_i^{-3}|\,\hat\eta_{ii}|^2\,.\end{aligned} \tag{1.44}$$

Now, F satisfies (1.43), i.e. the right-hand side of (1.44) is estimated from above by

$$\begin{aligned}&cF^{-1}\{F_k[\kappa_k^{-1}+\varepsilon\sigma]^{-2}\sigma_i^k\kappa_i^{-2}\,\hat\eta_{ii}\}^2 - F_k[\kappa_k^{-1}+\varepsilon\sigma]\{[\kappa_k^{-1}+\varepsilon\sigma]^{-2}\sigma_i^k\kappa_i^{-2}\,\hat\eta_{ii}\}^2\\ &\quad+F_k[\kappa_k^{-1}+\varepsilon\sigma]\{[\kappa_k^{-1}+\varepsilon\sigma]^{-2}\sigma_i^k\kappa_i^{-2}\,\hat\eta_{ii}\}^2 - F_k[\kappa_k^{-1}+\varepsilon\sigma]^{-2}\sigma_i^k\kappa_i^{-2}\kappa_i^{-1}|\,\hat\eta_{ii}|^2\\ &\quad= cF_\varepsilon^{-1}\left(\frac{\partial F_\varepsilon}{\partial\kappa_i}\,\hat\eta_{ii}\right)^2 - \frac{\partial F_\varepsilon}{\partial\kappa_i}\kappa_i^{-1}|\,\hat\eta_{ii}|^2\end{aligned} \tag{1.45}$$

and the lemma is proved.

The preceding considerations are also applicable if the κ_i are the principal curvatures of a hypersurface M with metric (g_{ij}). F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) with eigenvalues κ_i with respect to the metric.

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{1.46}$$

is then a contravariant tensor of second order. Sometimes, it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$h_j^i = g^{ik}h_{kj}\,. \tag{1.47}$$

Then

$$F_i^j = \frac{\partial F}{\partial h_j^i} \tag{1.48}$$

is also a mixed tensor with contravariant index j and covariant index i.

2. Notations and preliminary results

Let N be a complete $(n+1)$-dimensional Riemannian manifold and M a closed hypersurface. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), (R_{ijkl}), etc.. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function u on N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but, e.g. the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\varepsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\varepsilon} x_i^\varepsilon \tag{2.1}$$

with obvious generalizations to other quantities.

In local coordinates x^α and ξ^i the geometric quantities of the hypersurface M are connected through the following equations

$$x_{ij}^\alpha = -h_{ij}\nu^\alpha \tag{2.2}$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma\ . \tag{2.3}$$

The comma indicates ordinary partial derivatives.

In this implicit definition (2.2) the *second fundamental form* (h_{ij}) is taken with respect to $-\nu$.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha\ , \tag{2.4}$$

where we remember that ν_i^α is full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta \tag{2.5}$$

and the *Gauß equation*

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \bar{R}_{\alpha\beta\gamma\delta}\, x_i^\alpha\, x_j^\beta\, x_\kappa^\gamma\, x_l^\delta\ . \tag{2.6}$$

We assume that the domain $\bar{\Omega}$ is contained in a *normal Gaussian coordinate* neighbourhood $\mathscr{U} = (r_1, r_2) \times S_0$ with coordinates $(x^\alpha) = (r, x^i)$ such that

$$d\bar{s}^2 = dr^2 + \bar{g}_{ij}dx^i dx^j \tag{2.7}$$

where $r = x^0$, $\bar{g}_{ij} = \bar{g}_{ij}(r,x)$; here we use slightly ambiguous notation. S_0 is a compact n-dimensional Riemannian manifold homeomorphic to S^n and we identify S_0 and its image in N which is a closed hypersurface.

A point $p \in \mathscr{U}$ can be represented by its signed distance from S_0 and its base point $x \in S_0$, thus $p = (r,x)$.

Let $M \subset \mathscr{U}$ be a hypersurface which is a graph over S_0, i.e.

$$M = \{(r,x)\colon r = u(x),\ x \in S_0\}\ . \tag{2.8}$$

The induced metric of M, g_{ij}, can then be expressed as

$$g_{ij} = \bar g_{ij} + u_i u_j \tag{2.9}$$

with inverse

$$g^{ij} = \bar g^{ij} - \frac{u^i}{v}\frac{u^j}{v}\ , \tag{2.10}$$

where $(\bar g^{ij}) = (\bar g_{ij})^{-1}$ and

$$u^i = \bar g^{ij} u_j \qquad v^2 = 1 + \bar g^{ij} u_i u_j\ . \tag{2.11}$$

The normal vector v of M then takes the form

$$(v^\alpha) = v^{-1}(1, -u^i) \tag{2.12}$$

if x^0 is chosen appropriately.

From the Gauß formula we immediately deduce that the second fundamental form of M is given by

$$v^{-1} h_{ij} = -u_{ij} + \bar h_{ij}\ , \tag{2.13}$$

where

$$\bar h_{ij} = \frac{1}{2}\dot{\bar g}_{ij} = \frac{1}{2}\frac{\partial \bar g_{ij}}{\partial r} \tag{2.14}$$

is the second fundamental form of the level surfaces $\{r = \text{const}\}$, and where the second covariant derivatives of u are defined with respect to the induced metric.

Assume now, that $M = \operatorname{graph} u$ is strictly convex, then the principal curvatures of M with respect to the normal in the direction of the mean curvature vector Δx are positive. Thus, if we orient the coordinate axes r such that

$$\left\langle \frac{\partial}{\partial r}, \Delta x \right\rangle < 0\ , \tag{2.15}$$

then (h_{ij}) in formula (2.13) is positive definite.

Therefore, we choose r such that (2.15) is satisfied for the barrier M_1 – and hence also for M_2. Furthermore, we shall assume that r is positive in $\overline{\Omega}$.

Let $M_i = \operatorname{graph} u_i$, $i = 1,2$, then we conclude

$$u_1 \leqq u_2 \tag{2.16}$$

in view of our assumptions.

In fact, the strict inequality is valid in (2.16) unless $u_1 \equiv u_2$ and M_1 is a solution to our existence problem as can be deduced from the Harnack inequality.

In [8, Lemma 6.1] we proved that for convex graphs M the quantity v is uniformly bounded.

Lemma 2.1. *Let $M = \operatorname{graph} u|_{S_0}$ be a closed convex hypersurface represented in normal Gaussian coordinates, then the quantity $v = \sqrt{1 + |Du|^2}$ can be estimated by*

$$v \leqq c(|u|, S_0, \bar{g}_{ij}) . \tag{2.17}$$

The M_i are barriers for the pair (F, f). Let us remark that without loss of generality we may assume

$$F|_{M_1} < f \tag{2.18}$$

and

$$F|_{M_2} > f , \tag{2.19}$$

for, let $\eta \in C^\infty(\bar{\Omega})$ be a function with support in a small neighbourhood of $M_1 \cup M_2$ such that

$$\eta|_{M_1} > 0 \quad \text{and} \quad \eta|_{M_2} < 0 \tag{2.20}$$

and define for $\delta > 0$

$$f_\delta = f + \delta\eta . \tag{2.21}$$

Then, for small δ

$$f_\delta \geqq \frac{1}{2} f \tag{2.22}$$

and the M_i are barriers for (F, f_δ) satisfying the strict inequalities; since we shall derive $C^{4,\alpha}$-estimates independent of δ, we shall have proved the existence for f if we can prove it for f_δ.

Next, let us observe that it is sufficient to prove the existence for curvature functions $F \in (K)$ with

$$F_i \leqq \text{const} \quad \forall i , \tag{2.23}$$

for let F_ε be the elliptic regularizations of F, then $F_\varepsilon \in (K)$ satisfies (2.23), the M_i are barriers for (F_ε, f) for small ε in view of (2.18), (2.19) and if we can solve

$$F_\varepsilon|_M = f \tag{2.24}$$

with $M \subset \bar{\Omega}$, then we shall prove that all estimates are independent of ε.

We shall now demonstrate this for the lower bound on κ_i.

Lemma 2.2. *Let M be a strictly convex solution of* (2.24) *such that the principal curvatures of M, κ_i, can be bounded from above independent of ε, then there is $\varepsilon_0 > 0$ such that*

$$\kappa_i \geqq \varepsilon_0 \quad \forall \varepsilon . \tag{2.25}$$

Proof. We only use the simple estimate

$$F|_M \geqq F_\varepsilon|_M = f > 0 \tag{2.26}$$

and the fact that $F|_{\partial\Gamma_+}$ vanishes.

Thus, we shall assume in the following that $F \in (K)$ satisfies (2.23), and $f \in C^{2,\alpha}(\bar{\Omega})$ the inequalities (2.18) and (2.19).

3. C^2-estimates for solutions of an auxiliary problem

Let $M_0 = \operatorname{graph} u_0|_{S_0}$ be a supersolution for (F, f), i.e.

$$F|_{M_0} \geqq f \; . \tag{3.1}$$

Then, we want to prove that the auxiliary problem

$$F = f - \gamma e^{-\mu u}[u - u_0] \equiv \tilde{f} \tag{3.2}$$

has a smooth solution u satisfying

$$u_1 \leqq u \leqq u_0 \tag{3.3}$$

if μ, γ are sufficiently large.

In this section we shall derive a priori estimates for the C^2-norm of u or equivalently for the C^0-norm of the principal curvatures of M.

Let us first derive the elliptic equation for the second fundamental form.

Lemma 3.1. *Let M be a solution of the problem* (3.2), *then the second fundamental form satisfies*

$$\begin{aligned} -F^{kl}h_{ij;kl} &= F^{kl}h_{kr}h_l^r h_{ij} - F h_i^k h_{kj} - \tilde{f}_{\alpha\beta} x_i^\alpha x_j^\beta + \tilde{f}_\alpha \nu^\alpha h_{ij} \\ &\quad + F^{kl,rs} h_{kl;i} h_{rs;j} + 2F^{kl}\bar{R}_{\alpha\beta\gamma\delta} x_r^\alpha x_i^\beta x_k^\gamma x_j^\delta h_l^r - F^{kl}\bar{R}_{\alpha\beta\gamma\delta} x_r^\alpha x_k^\beta x_i^\gamma x_l^\delta h_j^r \\ &\quad - F^{kl}\bar{R}_{\alpha\beta\gamma\delta} x_r^\alpha x_k^\beta x_j^\gamma x_l^\delta h_i^r + F^{kl}\bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_{ij} \\ &\quad - F\bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta + F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\varepsilon}\{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_j^\varepsilon + \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta x_l^\varepsilon\} \; . \end{aligned} \tag{3.4}$$

Proof. We start with equation (3.2) and differentiate both sides covariantly twice. First, we obtain

$$F_i = F^{kl} h_{kl;i} \tag{3.5}$$

and

$$F_{ij} = F^{kl} h_{kl;ij} + F^{kl,rs} h_{kl;i} h_{rs;j} \; . \tag{3.6}$$

Next, we replace $h_{kl;ij}$ by $h_{ij;kl}$. Differentiating the Codazzi equation

$$h_{kl;i} = h_{ik;l} + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_k^\beta x_l^\gamma x_i^\delta \tag{3.7}$$

we obtain

$$\begin{aligned} h_{kl;ij} &= h_{ik;lj} + \bar{R}_{\alpha\beta\gamma\delta;\varepsilon} \nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_j^\varepsilon \\ &\quad + \bar{R}_{\alpha\beta\gamma\delta}\{\nu_j^\alpha x_k^\beta x_l^\gamma x_i^\delta + \nu^\alpha x_{kj}^\beta x_l^\gamma x_i^\delta + \nu^\alpha x_k^\beta x_{lj}^\gamma x_i^\delta + \nu^\alpha x_k^\beta x_l^\gamma x_{ij}^\delta\} \; . \end{aligned} \tag{3.8}$$

We now use the *Ricci identities*

$$h_{ik;lj} = h_{ik;jl} + h_{ak}R^a_{ilj} + h_{ai}R^a_{klj} \tag{3.9}$$

and differentiate once again the Codazzi equation

$$h_{ik;j} = h_{ij;k} + \overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_k x^\delta_j \tag{3.10}$$

to replace $h_{kl;ij}$ by $h_{ij;kl}$.

To replace $\tilde{f}_{ij}$ we use the chain rule

$$\tilde{f}_i = \tilde{f}_\alpha x^\alpha_i \qquad \tilde{f}_{ij} = \tilde{f}_{\alpha\beta} x^\alpha_i x^\beta_j + \tilde{f}_\alpha x^\alpha_{ij}\ . \tag{3.11}$$

Then, using the Gauß equation and Gauß formula, the symmetry properties of the Riemann curvature tensor and the homogeneity of F, i.e.

$$F = F^{kl}h_{kl} \tag{3.12}$$

we deduce the equation (3.4).

Since the mixed tensor h^i_j is a more natural geometric object, let us look at the differential equation for h^i_i.

Lemma 3.2. *The differential equation for* h^i_i (*no summation over* i) *has the form*

$$\begin{aligned}
-F^{kl}h^i_{i;kl} &= F^{kl}h_{kr}h^r_l h^i_i - F h^k_i h^i_k - \tilde{f}_{\alpha\beta}x^\alpha_i x^\beta_j g^{ji} + \tilde{f}_\alpha \nu^\alpha h^i_i \\
&\quad + F^{kl,rs}h_{kl;i}h_{rs;j}g^{ji} + 2F^{kl}\overline{R}_{\alpha\beta\gamma\delta}x^\alpha_r x^\beta_i x^\gamma_k x^\delta_j g^{ji}h^r_l - F^{kl}\overline{R}_{\alpha\beta\gamma\delta}x^\alpha_r x^\beta_k x^\gamma_i x^\delta_l h^{ri} \\
&\quad - F^{kl}\overline{R}_{\alpha\beta\gamma\delta}x^\alpha_r x^\beta_k x^\gamma_j x^\delta_l h^{ri} + F^{kl}\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_k \nu^\gamma x^\delta_l h^i_i \\
&\quad - F\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i \nu^\gamma x^\delta_j g^{ji} + F^{kl}\overline{R}_{\alpha\beta\gamma\delta;\varepsilon}\{\nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\varepsilon_j + \nu^\alpha x^\beta_i x^\gamma_k x^\delta_j x^\varepsilon_l\}g^{ji}\ .
\end{aligned} \tag{3.13}$$

Consider now the quantity $v = \sqrt{1 + |Du^2|}$. We know that v is uniformly bounded because M is convex, and we shall further exploit this fact by using v as a comparison function.

Lemma 3.3. *Let* $M = \operatorname{graph} u|_{S_0}$ *be a strictly convex solution of* (3.2), *then* v *satisfies the elliptic equation*

$$\begin{aligned}
-F^{ij}v_{ij} &= -F^{ij}h_{ik}h^k_j v - 2v^{-1}F^{ij}v_i v_j + r_{\alpha\beta}\nu^\alpha\nu^\beta F v^2 \\
&\quad + F^{ij}\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k r_\varepsilon x^\varepsilon_m g^{mk} v^2 + 2F^{ij}r_{\alpha\beta}h^k_i x^\alpha_k x^\beta_j v^2 \\
&\quad + F^{ij}r_{\alpha\beta\gamma}\nu^\alpha x^\beta_i x^\gamma_j v^2 + \tilde{f}_\alpha x^\alpha_m g^{mk} r_\beta x^\beta_k v^2\ .
\end{aligned} \tag{3.14}$$

Proof. From (2.12) we conclude

$$v = (r_\alpha \nu^\alpha)^{-1} \,. \tag{3.15}$$

Differentiating this equation we obtain

$$v_i = -v^2\{r_{\alpha\beta}\nu^\alpha x_i^\beta + r_\alpha \nu_i^\alpha\} \tag{3.16}$$

and

$$v_{ij} = 2v^{-1}v_i v_j - v^2\{r_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma + r_{\alpha\beta}\nu_j^\alpha x_i^\beta + r_{\alpha\beta}\nu^\alpha x_{ij}^\beta + r_{\alpha\beta}\nu_i^\alpha x_j^\beta + r_\alpha \nu_{ij}^\alpha\} \,. \tag{3.17}$$

Inserting the last relation in the left-hand side of (3.14) and simplifying the resulting expression with the help of the Weingarten and Codazzi equations we arrive at the desired conclusion.

Lemma 3.4. *For convex hypersurfaces which stay in a compact domain we have*

$$|F^{ij} r_{\alpha\beta} h_i^k x_k^\alpha x_j^\beta| \leqq cF \,. \tag{3.18}$$

Proof. Choose a coordinate system ξ^i such that in a fixed but arbitrary point in M

$$g_{ij} = \delta_{ij}, \qquad h_{ij} = \kappa_i \delta_{ij} \,. \tag{3.19}$$

Then,

$$|F^{ij} r_{\alpha\beta} h_i^k x_k^\alpha x_j^\beta| \leqq \sum_i |F^{ii} h_i^i| \sup|D^2 r| = F^{ij} h_{ij} \sup|D^2 r| = F \sup|D^2 r| \,. \tag{3.20}$$

Next, consider the function $\tilde\psi = \psi|_M$. It satisfies the elliptic equation

$$-F^{ij}\tilde\psi_{ij} = \psi_\alpha \nu^\alpha F - F^{ij}\psi_{\alpha\beta} x_i^\alpha x_j^\beta \,, \tag{3.21}$$

where we used the homogeneity of F.

Futhermore, in view of the strict convexity of ψ

$$\psi_{\alpha\beta} \geqq c\bar g_{\alpha\beta} \,, \tag{3.22}$$

i.e.

$$F^{ij}\psi_{\alpha\beta} x_i^\alpha x_j^\beta \geqq cF^{ij} g_{ij} \geqq cF(1,\ldots,1) \tag{3.23}$$

with a positive constant c. The second estimate in (3.23) follows from

Lemma 3.5. *Let* $F \in C^2(\Gamma_+)$ *be homogeneous of degree* 1, *monotone increasing and concave*, *then*

$$F^{ij} g_{ij} \geqq F(1,\ldots,1) \,. \tag{3.24}$$

A proof can be found in [14, Lemma 3.2].

We are now ready to prove the a priori estimate for the second derivatives of u.

Lemma 3.6. *Let F be of class (K) and let M be a strictly convex solution of* (3.2), (3.3), *then the principal curvatures of M can be a priori bounded from above. More precisely, let $|A|$ denote the length of the second fundamental form of M and let $|A_0|$ be the corresponding quantity for M_0, then the estimate*

$$|A|^2 \leqq c(1 + |A_0|) , \tag{3.25}$$

is valid, where the constant c is larger than 1 *and depends on the C^2-norms of f and ψ, $\inf_{\overline{\Omega}} f, \mu, \gamma$, and on geometric quantities of the ambient space in the domain $\overline{\Omega}$.*

Proof. The proof follows the lines of the proof of a corresponding lemma in [8, Lemma 8.2].

Let φ be defined by

$$\varphi = \sup\{h_{ij}\eta^i\eta^j : \|\eta\| = 1\} \tag{3.26}$$

and w by

$$w = \log\varphi + \log v + \lambda\tilde{\psi} \tag{3.27}$$

where λ is a large positive parameter. We claim that w is bounded.

Let x_0 be a point in M such that

$$\sup_M w \leqq w(x_0) . \tag{3.28}$$

We then can introduce a Riemannian normal coordinate system ξ^i at $x_0 \in M$ such that at x_0 we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h^n_n . \tag{3.29}$$

Let $\eta = (\eta^i)$ be the contravariant vector defined by

$$\eta = (0, \dots, 0, 1) \tag{3.30}$$

and set

$$\tilde{\varphi} = \frac{h_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j} . \tag{3.31}$$

$\tilde{\varphi}$ is well defined in a neighbourhood of ξ_0.

Now, define $\tilde{w}$ by replacing φ by $\tilde{\varphi}$ in (3.27); then $\tilde{w}$ assumes its maximum at ξ_0. Moreover, at ξ_0 we have

$$\tilde{\varphi}_i = h^n_{n;i} \quad \text{and} \quad \varphi_{ij} = h^n_{n;ij} \tag{3.32}$$

i.e., $\tilde{\varphi}$ satisfies at ξ_0 the same differential equation (3.13) as h^n_n. For the sake of greater clarity, let us therefore treat h^n_n like a scalar and pretend that w is defined by

$$w = \log h^n_n + \log v + \lambda\tilde{\psi} . \tag{3.33}$$

Applying the maximum principle at ξ_0, we deduce from (3.13), (3.14) and (3.21)

$$0 \leqq -Fh^n_n + c + F^{ij}g_{ij}c - \tilde{f}_{\alpha\beta}x^\alpha_n x^\beta_k g^{kn}(h^n_n)^{-1} + \lambda \quad {}_\alpha \nu^\alpha F - \lambda F^{ij}\psi_{\alpha\beta}x^\alpha_i x^\beta_j - F^{ij}(\log v)_i(\log v)_j + F^{ij}(\log h^n_n)_i(\log h^n_n)_j + F^{kl,rs}h_{kl,n}h_{rs,m}g^{mn}(h^n_n)^{-1} \tag{3.34}$$

where we have estimated bounded terms by a positive constant c and assumed that $h^n_n \geqq 1$.

Now, the last term in the preceding inequality is estimated from above by

$$cF^{-1}\tilde{f}_n\tilde{f}_m g^{mn} - (h^n_n)^{-2}F^{ij}h_{in;n}h_{jn;m}g^{mn} , \tag{3.35}$$

cf. Lemma 1.2. Moreover, because of the Codazzi equation we have

$$h_{in;n} = h_{nn;i} + \overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_n x^\gamma_i x^\delta_n \tag{3.36}$$

and hence, when we abbreviate the curvature term by $\overline{R}_i$, we conclude that the crucial term in (3.35) is equal to

$$-(h^n_n)^{-2}F^{ij}(h^n_{n;i} + \overline{R}_i)(h^n_{n;j} + \overline{R}_j) . \tag{3.37}$$

Thus, the terms in (3.34) containing the derivatives are estimated from above by

$$cF^{-1}\tilde{f}_n\tilde{f}_m g^{mn} - F^{ij}(\log v)_i(\log v)_j - 2(h^n_n)^{-1}F^{ij}(\log h^n_n)_i\overline{R}_j . \tag{3.38}$$

Moreover, at ξ_0 Dw vanishes, i.e.

$$D\log h^n_n = -D\log v - \lambda Du \tag{3.39}$$

and (3.38) is further estimated from above by

$$cF^{-1}\tilde{f}_n\tilde{f}_m g^{mn} + (h^n_n)^{-1}c\lambda F^{ij}g_{ij} , \tag{3.40}$$

where we assumed $\lambda \geqq 1$.

Summarizing, we deduce from (3.34)

$$0 \leqq \{-Fh^n_n + c + \lambda \quad {}_\alpha \nu^\alpha F - \tilde{f}_{\alpha\beta}x^\alpha_n x^\beta_m g^{mn}(h^n_n)^{-1} + cF^{-1}\tilde{f}_n\tilde{f}_m g^{mn}\} + \{cF^{ij}g_{ij} + (h^n_n)^{-1}c\lambda F^{ij}g_{ij} - \lambda F^{ij}\psi_{\alpha\beta}x^\alpha_i x^\alpha_j\} . \tag{3.41}$$

We now choose λ very large and conclude in view of (3.23) that the second term in (3.41) is negative provided h^n_n is large enough.

Let us now have a closer look at the crucial term involving the second derivatives of $\tilde{f}$. In the Gaussian coordinate system (r, x^i) we write $\tilde{f}$ as

$$\tilde{f} = f - \gamma e^{-\mu u}[u - u_0] \tag{3.42}$$

and deduce that we only have to worry about the second derivatives of u_0 with respect to the metric in the ambient space. Let us abbreviate their norm in $\overline{\Omega}$ with $\|D^2u_0\|_{\overline{\Omega}}$, then we shall show in Lemma 3.7 below

$$\|D^2u_0\|_{\overline{\Omega}} \leqq c(1 + |A_0|_{M_0}) \, , \tag{3.43}$$

where

$$|A_0|_{M_0} = \sup_{M_0} |A_0| \, , \tag{3.44}$$

and the constant c depends on $\overline{\Omega}$ and some geometric quantities of the ambient space restricted to $\overline{\Omega}$.

Hence, we deduce from (3.41) that at $x_0 \in M$ the estimate

$$|h^n_n|^2 \leqq c(1 + |A_0|_{M_0}) \tag{3.45}$$

is valid, where c depends on the quantities mentioned in Lemma 3.6; here, we also used the relation

$$F = \tilde{f} \geqq f \, . \tag{3.46}$$

To complete the proof of the lemma observe that by the very definition of $w(x_0)$ we have at x_0

$$|A| \leqq c(1 + h^n_n) \tag{3.47}$$

from which we infer the estimate (3.25) in view of (3.45).

Lemma 3.7. *Let* $M = \operatorname{graph} u|_{S_0} \subset \overline{\Omega}$ *be a closed hypersurface, where we assume that* $\overline{\Omega}$ *is contained in a normal Gaussian coordinate neighbourhood* $\mathscr{U}$, *then, when viewing* $u = u(r,x) = u(x)$ *as being defined in* $\overline{\Omega}$, *we have*

$$\|D^2u\|_{\overline{\Omega}} \leqq c(1 + |A|_M), \quad |A|_M = \sup_M |A| \tag{3.48}$$

where c depends on $v = \sqrt{1 + |Du|^2}$, $\bar{g}_{ij}$, *and on* $\bar{h}_{ij} = \frac{1}{2}\dot{\bar{g}}_{ij}$.

Proof. Let (r, x^i) be the normal Gaussian coordinates, then the (x^i) are also coordinates for $M = \operatorname{graph} u$. The metric $\bar{g}_{\alpha\beta}$ has the form

$$d\bar{s}^2 = dr^2 + \bar{g}_{ij}(x,r)dx^i dx^j \tag{3.49}$$

and the metric of M is

$$g_{ij} = u_i u_j + \bar{g}_{ij}(x,u) \, . \tag{3.50}$$

Indicate covariant derivatives with respect to (3.49) by a semicolon, with respect to (3.50) simply by indices, and normal partial derivatives by a comma.

The only non-zero covariant second derivatives of u in N are of the form

$$u_{;ij} = u_{,ij} - \bar{\Gamma}^k_{ij} u_k \, , \tag{3.51}$$

where $\bar{\Gamma}^k_{ij}$ are Christoffel symbols in N.

We evaluate (3.51) at different points in $\overline{\Omega}$, first in (x,u) and secondly in (x,r), then we have

$$u_{;ij}|_{(x,u)} = u_{;ij}|_{(x,r)} + \{\overline{\Gamma}^k_{ij}|_{(x,r)} - \overline{\Gamma}^k_{ij}|_{(x,u)}\}u_k \ . \tag{3.52}$$

Thus, taking (2.13) into account, we conclude that (3.48) will be proved if we can show

$$u_{;ij} = v^2 u_{ij} - v^2 \bar{h}_{ij}\|Du\|^2 + v^2 \bar{h}_{ik}u^k u_j + v^2 \bar{h}_{jk}u^k u_i \ , \tag{3.53}$$

where $\|Du\|$ is the norm with respect to g_{ij} and the indices are also raised with respect to this metric.

To prove (3.53) we choose a coordinate system such that at (x,u) $\overline{\Gamma}^k_{ij} = 0$, or equivalently,

$$\frac{\partial \bar{g}_{ij}}{\partial x^k} = 0 \quad \forall (i,j,k) \tag{3.54}$$

and obtain

$$u_{;ij} = u_{,ij} \tag{3.55}$$

and by straightforward computation

$$u_{ij} = u_{,ij} - \Gamma^k_{ij}u_k = [1 - \|Du\|^2]u_{,ij} + \bar{h}_{ij}\|Du\|^2 - \bar{h}_{ik}u^k u_j - \bar{h}_{jk}u^k u_i \tag{3.56}$$

hence the result.

4. Existence of solutions to the auxiliary problem

Assuming the conditions of Theorem 0.4 to be valid, let $M = \operatorname{graph} u_0|_{S_0} \subset \overline{\Omega}$ be a strictly convex supersolution of the pair (F, f) of class $C^{4,\alpha}$, where in addition F is supposed to satisfy

$$\frac{\partial F}{\partial \kappa_i} \leqq c_0 \quad \forall i \ , \tag{4.1}$$

then we shall prove

Theorem 4.1. *The auxiliary problem*

$$F|_M = f - \gamma e^{-\mu u}[u - u_0] \tag{4.2}$$

has a strictly convex solution $M = \operatorname{graph} u|_{S_0} \subset \overline{\Omega}$ *of class* $C^{4,\alpha}$ *such that*

$$u_1 \leqq u \leqq u_0 \tag{4.3}$$

provided the positive constants $\mu = \mu(f, \Omega)$ *and* $\gamma = \gamma(\mu, c_0, \Omega)$ *are sufficiently large. Here, the reference that a term depends on* Ω *should also indicate that geometrical quantities of the ambient space and of the barriers are involved.*

Proof. Extend

$$(4.4)\qquad f_0 = F|_{M_0}$$

to $\overline{\Omega}$ by setting

$$(4.5)\qquad f_0(x,r) = f_0(x), \quad x \in S_0 \, ,$$

and consider the convex combination

$$(4.6)\qquad f_t = tf + (1-t)f_0, \quad 0 \leqq t \leqq 1 \, .$$

We shall show that the problem

$$(4.7)\qquad \begin{cases} F|_{M_t} = f_t - \gamma e^{-\mu u_t}[u_t - u_0] \\ u_1 \leqq u_t \leqq u_0 \\ u_t \in C^{4,\alpha}(S_0) \end{cases}$$

has a solution for all $t \in [0,1]$ by using the continuity method. There is a slight ambiguity in the notations for $t = 1$, but that should not cause any confusion.

Let Λ be the set of all $t \in [0,1]$ such that (4.7) has a solution, then, Λ is non-empty for $0 \in \Lambda$ and we shall show that Λ is both open and closed.

(i) Λ *is closed*, for, let $t \in \Lambda$, then we have

$$(4.8)\qquad |u_t|_{2,S_0} \leqq \text{const}$$

independent of t if γ, μ are sufficiently large, cf. Sect. 3. Hence, we are able to apply the $C^{2,\alpha}$-estimates, cf. e.g. [10], because the operator is now uniformly elliptic and is confined to a compact subset of Γ_+; note that

$$(4.9)\qquad f_t - \gamma e^{-\mu u_t}[u_t - u_0] \geqq f_t \geqq \varepsilon_0 > 0 \, .$$

But then, the Schauder theory can be applied leading to uniform $C^{4,\alpha}$-estimates, i.e. Λ is closed.

(ii) Λ *is open*. Let $t_0 \in \Lambda$ and define $\tilde{u} = u_{t_0}$. For brevity set

$$(4.10)\qquad \tilde{f} = f_t - \gamma e^{-\mu r}[r - u_0] \, ,$$

where we drop the subscript t of $\tilde{f}$.

As we shall prove in Lemma 4.8 below, the linearization

$$(4.11)\qquad L\varphi = \frac{d}{d\varepsilon}[F(\tilde{u} + \varepsilon\varphi) - \tilde{f}(\tilde{u} + \varepsilon\varphi)]\Big|_{\varepsilon=0}$$

is an elliptic operator of the form

$$(4.12)\qquad L\varphi = -a^{ij}\varphi_{ij} + b^i D_i \varphi + c\varphi$$

with $C^{1,\alpha}$ coefficients such that

$$(4.13)\qquad c = c(x) \geqq \varepsilon_0 > 0 \, .$$

Thus, L is a homeomorphism from $C^{3,\alpha}(S_0)$ onto $C^{1,\alpha}(S_0)$ and in view of the inverse function theorem we obtain the existence of solutions $u_t \in C^{3,\alpha}(S_0)$ of the equation

$$F = \tilde{f} \tag{4.14}$$

if $|t - t_0|$ is small, but these solutions are then of class $C^{4,\alpha}$.

We claim furthermore

Lemma 4.2.

$$u_1 \leqq u_t \leqq u_0 \tag{4.15}$$

if $|t - t_0|$ is small and μ, γ are sufficiently large.

Proof. We first observe that u_1 is a subsolution for $(F, \tilde{f})$ and u_0 a supersolution, because in the case of u_1

$$F|_{M_1} \leqq f = tf + (1-t)f \tag{4.16}$$

and

$$\begin{aligned} f(x, u_1) &\leqq f(x, u_0) - \gamma e^{-\mu u_1}[u_1 - u_0] \\ &\leqq f_0(x) - \gamma e^{-\mu u_1}[u_1 - u_0] \, . \end{aligned} \tag{4.17}$$

The last inequality is merely a restatement of the fact that u_0 is a supersolution for (F, f), while the first inequality is due to the monotonicity of

$$\varphi(r) = f(x, r) + \gamma e^{-\mu r}[r - u_0] \tag{4.18}$$

in the interval $u_1 \leqq r \leqq u_0$ for large γ. Hence, u_1 is a subsolution.

To prove that u_0 is a supersolution we estimate

$$\begin{aligned} \tilde{f}(x, u_0) &\leqq tf(x, u_0) + (1-t) f_0(x) \\ &\leqq tf_0(x) + (1-t) f_0 = F|_{M_0} \, . \end{aligned} \tag{4.19}$$

If one of the inequalities in (4.15) is strict for $t = t_0$ in S_0, then it will also be valid for small $|t - t_0|$ by continuity. Thus, suppose that one of the inequalities in (4.15) is not strict for $t = t_0$, e.g. assume $u_1 = \tilde{u}$ at some point $x_0 \in S_0$. Then, the Harnack inequality or the strict maximum principle would yield

$$u_1 \equiv \tilde{u} \, . \tag{4.20}$$

But then the convex combination

$$\tau u_1 + (1-\tau) u_t, \quad 0 \leqq \tau \leqq 1 \tag{4.21}$$

would be admissible functions for small $|t - t_0|$, i.e. their graphs would be strictly convex hypersurfaces and we could apply the maximum principle to

$$(4.22)\qquad 0 = F(u_1) - F(u_t) + \tilde{f}(u_t) - \tilde{f}(u_1) = \int_0^1 \frac{d}{d\tau}\{\cdots\} = -a^{ij}\varphi_{ij} + b^i\varphi_i + c\varphi$$

where $\varphi = u_1 - u_t$ and $c = c(x) > 0$, since for $t = t_0$ the coefficients of the right-hand side are exactly the coefficients of the linearization in (4.12), thus we conclude

$$(4.23)\qquad u_1 \leqq u_t\,.$$

By the same arguments we obtain

$$(4.24)\qquad u_t \leqq u_0$$

for small $|t - t_0|$.

So far, the parameter γ still depends on $f_0 = F(u_0)$ because of our definition of f_t, but we want γ to be independent of u_0. This can be easily derived by applying the previous arguments to the following situation: Let γ_0 be a constant such that the linearization of the operator

$$(4.25)\qquad F = f - \gamma e^{-\mu u}[u - u_0]$$

is injective provided $u \leqq u_0$ and $\gamma \geqq \gamma_0$, where $\gamma_0 = \gamma_0(\mu, c_0, f, \Omega)$, cf. Lemma 4.8. Then, we know that (4.25) has a solution u with $u_1 \leqq u \leqq u_0$ for large γ where γ might depend on $F(u_0)$.

Now, let $\bar{\gamma} \geqq \gamma_0$ be arbitrary and Λ be the set of $\gamma \geqq \bar{\gamma}$ such that (4.25) has a solution u with $u_1 \leqq u \leqq u_0$. Λ is not empty; let $\gamma^* = \inf \Lambda$, then $\gamma^* \in \Lambda$ because of the a priori estimates, and we also conclude $\gamma^* = \bar{\gamma}$ because of the inverse function theorem. Hence, we have shown that the parameter γ can be chosen independently of u_0.

To complete the proof of Theorem 4.1 it remains to verify that the linearized operator is injective.

To achieve this, let us first prove some preliminary lemmata. Let $M \subset N$ be a strictly convex closed hypersurface, $\eta = (\eta^\alpha)$ a vectorfield defined in a neighbourhood $\mathscr{U}$ of M and $\varphi \in C^2(\mathscr{U})$. Then, consider the flow $x = x(t)$ with velocity

$$(4.26)\qquad \dot{x} = \varphi\eta$$

where $x(0)$ is an embedding of the hypersurface M. For small $|t|$ there exists a smooth flow $x(t)$ such that each $x(t)$ is an embedding of a strictly convex hypersurface $M(t)$.

Let (ξ^i) be a coordinate system for $M(t)$. We are interested in the evolution of g_{ij}, h_{ij}, v, and F.

Lemma 4.3 (Evolution of the metric). *The evolution equation for g_{ij} is*

$$(4.27)\qquad \dot{g}_{ij} = \varphi_i\langle\eta, x_j\rangle + \varphi_j\langle\eta, x_i\rangle + \varphi\eta_{\alpha\beta}[x_i^\alpha x_j^\beta + x_j^\alpha x_i^\beta]$$

where $\eta_{\alpha\beta} = \eta_{\alpha;\beta}$.

Proof. Differentiating

$$g_{ij} = \langle x_i, x_j \rangle \tag{4.28}$$

with respect to t yields

$$\dot g_{ij} = \langle \dot x_i, x_j \rangle + \langle x_i, \dot x_j \rangle \tag{4.29}$$

and

$$\dot x_i^\alpha = \varphi_i \eta^\alpha + \varphi \eta_i^\alpha = \varphi_i \eta^\alpha + \varphi\, \eta_\beta^\alpha x_i^\beta \ , \tag{4.30}$$

hence the result.

Lemma 4.4 (Evolution of the normal). The normal vector ν evolves according to

$$\dot\nu = -g^{kl}[\varphi_l \langle \nu, \eta \rangle + \varphi \eta_{\alpha\beta} \nu^\alpha x_l^\beta] x_k \ . \tag{4.31}$$

Proof. Since ν is a unit vector we have $\dot\nu \in T(M)$. Furthermore, differentiating

$$0 = \langle \nu, x_i \rangle \tag{4.32}$$

with respect to t, we deduce

$$\langle \dot\nu, x_i \rangle = -\langle \nu, \dot x_i \rangle = -\varphi_i \langle \nu, \eta \rangle - \varphi \eta_{\alpha\beta} \nu^\alpha x_i^\beta \ , \tag{4.33}$$

hence the result.

Lemma 4.5 (Evolution of the second fundamental form). The second fundamental form evolves according to

(4.34)
$$\begin{aligned}\dot h_i^j = &-\varphi_i^j \langle \nu, \eta \rangle - g^{jl} \langle \eta, x_l \rangle h_i^k \varphi_k - \varphi^j \langle \eta, x_k \rangle h_i^k - \varphi^j \langle \nu, \eta_i \rangle - \varphi_i g^{kj} \langle \nu, \eta_k \rangle \\ &+ \varphi\{-\eta_{\alpha\beta} x_l^\alpha x_k^\beta h_i^k g^{lj} - \eta_{\alpha\beta} x_k^\alpha x_l^\beta h_i^k g^{lj} + \eta_{\alpha\beta} \nu^\alpha \nu^\beta h_i^j - \eta_{\alpha\beta\gamma} \nu^\alpha x_k^\beta x_i^\gamma g^{kj}\} \\ &- \varphi \bar R_{\alpha\beta\gamma\delta} x_k^\alpha \nu^\beta x_i^\gamma \eta^\delta g^{kj} \ .\end{aligned}$$

Proof. We use the *Ricci identities* to interchange the covariant derivatives of ν with respect to t and ξ^i

$$\begin{aligned}\frac{d}{dt}(\nu_i^\alpha) &= (\dot\nu^\alpha)_i - \bar R^\alpha_{\beta\gamma\delta} \nu^\beta x_i^\gamma \dot x^\delta \\ &= -g^{kl}\{\varphi_{lj} \langle \nu, \eta \rangle + \varphi_l \langle \nu_i, \eta \rangle + \varphi_l \langle \nu, \eta_i \rangle + \varphi_i \langle \nu, \eta_l \rangle\} x_k^\alpha \\ &\quad - \varphi g^{kl}\{\eta_{\delta\beta\gamma} \nu^\delta x_l^\beta x_i^\gamma + \eta_{\delta\beta} \nu_i^\delta x_l^\beta + \eta_{\delta\beta} \nu^\delta x_{li}^\beta\} x_k^\alpha \\ &\quad - g^{kl}\{\varphi_l \langle \nu, \eta \rangle + \varphi \eta_{\delta\beta} \nu^\delta x_l^\beta\} x_{ki}^\alpha - \bar R^\alpha_{\beta\gamma\delta} \nu^\beta x_i^\gamma \eta^\delta \varphi \ .\end{aligned} \tag{4.35}$$

For the second equality we used (4.31).

On the other hand, in view of the Weingarten equation, we have

$$\frac{d}{dt}(\nu_i^\alpha) = \frac{d}{dt}(h_i^k x_k^\alpha) = \dot h_i^k x_k^\alpha + h_i^k \dot x_k^\alpha \ . \tag{4.36}$$

Multiplying the resulting equation with $\bar{g}_{\alpha\beta}x_j^\beta$ we conclude

$$
\begin{aligned}
(4.37)\qquad \dot{h}_i^k g_{kj} &+ h_i^k \varphi_k \langle \eta, x_j\rangle + h_i^k \varphi \eta_{\alpha\beta} x_j^\alpha x_k^\beta \\
&= -\varphi_{ij}\langle \nu, \eta\rangle - \varphi_j \langle \nu_i, \eta\rangle - \varphi_j \langle \nu, \eta_i\rangle - \varphi_i \langle \nu, \eta_j\rangle \\
&\quad - \varphi\{\eta_{\delta\beta\gamma}\nu^\delta x_j^\beta x_i^\gamma + \eta_{\delta\beta} x_l^\delta x_j^\beta h_i^l - \eta_{\delta\beta}\nu^\delta \nu^\beta h_{ij}\} \\
&\quad - \varphi \overline{R}_{\alpha\beta\gamma\delta} x_j^\alpha \nu^\beta x_i^\gamma \eta^\delta
\end{aligned}
$$

or equivalently (4.34).

Lemma 4.6 (Evolution of F). *F evolves according to*

$$
\begin{aligned}
(4.38)\qquad \dot{F} = &-\langle \nu, \eta\rangle F^{ij}\varphi_{ij} - 2F^{ij}\varphi_i \langle \nu, \eta_j\rangle - 2F^{ij} h_i^k \varphi_j \langle \eta, x_k\rangle \\
&+ \varphi\{F\eta_{\alpha\beta}\nu^\alpha \nu^\beta - 2F^{ij}\eta_{\alpha\beta} x_i^\alpha x_k^\beta h_j^k - F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma \\
&- F^{ij}\overline{R}_{\alpha\beta\gamma\delta} x_i^\alpha \nu^\beta x_j^\gamma \eta^\delta\}\,.
\end{aligned}
$$

Proof. We have in view of (4.34)

$$
\begin{aligned}
(4.39)\qquad \frac{dF}{dt} = F_j^i \dot{h}_i^j = &-\langle \nu, \eta\rangle F^{ij}\varphi_{ij} - 2F^{ij}\varphi_i \langle \nu, \eta_j\rangle - 2F^{ij} h_i^k \varphi_j \langle \eta, x_k\rangle \\
&+ \varphi\{F\eta_{\alpha\beta}\nu^\alpha \nu^\beta - 2F^{ij}\eta_{\alpha\beta} x_i^\alpha x_k^\beta h_j^k - F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma \\
&- F^{ij}\overline{R}_{\alpha\beta\gamma\delta} x_i^\alpha \nu^\beta x_j^\gamma \eta^\delta\}
\end{aligned}
$$

where we used the homogeneity of F and the fact that F^{ij} and h_{ij} can be diagonalized simultaneously.

Remark 4.7. Let us note that the coefficient of φ in (4.38) can be bounded by $c_1(F + F^{ij}g_{ij})$ with a uniform constant c_1 as long as the flow stays in a compact domain, cf. Lemma 3.4.

Let us now compute the linearization of the operator $F - \tilde{f}$.

Lemma 4.8. *Let $M = \operatorname{graph} u|_{S_0} \subset \overline{\Omega}$ be a solution of*

$$
(4.40)\qquad F = f - \gamma e^{-\mu u}[u - u_0] \equiv \tilde{f}
$$

with $u \leqq u_0$, and where F satisfies (4.1)*. Then, for $\varphi \in C^{2,\alpha}(S_0)$, we have*

$$
(4.41)\qquad \frac{d}{dt}[F(u + t\varphi) - \tilde{f}(u + t\varphi)]\bigg|_{t=0} = -a^{ij}\varphi_{ij} + b^i \varphi_i + c\varphi\,,
$$

where the coefficients are of class $C^{1,\alpha}$, $a^{ij} > 0$, $c = c(x) > 0$ provided μ and γ are large, $\mu = \mu(f, \overline{\Omega})$ and $\gamma = \gamma(\mu, c_0, f, \overline{\Omega})$; c_0 is the constant in (4.1)*. The covariant derivatives in* (4.41) *are calculated with respect to the induced metric of M.*

Proof. For small $|t|$, the graphs of the functions $(u + t\varphi)$ are solutions of the flow

$$\dot{x}^\alpha = \varphi r^\alpha \tag{4.42}$$

with initial hypersurface M, where r is the *radial* function in the normal Gaussian coordinate system (r, x^i).

Thus, formula (4.39) is applicable with η replaced by grad r and we see that the linearized operator is of the form (4.41) with

$$a^{ij} = v^\alpha r_\alpha F^{ij} = v^{-1} F^{ij} . \tag{4.43}$$

To estimate the coefficient of φ we make use of the observation in Remark 4.7 to obtain

$$\begin{aligned} c = c(x,u) &\geqq -c_1[F + F^{ij} g_{ij}] - \left.\frac{d\tilde{f}}{dt}\right|_{t=0} \varphi^{-1} \\ &\geqq -c_1 \, (F + nc_0) - \left.\frac{\partial \tilde{f}}{\partial r}\right|_{r=u} \end{aligned} \tag{4.44}$$

where

$$\left.\frac{\partial \tilde{f}}{\partial r}\right|_{r=u} = \frac{\partial f}{\partial r} + \mu\gamma e^{-\mu u}[u - u_0] - \gamma e^{-\mu u} . \tag{4.45}$$

Hence, we conclude that the right-hand side in (4.44) is estimated from below by

$$\gamma e^{-\mu u} + \gamma(\mu - c_1) e^{-\mu u}[u_0 - u] - c_1 f - nc_1 c_0 - \frac{\partial f}{\partial r} \tag{4.46}$$

which is strictly positive if we choose $\mu \geqq c_1$ and γ large enough.

5. Existence of solutions to the original problem

We know that for each supersolution u_0 of (F, f) the auxiliary problem

$$F = f - \gamma e^{-\mu u}[u - u_0] \tag{5.1}$$

has a solution u with

$$u_1 \leqq u \leqq u_0 . \tag{5.2}$$

Moreover, u is also a supersolution of (F, f). We now define successively

$$u_2 = \text{the upper barrier} \tag{5.3}$$

and for $k \geqq 3$, u_k as the solution of

$$\begin{cases} F = f - \gamma e^{-\mu u_k}[u_k - u_{k-1}] \\ u_1 \leqq u_k \leqq u_{k-1} . \end{cases} \tag{5.4}$$

Thus, we obtain a monotone falling sequence of functions u_k which converge on S_0 to some function u. The hypersurfaces $M_k = \operatorname{graph} u_k|_{S_0}$ are strictly convex, i.e. we have uniform C^1-estimates, and from the estimates in Sect. 3, we shall conclude that we also have uniform C^2-estimates, or equivalently, uniform estimates for $|A_k|$, where again we note that

$$F|_{M_k} \geqq f \quad \forall k \geqq 2\,. \tag{5.5}$$

To obtain the uniform estimates for $|A_k|$, we use the estimate (3.25) which yields

$$|A_k|^2_{M_k} \leqq c(1 + |A_{k-1}|_{M_{k-1}}) \quad \forall k \geqq 3\,, \tag{5.6}$$

where

$$|A_k|_{M_k} = \sup_{M_k} |A_k|\,. \tag{5.7}$$

Set

$$r_k = 1 + |A_k|_{M_k}\,, \tag{5.8}$$

then, we deduce from (5.6)

$$r_k \leqq c r_{k-1}^{1/2} \quad \forall k \geqq 3 \tag{5.9}$$

with a different constant c, and hence, by iteration

$$r_k \leqq c^{\sum_{i=0}^{k-2} q^i} r_2^{1/2}\,, \tag{5.10}$$

where $q = \frac{1}{2}$, and hence

$$r_k \leqq c^2 r_2^{1/2} \quad \forall k \geqq 3\,. \tag{5.11}$$

Therefore, the u_k are uniformly bounded in $C^{4,\alpha}(S_0)$ and the graph of the limit function u is a solution to our problem.

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