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Existence, Regularity, and Boundary Behaviour of Generalized Surfaces of Prescribed Mean Curvature*

Claus Gerhardt

0. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqq 2$, with Lipschitz boundary $\partial\Omega$, let A be the minimal surface operator

$$A = -D^i\{p_i[1+|p|^2]^{-\frac{1}{2}}\}^1 \tag{1}$$

and let $H = H(x,t)$ be locally Lipschitz in $\mathbb{R}^n \times \mathbb{R}$ with

$$\frac{\partial H}{\partial t} \geqq 0. \tag{2}$$

Then the classical Dirichlet problem for surfaces of prescribed mean curvature to given boundary values $\varphi \in C^0(\partial\Omega)$ consists in determining a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying

$$Au + H(x,u) = 0 \tag{3}$$

and

$$u|_{\partial\Omega} = \varphi. \tag{4}$$

Furthermore, assuming the boundary and the data to be sufficiently smooth, the solution is supposed to be smooth up to the boundary.

It is well-known that this problem is not solvable in general unless we at least impose the condition

$$|H(x,\varphi(x))| \leqq (n-1)\, H_{n-1}(x) \qquad \forall\, x \in \partial\Omega \tag{5}$$

on the mean curvature H_{n-1} of $\partial\Omega$. For reference see the paper of Serrin [48; Chap. III. 18].

On the other hand, we shall show that the variational problem

$$J(v) = \int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\, dx + \int_\Omega \int_0^v H(x,t)\, dt\, dx + \int_{\partial\Omega} |v-\varphi|\, d\mathscr{H}_{n-1} \to \min \quad \text{in } BV(\Omega) \tag{6}$$

is solvable without assuming any further condition on $\partial\Omega$ provided that

$$H_0 = H(.,0) \tag{7}$$

satisfies

$$\left|\int_A H_0\, dx\right| \leqq (1-\varepsilon_0)\, \mathbf{M}(\partial A) \tag{8}$$

for some positive constant ε_0 independent from A, where A is any measurable subset of Ω, and $\mathbf{M}(\partial A)$ denotes the mass of ∂A in the sense of [8; Chap. 4.1.7].

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1 Here and in the following we make the convention that we sum over repeated indices from 1 to n.

Since every solution u of the variational problem (6) belongs to $C^2(\Omega)$ and satisfies (3) (a proof is given below) we cannot expect in general that the boundary values of u agree with the initial data φ. A counterexample has been constructed by Santi [46], cf. the paper of Nitsche [44] also. Moreover, the following proposition is valid [2].

Proposition 1. *Let H satisfy the assumptions* (2) *and*

$$\begin{aligned} &\sup_{t\in\mathbb{R}}\{\inf_{x\in\Omega} H(x,t)\} = +\infty \\ &\inf_{t\in\mathbb{R}}\{\sup_{x\in\Omega} H(x,t)\} = -\infty . \end{aligned} \tag{9}$$

Assume $\partial\Omega$ to be of class C^2. Then for any $\varphi\in L^1(\partial\Omega)$ the variational problem (6) *has a bounded solution u.*

Obviously, the boundary values of u cannot coincide with φ if $\varphi\notin L^\infty(\partial\Omega)$.

However, in the case $H(x,t)=H(x)$ and for smooth $\partial\Omega$ Giaquinta [18] and Miranda [42] ($H=0$) proved

Proposition 2. *Suppose the conditions* (5) *and* (8) *to be satisfied, and let $\varphi\in C^0(\partial\Omega)$. Then the variational problem* (6) *has a unique solution $u\in C^2(\Omega)\cap C^0(\bar\Omega)$ such that*

$$u|_{\partial\Omega}=\varphi . \tag{10}$$

Their results also hold locally, i.e. the (unique) solution u is continuous up to those boundary parts $\Gamma\subset\partial\Omega$, which are smooth, and coincides there with φ provided that (5) is satisfied on Γ. However, they cannot prove with their methods that u is smooth up to those boundary parts Γ, if $\varphi\in C^2(\Gamma)$.

A partial result in this direction has been obtained by Lichnewsky (oral communication) who demonstrated

Proposition 3. *Let H be identically zero, and let $\Gamma\subset\overline{\operatorname{conv}(\Omega)}\cap\partial\Omega$ be of positive $(n-1)$-dimensional Hausdorff measure. Assume, moreover, that φ satisfies a uniform bounded slope condition on Γ, i.e. for any point $x_0\in\Gamma$ there are linear mappings $\pi^\pm_{x_0}$ such that*

$$\pi^-_{x_0}(x-x_0)\leqq\varphi(x)-\varphi(x_0)\leqq\pi^+_{x_0}(x-x_0)\qquad \forall\, x\in\Omega \tag{11}$$

and

$$|D\pi^\pm_{x_0}|\leqq K \qquad \textit{independent from } x_0 . \tag{12}$$

Then the variational problem (6) *has a unique solution u which satisfies a bounded slope condition up to Γ and coincides with φ there.*

The assumption that Γ is to be convex seems to be quite unnatural. We shall show that the condition

$$0\leqq H_{n-1}(x)\qquad \forall\, x\in\Gamma^{\,3} \tag{13}$$

or more generally

$$|H(x,\varphi(x))|\leqq H_{n-1}(x)\qquad \forall\, x\in\Gamma \tag{14}$$

is sufficient to prove the existence of a unique solution to the variational problem (6) which is smooth up to Γ and coincides with φ there, if φ is supposed to be

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[2] A proof is given in [15].

[3] In the meantime A. Lichnewsky told me that he could improve his methods of proof so that his result would be valid provided that the outer curvature of Γ (in the sense of [49]) is strictly positive.

smooth. Moreover, we shall show uniqueness of the solution and partial coincidence with the boundary data even for boundary values φ belonging to $L^1(\partial\Omega)$ provided that φ is continuous at a point $x_0 \in \Gamma$ and (14) is fulfilled a.e. in Γ.

Since we are interested in the presence of an obstacle ψ, we shall consider the variational problems

$$J(v) \to \min \quad \text{in } H^{1,1}(\Omega) \cap \{v \geqq \psi\} \tag{15}$$

and

$$J(v) \to \min \quad \text{in } BV(\Omega) \cap \{v \geqq \psi\}, \tag{15'}$$

where J is the same functional as in (6), and where ψ satisfies

$$\psi \in H^{1,\infty}(\Omega), \qquad \psi|_{\partial\Omega} \leqq \varphi. \tag{16}$$

In the following we shall show that the variational problem (15′) has a solution $u \in C^{0,1}(\Omega) \cap H^{1,1}(\Omega)$ which is uniquely determined up to an additive constant in the function class $H^{1,1}(\Omega)$. Hence, this solution also solves problem (15). Conversely, any solution of (15) minimizes the functional J in the larger space $BV(\Omega) \cap \{v \geqq \psi\}$ which can easily be deduced by approximation (cf. [14; Lemma A 1 and Lemma A 2]).

In contrast to the case when the obstacle is absent (cf. [59]) we could not prove that any solution of (15′) is of class $H^{1,1}(\Omega)$. Thus, we had to restrict ourselves to the variational problem (15), since we shall use the fact that the solutions are unique up to an additive constant, and this conclusion, however, may not be valid in the more general case.

In order to formulate our results more easily, let us introduce the following definition

Definition 1. Let Γ be a closed subset of $\partial\Omega$. Then we shall denote by U_Γ any open subset of Ω which satisfies

$$\bar{U}_\Gamma \cap \partial\Omega = \Gamma. \tag{17}$$

The main theorems which we shall prove are

Theorem 1. *Let Γ_0 be an open subset of $\partial\Omega$ being of class C^2. Assume that $\varphi \in L^1(\partial\Omega)$ belongs to $C^0(\Gamma_0)$ and that H satisfies besides the conditions* (2) *and* (8)

$$|H(x, \varphi(x))| \leqq (n-1)\, H_{n-1}(x) \qquad \forall\, x \in \Gamma_0. \tag{18}$$

Then the variational problem (15) *has a unique solution $u \in C^{0,1}(\Omega) \cap H^{1,1}(\Omega)$ such that*

$$u = \varphi \quad \text{on } \Gamma_0 \tag{19}$$

and

$$u \in C^0(\bar{U}_\Gamma) \qquad \forall\, \Gamma \subset\subset \Gamma_0. \tag{20}$$

Furthermore, for smooth data we shall prove

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied and suppose φ to be in $C^2(\Gamma_0)$. Then we have*

$$u \in H^{1,\infty}(U_\Gamma) \qquad \forall\, \Gamma \subset\subset \Gamma_0. \tag{21}$$

Moreover, the smoothness of u increases with that of ψ.

Theorem 3. *Any solution u of the variational problem* (15) *is of class $H^{2,p}_{\mathrm{loc}}(\Omega)$ for any p, $n < p \leqq \infty$, provided that $\varphi \in L^1(\partial\Omega)$, $\psi \in H^{2,p}(\Omega)$, and that H satisfies the* *conditions* (2) *and* (8). *Furthermore, under the assumptions of Theorem* 2 *we can prove*

$$u \in H^{2,p}(U_\Gamma) \qquad \forall\, \Gamma \subset\subset \Gamma_0 \tag{22}$$

for any p, $n<p<\infty$, *if* ψ *belongs to* $H^{2,p}(\Omega)$.

Finally, we shall prove the following generalization of Theorem 1.

Theorem 4. *Let* Γ_0 *be as Theorem* 1, *and suppose that* $\varphi \in L^1(\partial\Omega)$ *is continuous in* $x_0 \in \Gamma_0$. *Assume, furthermore, that* H *satisfies besides the conditions* (2) *and* (8)

$$|H(x,\varphi(x))| \leqq (n-1)\, H_{n-1}(x) \qquad \text{for a.e. } x \in \Gamma_0. \tag{23}$$

Then the variational problem (15) *has a unique solution* $u \in C^{0,1}(\Omega) \cap H^{1,1}(\Omega)$ *such that* u *is continuous in* $\Omega \cup \{x_0\}$ *and satisfies*

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = \varphi(x_0). \tag{24}$$

1. A Priori Estimates for $|u|$

In this section we are going to prove local and global estimates for the modulus of solutions to the variational problem (15′). Interior estimates have already been proved by De Giorgi [21], Miranda [42] in the case $H=0$, and by Massari [37] and Giaquinta [18] in the case $H=H(x)$. Unfortunately, these estimates come from a contradiction argument and are no explicit bounds in terms of known quantities. A result in this direction has been obtained by Lichnewsky [36], using Serrin's methods (cf. [47]), for weak solutions $u \in H^{1,1}(\Omega)$ of the equation

$$Au + H(x) = 0 \tag{25}$$

where $H \in L^p(\Omega)$ for $p>n$.

The proof we present here is analogous to that one we used in [14] to demonstrate the boundedness of solutions to the capillarity problem.

We assume in this section that Ω is a bounded domain with Lipschitz boundary, that H is measurable in x and continuous in t, that it satisfies (2) and either (8) or

$$H_0 \in L^p(\Omega) \qquad \text{with } p>n. \tag{26}$$

Then we obtain

Lemma 1. *Let* Γ_0 *be an relatively open part of* $\partial\Omega$, *and let* $\varphi \in L^1(\partial\Omega) \cap L^\infty(\Gamma_0)$ *be given. Furthermore, let* $\Gamma \subset \Gamma_0$ *be any closed subset and* U_Γ *be any open set appearing in Definition* 1.

Then, under the preceding assumptions, any solution $u \in BV(\Omega)$ *of the variational problem* (15′) *can be estimated in* U_Γ *by*

$$\max\{\inf_{x\in\Omega} \min(\psi,0), -c_1\} \leqq u \leqq \max\{\sup_{x\in\Omega} \max(\psi,0), c_1\} \tag{27}$$

where c_1 *depends on* U_Γ, $\|\varphi\|_{L^\infty(\Gamma_0)}$, $\|u\|_1$, n, Ω, *and either on* $\|H_0\|_p$ *or on* ε_0.

Here, we denote by $\|\cdot\|_q$ the norm in $L^q(\Omega)$ for $1 \leqq q \leqq \infty$.

Proof of Lemma 1. Let k be a positive number greater than

$$\max\{\sup_x \max(\psi,0), \|\varphi\|_{L^\infty(\Gamma_0)}\},$$

and let η be a smooth function such that

$$0 \leqq \eta \leqq 1, \quad \eta|_\Gamma = 1, \tag{28}$$

and

$$\operatorname{supp} \eta \cap \partial\Omega \subset \Gamma_0. \tag{29}$$

Then, $u_k = (1-\eta)\, u + \min(\eta\, u, k)$ belongs to $BV(\Omega) \cap \{v \geqq \psi\}$, and from the minimum property of u we get

$$J(u) \leqq J(u_k). \tag{30}$$

Hence, using the notation $A(k, \eta) = \{x \in \Omega\colon \eta\, u > k\}$ and supposing for a moment u to be smooth, we obtain

$$\begin{aligned} &\int_{A(k,\eta)} (1+|Du|^2)^{\frac{1}{2}}\, dx + \int_\Omega \int_{u_k}^{u} H(x,t)\, dt\, dx + \int_{\partial\Omega} \{|u-\varphi| - |u_k - \varphi|\}\, d\mathscr{H}_{n-1} \\ &\quad \leqq \int_{A(k,\eta)} (1+|D[(1-\eta)\, u]|^2)^{\frac{1}{2}}\, dx \\ &\quad \leqq \int_{A(k,\eta)} (1+(1-\eta)^2\, |Du|^2)^{\frac{1}{2}}\, dx + \int_{A(k,\eta)} (1+u^2\, |D\eta|^2)^{\frac{1}{2}}\, dx, \end{aligned} \tag{31}$$

where we used the estimate $(1+|a+b|^2)^{\frac{1}{2}} \leqq (1+|a|^2)^{\frac{1}{2}} + (1+|b|^2)^{\frac{1}{2}}$. Furthermore, taking the estimate

$$\begin{aligned} (1+t^2)^{\frac{1}{2}} - \{1+(1-\eta)^2\, t^2\}^{\frac{1}{2}} &\geqq t - \{1+(1-\eta)\, t\} \\ &\geqq \eta\, t - 1 \end{aligned} \tag{32}$$

and the relation

$$|u-\varphi| - |u_k - \varphi| = \max(\eta\, u - k, 0) \tag{33}$$

(which is valid since $k \geqq \|\varphi\|_{L^\infty(\Gamma_0)}$) into account, (31) yields

$$\begin{aligned} &\int_{A(k,\eta)} |D(\eta\, u)|\, dx + \int_\Omega \int_{u_k}^{u} H(x,t)\, dt\, dx + \int_{\partial\Omega} \max(\eta\, u - k, 0)\, d\mathscr{H}_{n-1} \\ &\quad \leqq 2 \cdot |A(k,\eta)| + 2 \cdot |D\eta|_\Omega \cdot \int_{A(k,\eta)} u\, dx, \end{aligned} \tag{34}$$

where $|A(k,\eta)|$ denotes the Lebesgue measure of $A(k,\eta)$.

Finally, setting $w = \max(\eta\, u - k, 0)$ and observing that in view of (2)

$$\int_{u_k}^{u} H(x,t)\, dt \geqq H_0 (u - u_k) = H_0 \cdot w, \tag{35}$$

we get the inequality

$$\int_\Omega |Dw|\, dx + \int_\Omega H_0\, w\, dx + \int_{\partial\Omega} w\, d\mathscr{H}_{n-1} \leqq 2 \cdot |A(k,\eta)| + 2 \cdot |D\eta|_\Omega \cdot \int_{A(k,\eta)} u\, dx \tag{36}$$

which will also be valid for $u \in BV(\Omega)$ using an approximation argument (cf. [14; Lemma A 4]).

To estimate the integral $\int_\Omega H_0\, w\, dx$, we use either the assumption (8) which yields (cf. [19])

$$\int_\Omega H_0\, w\, dx \geqq -(1-\varepsilon_0) \int_\Omega |Dw|\, dx - (1-\varepsilon_0) \int_{\partial\Omega} w\, d\mathscr{H}_{n-1} \tag{37}$$

or the Hölder inequalities

$$\left|\int_\Omega H_0 \cdot w\,dx\right| \leqq \|w\|_{n^*} \cdot \Big(\int_{A(k,\eta)} |H_0|^n\,dx\Big)^{1/n} \leqq \|w\|_{n^*} \cdot \|H_0\|_p \cdot |A(k,\eta)|^{(p-n)/n\cdot p}, \tag{38}$$

where we denote by n^* the conjugate exponent, $\dfrac{1}{n^*} = 1 - \dfrac{1}{n}$.

Taking only the estimate (38) into account, since the reasoning would be more easily in the case of applying (37), we deduce from (36)

$$\int_\Omega |Dw|\,dx + \int_\Omega w\,dx - \{\|H_0\|_p \cdot |A(k,\eta)|^{(p-n)/n\cdot p} + |A(k,\eta)|^{1/n}\}\,\|w\|_{n^*} \leqq 2\cdot|A(k,\eta)| + 2\cdot|D\eta|_\Omega \cdot \int_{A(k,\eta)} u\,dx. \tag{39}$$

Here, we used $\|w\|_1 \leqq \|w\|_{n^*} \cdot |A(k,\eta)|^{1/n}$.

Now, applying the Sobolev Imbedding Theorem and using the fact that

$$|A(k,\eta)| \leqq \frac{1}{k}\cdot\int_\Omega |\eta u|\,dx \leqq \frac{1}{k}\cdot\int_\Omega |u|\,dx \tag{40}$$

we derive from (39)

$$\|w\|_{n^*} \leqq c_2 \cdot \Big\{|A(k,\eta)| + \int_{A(k,\eta)} u\,dx\Big\} \tag{41}$$

for $k \geqq k_0$, where k_0 and c_2 depend on η, $\|u\|_1$, $\|H_0\|_p$, and known quantities.

Using the Hölder inequalities once again, (41) yields

$$\int_{A(k,\eta)} (\eta u - k)\,dx \leqq c_2 \cdot \Big\{|A(k,\eta)|^{1+1/n} + |A(k,\eta)|^{1/n}\cdot \int_{A(k,\eta)} u\,dx\Big\} \tag{42}$$

or finally

$$(h-k)\cdot|A(h,\eta)| \leqq c_2 \cdot \Big\{|A(k,\eta)|^{1+1/n} + |A(k,\eta)|^{1/n}\cdot \int_{A(k,\eta)} u\,dx\Big\} \tag{43}$$

for $h > k \geqq k_0$.

To complete the proof of Lemma 1, we apply a result due to Stampacchia [51; Lemme 4.1] which can be stated as follows

Lemma 2. *Let the positive constants c_3, k_0, and γ be given. Then we deduce from the inequality*

$$(h-k)\cdot|A(h,\eta)| \leqq c_3 \cdot |A(k,\eta)|^\gamma, \qquad h > k \geqq k_0 > 0, \tag{44}$$

that

$$h^{\frac{1}{1-\gamma}} \cdot |A(h,\eta)| \leqq 2^{\frac{1}{(1-\gamma)^2}} \cdot \Big\{c_3^{\frac{1}{1-\gamma}} + (2k_0)^{\frac{1}{1-\gamma}} \cdot |A(k_0,\eta)|\Big\} \tag{45}$$

if $\gamma < 1$, or, if $\gamma > 1$,

$$|A(k_0 + d, \eta)| = 0 \tag{46}$$

where

$$d = c_3 \cdot [|A(k_0,\eta)|]^{\gamma-1} \cdot 2^{\frac{\gamma}{\gamma-1}}. \tag{47}$$

We shall use an iteration procedure to increase the exponent in (43) such that we can apply (46). Let us indicate the first step. Combining (43) and (45) we conclude that

$$(u\eta)^+ \in L^q(\Omega), \qquad (u\eta)^+ = \max(u\eta, 0), \tag{48}$$

for any $q < \frac{n}{n-1}$, where $\|(u\eta)^+\|_q$ depends on q, η, and known quantities. Since the inequality (43) is valid for any cut-off function ζ satisfying (29), we then choose ζ such that

$$\zeta|_{\operatorname{supp}\eta} = 1. \tag{49}$$

Hence, we deduce

$$\int_{A(k,\eta)} u\,dx \leqq \|(u\zeta)^+\|_q \cdot |A(k,\eta)|^{1-1/q}. \tag{50}$$

Inserting this estimate in (43) yields

$$(h-k)\cdot|A(h,\eta)| \leqq c_2(q,\zeta)\cdot\{|A(k,\eta)|^{1+1/n} + |A(k,\eta)|^{1-1/q+1/n}\} \tag{51}$$

for $1 < q < \frac{n}{n-1}$.

Evidently, we can increase the exponent of $|A(k,\eta)|$ by a finite iteration to some $\gamma > 1$, hence we conclude that u is bounded from above in U_r by an estimate of the form (27).

Though, u is obviously bounded from below by ψ, it would be worth to get the sharper estimate (27), for by this we had also derived a bound for solutions to the free problem

$$J(v) \to \min \quad \text{in } BV(\Omega) \tag{52}$$

setting formally $\psi = -\infty$.

In order to obtain the lower bound, we choose $k \geqq \|\varphi\|_{L^\infty(\Gamma_0)}$ and insert in (30) $u_k = (1-\eta)\,u + \max(\eta u, -k)$.

The proof of Lemma 1 would then be completed by similar considerations as above.

2. Existence of Solutions in $BV(\Omega)$

In the case that H does not depend on t Giaquinta [19] demonstrated the existence of solutions in $BV(\Omega)$ to the problem (6) provided that H satisfies the condition (8) (cf. [19] for a discussion of this condition).

If H depends on t, and if the conditions (2) and (8) are fulfilled, then the proof of the existence of solutions to (15′) is almost the same.

Theorem 5. *Let H satisfy the conditions* (2) *and* (8), *and let $\varphi \in L^1(\partial\Omega)$. Then the variational problem* (15′) *has a solution $u \in BV(\Omega)$ such that*

$$\int_\Omega |Du|\,dx + \int_\Omega |u|\,dx$$

is bounded by a constant depending only on ε_0, Ω, $\int_{\partial\Omega} |\varphi|\,d\mathscr{H}_{n-1}$, $\sup_\Omega \max(\psi, 0)$, and $\int_\Omega H(x, \sup_\Omega \max(\psi,0))\,dx$.

Proof. Let $\mathbf{K}$ be the convex set

$$\mathbf{K}=BV(\Omega)\cap\{v\geqq\psi\}, \tag{53}$$

and let B be any ball containing $\bar{\Omega}$. We are going to use an advice of Santi [46] and extend φ to some function in $H^{1,1}(B-\bar{\Omega})$ having boundary values zero on ∂B and which we denote by φ, too. This extension is possible in view of a result due to Gagliardo [16].

Then, defining $\tilde{H}$ and $\tilde{\mathbf{K}}$ by

$$\tilde{H}(x,t)=\begin{cases}H(x,t), & \text{if } x\in\bar{\Omega}\\ 0, & \text{if } x\in\mathbb{R}^n-\bar{\Omega}\end{cases} \tag{54}$$

and

$$\tilde{\mathbf{K}}=\{v\in BV(B)\colon v\in\mathbf{K}\cap\{v|_{B-\bar{\Omega}}=\varphi\}\}, \tag{55}$$

we conclude that

$$\tilde{J}(v)=\int_B(1+|Dv|^2)^{\frac{1}{2}}\,dx+\int_B\int_0^v\tilde{H}(x,t)\,dt\,dx \tag{56}$$

equals

$$J(v)+\int_{B-\bar{\Omega}}(1+|D\varphi|^2)^{\frac{1}{2}}\,dx. \tag{57}$$

Hence, it is equivalent to solve (15′) or

$$\tilde{J}(v)\to\min\quad\text{in }\tilde{\mathbf{K}}. \tag{58}$$

Let v_ε be a mimizing sequence of (58). We shall show that the v_ε's are uniformly bounded in $BV(B)$. To prove this, let us remark that in view of (2)

$$\int_0^{v_\varepsilon}\tilde{H}(x,t)\,dt\geqq\tilde{H}_0\cdot v_\varepsilon \tag{59}$$

where $\tilde{H}_0=\tilde{H}(.,0)$. Taking the condition (8) into account we obtain

$$\int_\Omega H_0\cdot v_\varepsilon\,dx\geqq-(1-\varepsilon_0)\cdot\int_\Omega|Dv_\varepsilon|\,dx-(1-\varepsilon_0)\cdot\int_{\partial\Omega}|v_\varepsilon|\,d\mathscr{H}_{n-1} \tag{60}$$

(cf. [19]), or finally

$$\begin{aligned}\int_\Omega H_0\cdot v_\varepsilon\,dx&\geqq-(1-\varepsilon_0)\cdot\Big\{\int_B|Dv_\varepsilon|\,dx-\int_{B-\bar{\Omega}}|D\varphi|\,dx+\int_{\partial\Omega}|\varphi|\,d\mathscr{H}_{n-1}\Big\}\\&\geqq-(1-\varepsilon_0)\cdot\int_B|Dv_\varepsilon|\,dx-c_4\cdot\int_{\partial\Omega}|\varphi|\,d\mathscr{H}_{n-1}.\end{aligned} \tag{61}$$

Thus, we conclude

$$\tilde{J}(v_\varepsilon)\geqq\varepsilon_0\cdot\int_B|Dv_\varepsilon|\,dx-c_4\cdot\int_{\partial\Omega}|\varphi|\,d\mathscr{H}_{n-1}. \tag{62}$$

Since $\tilde{J}(v_\varepsilon)$ is estimated from above by

$$J(\sup_\Omega\max(\psi,0))+\int_{B-\bar{\Omega}}(1+|D\varphi|^2)^{\frac{1}{2}}\,dx, \tag{63}$$

(62) implies that v_ε is a bounded sequence in $BV(B)$. Hence, a subsequence, which we again denote by v_ε, converges in $L^q(B)$ for any q, $1\leqq q<\frac{n}{n-1}$, to some function u.

To complete the proof of the Theorem, let us show that $\tilde{J}$ is lower semicontinuous with respect to convergence in $BV(B)$. Since the lower semicontinuity of the area functional is well-known, it remains to prove

$$\liminf \int_\Omega \int_0^{v_\varepsilon} H(x,t)\,dt\,dx \geqq \int_\Omega \int_0^{u} H(x,t)\,dt\,dx. \tag{64}$$

The validity of (64) can be easily deduced from the inequality

$$\int_0^{v_\varepsilon} \{H(x,t)-H(x,0)\}\,dt \geqq 0 \tag{65}$$

using Fatou's Lemma and the relation

$$\int_\Omega H_0 \cdot v_\varepsilon\,dx \to \int_\Omega H_0 \cdot u\,dx \tag{66}$$

which is obviously satisfied since $H_0 \in L^n(\Omega)$ and v_ε converges weakly in $L^{n/n-1}(\Omega)$ to u.

3. Interior Regularity of u

In the preceding sections we made no use of the Lipschitz continuity of H. However, this property becomes important for proving the regularity of solutions to the variational problem (15′).

The interior regularity of u will follow from a general theorem concerning the regularity of solutions $w \in BV(\Omega)$ of the variational problem

$$\begin{gathered} L(v) = \int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\,dx + \int_\Omega \int_0^{v} H(x,t)\,dt\,dx \to \min \\ \text{in } BV(\Omega) \cap \{v \geqq \psi\} \cap \{v|_{\partial\Omega} = w|_{\partial\Omega}\}. \end{gathered} \tag{67}$$

Theorem 6. *Let w be a locally bounded solution in $BV(\Omega)$ of the variational problem (67), and let $H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R})$ be strictly increasing in t. Then w is locally Lipschitz in Ω provided that $\psi \in C^{0,1}(\bar{\Omega})$. Precisely, we have the estimate*

$$|Dw|_{\Omega'} \leqq c_5(|w|_{\Omega''}, |D\psi|_{\Omega''}, |DH|_{\Omega''}) \qquad \forall\, \Omega' \subset\subset \Omega'' \subset\subset \Omega. \tag{68}$$

Proof. We shall use the results of [11; Theorem 1] concerning the existence of surfaces of prescribed mean curvature over obstacles together with an extended version of the a priori estimates of Ladyzhenskaya and Uraltseva [33].

Let $x_0 \in \Omega$ be given, and denote by $B_R = B_R(x_0)$ a ball of radius R with center x_0. Furthermore, we assume that w and ψ are extended into $\mathbb{R}^n$ as functions with compact support, and we let w_ε resp. ψ_ε be the mollifactions of w resp. ψ with a common mollifier. Then, we consider the Dirichlet problem

$$\begin{gathered} A v_\varepsilon^* + H(x, v_\varepsilon^*) = 0 \quad \text{in } B_R \\ v_\varepsilon^*|_{\partial B_R} = w_\varepsilon. \end{gathered} \tag{69}$$

Obviously, if we choose R sufficiently small, $R < R_0$, (69) has a solution $v_\varepsilon^* \in C^2(\bar{B}_R)$, as we deduce from [48]. We only have to choose R_0 such, that the inequalities

$$|H(x, w_\varepsilon(x))| \leqq \frac{n-1}{R_0} \tag{70}$$

and

$$|H(x,0)| \leqq \frac{n-1}{R_0} \tag{71}$$

are satisfied in $B_{R_0}(x_0)$. The first estimate yields to an a priori estimate for $|Dv_\varepsilon^*|_{\partial B_R}$, while the second one gives a bound for $|v_\varepsilon^*|_{B_R}$, in view of the monotonicity of $H(x,.)$: Observe that for any positive function δ we have

$$A\delta + H(x,\delta) \geqq A\delta + H_0 \tag{72}$$

and

$$A(-\delta) + H(x,-\delta) \leqq A(-\delta) + H_0. \tag{73}$$

Moreover, we may conclude that

$$|v_\varepsilon^*|_{B_R} \leqq c_6 = c_6(|w|_{B_{R_0}}, R, |H|_{B_{R_0}}) \tag{74}$$

and

$$|Dv_\varepsilon^*|_{B_R} \leqq c_7 = c_7(c_6, |D^2 w_\varepsilon|_{B_R}, |DH|_{B_R}). \tag{75}$$

Since these estimates hold uniformly in τ if we replace H by $\tau \cdot H$ in (69), where τ is a real parameter, $0 \leqq \tau \leqq 1$, we derive from [11] that the variational inequality

$$\begin{aligned} &\langle Av_\varepsilon + H(x,v_\varepsilon), v - v_\varepsilon\rangle \geqq 0 \quad \forall v \in K, \\ &K = \{v \in H^{1,\infty}(B_R) \colon v \geqq \psi_\varepsilon, v|_{\partial B_R} = w_\varepsilon\}, \end{aligned} \tag{76}$$

has a solution $v_\varepsilon \in K \cap H^{2,p}(B_R)$ for any $p, n < p < \infty$, such that

$$|v_\varepsilon|_{B_R} \leqq c_8 = c_8(c_6, |\psi|_{B_R}). \tag{77}$$

As we shall prove in the Appendix, we have the interior gradient estimates

$$|Dv_\varepsilon|_{\Omega'} \leqq c_9 = c_9(c_8, |D\psi|_{B_R}, |DH|_{B_R}, \Omega') \qquad \forall \Omega' \subset\subset B_R. \tag{78}$$

Moreover, we know (cf. [11; Section 4]) that v_ε minimizes the functional

$$\int_{B_R} (1+|Dv|^2)^{\frac{1}{2}} dx + \int_{B_R} \int_0^v H(x,t)\,dt\,dx + \int_{\partial B_R} |v - w_\varepsilon|\, d\mathcal{H}_{n-1} \tag{79}$$

in $BV(B_R) \cap \{v \geqq \psi_\varepsilon\}$.

Hence, setting

$$\tilde{v}_\varepsilon = \begin{cases} v_\varepsilon & \text{in } B_R \\ w_\varepsilon & \text{in } \Omega - B_R \end{cases} \tag{80}$$

we derive

$$\tilde{L}(\tilde{v}_\varepsilon) \leqq \tilde{L}(w_\varepsilon), \tag{81}$$

where

$$\tilde{L}(v) = L(v) - \int_{\Omega - B_R} \int_0^v H(x,t)\,dt\,dx. \tag{81a}$$

From [14; Lemma A 1] and from Lebesgue's theorem of dominated convergence we conclude that the right side of (81) tends to $\tilde{L}(w)$, if ε goes to zero. On the other hand, from the estimate (77), (78), and from the definition of $\tilde{v}_\varepsilon$ we deduce that the $\tilde{v}_\varepsilon$'s converge in $BV(\Omega)$ to some function $v_0, v_0 \geqq \psi$, which is locally Lipschitz in B_R and coincides with w in $\Omega - \bar{B}_R$. Moreover, we immediately derive on account of known lower semicontinuity properties of the integrals we deal with

$$\tilde{L}(v_0) \leqq \liminf \tilde{L}(\tilde{v}_\varepsilon) \leqq \tilde{L}(w). \tag{82}$$

Hence, we obtain

$$L(v_0) \leqq L(w), \tag{82a}$$

and we conclude that v_0 is equal to w, since $v_0|_{\partial\Omega} = w|_{\partial\Omega}$ and the variational problem (67) has no distinct solutions.

In order to prove the existence of a regular solution to the variational problem (15′), let ε be an arbitrary positive number and define $H_\varepsilon(x,t) = H(x,t) + \varepsilon \cdot t$. The functional J_ε is similar defined replacing H by H_ε in the definition of J. Then, since $H_\varepsilon(x,0) = H(x,0)$ and H_ε is strictly increasing in t, we conclude that the variational problem

$$J_\varepsilon(v) \to \min \quad \text{in } BV(\Omega) \cap \{v \geqq \psi\}$$

has a unique solution $u_\varepsilon \in C^{0,1}(\Omega) \cap H^{1,1}(\Omega)$, such that the terms

$$\int_\Omega |Du_\varepsilon|\,dx + \int_\Omega |u_\varepsilon|\,dx$$

and

$$|Du_\varepsilon|_{\Omega'} \quad \forall \Omega' \subset\subset \Omega$$

are uniformly bounded with respect to ε. Evidently, the u_ε's form a minimizing sequence for the variational problem (15′).

Moreover, if we look at the proof of Theorem 5 and set

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega \\ \varphi & \text{in } B - \Omega, \end{cases}$$

where we use the notations of Section 2, then the $\tilde{u}_\varepsilon$'s are a minimizing sequence for the variational problem (58). Thus, following the proof of Theorem 5, we deduce that a subsequence of the $\tilde{u}_\varepsilon$'s converges in $L^1(B)$ to some function $\tilde{u}$ which is locally Lipschitz in Ω, and which solves the variational problem (58). Hence, $u = \tilde{u}|_\Omega$ is a regular solution of (15′).

4. Proof of Theorem 1

We are now ready to prove the assertions of Theorem 1. Assume that the conditions of the theorem are satisfied, and let u be a solution to the variational problem (15). Furthermore, let Γ resp. Γ' be closed resp. open subsets of Γ_0 such that

$$\Gamma' \subset\subset \Gamma \subset \Gamma_0. \tag{83}$$

Since $\varphi|_{\Gamma_0}$ is continuous we can find approximating sequences $\varphi_\varepsilon^+, \varphi_\varepsilon^- \in C^2(\Gamma_0)$ satisfying

$$\varphi_\varepsilon^- \leqq \varphi \leqq \varphi_\varepsilon^+ \quad \text{on } \Gamma \tag{84}$$

and

$$\lim \varphi_\varepsilon^+(x) = \lim \varphi_\varepsilon^-(x) = \varphi(x) \quad \forall x \in \Gamma'. \tag{85}$$

Applying the results of Serrin [48] we shall construct barriers δ_ε^+, δ_ε^- such that

$$\delta_\varepsilon^+, \delta_\varepsilon^- \in C^{1,1}(\bar{U}_\Gamma) \tag{86}$$

for some U_Γ,

$$\delta_\varepsilon^- \leqq \varphi_\varepsilon^- \leqq \varphi_\varepsilon^+ \leqq \delta_\varepsilon^+ \qquad \text{on } \Gamma, \tag{87}$$

$$\delta_\varepsilon^- = \varphi_\varepsilon^-, \qquad \delta_\varepsilon^+ = \varphi_\varepsilon^+ \qquad \text{on } \Gamma', \tag{88}$$

and

$$\delta_\varepsilon^-(x) \leqq u(x) \leqq \delta_\varepsilon^+(x) \qquad \forall x \in U_\Gamma. \tag{89}$$

Suppose for a moment that we had constructed the barriers δ_ε^+ and δ_ε^-. Then, we deduce from the inequality (89) in view of (85)

$$\lim_{\substack{x \in \Omega \\ x \to x_0}} u(x) = \varphi(x_0) \qquad \forall x_0 \in \Gamma'. \tag{90}$$

Hence, u is continuous up to Γ', and it is the unique solution of the variational problem (15), since it has been chosen arbitrarily and the solutions of (15) only differ by an additive constant. The assertions of Theorem 1 now follow easily taking the interior regularity of u into account.

To construct the barriers, let us consider a closed cylinder Z_ρ of radius ρ where we assume that $Z_\rho \cap \partial\Omega = \Gamma$. Furthermore, we choose ρ_1, ρ_2 with $\rho < \rho_1 < \rho_2$, such that the cylinders Z_{ρ_1} and Z_{ρ_2}, having the same axis as Z_ρ, intersect $\partial\Omega$ in two subsets Γ_1 and Γ_2 with

$$\Gamma \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma_0. \tag{91}$$

We let $U(\Gamma, r)$ be an open subset of Ω which is bounded by Γ, the level surface $\Gamma_r = \{x \in \Omega: \operatorname{dist}(x, \partial\Omega) = r\}$, and by ∂Z_ρ. The sets $U(\Gamma_1, r)$ and $U(\Gamma_2, r)$ are similar defined.

Assume for a moment that Γ_0 is of class C^3. Then there exists a number d_0 which depends on the principal curvatures of Γ_2 and on the slope of the cylinder walls with respect to $\partial\Omega$ such that the distance function $d(x) = \operatorname{dist}(x, \partial\Omega)$ belongs to $C^2(\bar{U}(\Gamma_1, d_0))$.

Let φ^* be of class C^2 in $U(\Gamma_1, d_0)$ and assume that

$$-H(x, \varphi^*(x)) \leqq (n-1) \cdot H_{n-1}(x) \qquad \forall x \in \Gamma_1. \tag{92}$$

According to Serrin [48; Thm. 10.1] we can find to every given pair of sufficiently large positive numbers α, M a real number r, $0 < r < d_0$, and a real function $h \in C^2([0, r])$ with

$$h(0) = 0, \qquad h(r) = M, \qquad h' \geqq \alpha \tag{93}$$

such that $\delta^+(x) = \varphi^*(x) + h(d(x))$ satisfies in $U(\Gamma, r)$ the inequality

$$A\delta^+ + H(x, \delta^+) \geqq 0. \tag{94}$$

On the other hand, if we suppose that

$$H(x, \varphi^*(x)) \leqq (n-1) \cdot H_{n-1}(x) \qquad \forall x \in \Gamma_1 \tag{95}$$

then $\delta^-(x) = \varphi^*(x) - h(d(x))$ satisfies the relation

$$A\delta^- + H(x, \delta^-) \leqq 0 \qquad \text{in } U(\Gamma, r). \tag{96}$$

Now, to construct δ_ε^+ we choose $m \geqq \sup_{U(\Gamma, d_0)} |u| + \sup_{\Gamma_1} |\varphi|$ and let $\varphi^* \in C^2(U(\Gamma_1, d_0))$ be a smooth function satisfying

$$\varphi^* \geqq \varphi_\varepsilon^+ \quad \text{on } \Gamma, \tag{97}$$

$$\varphi^* = \varphi_\varepsilon^+ \quad \text{on } \Gamma', \tag{98}$$

and

$$\varphi^* \geqq m \quad \text{on } [\partial U(\Gamma, r) \cup \Gamma_1] - \Gamma, \tag{99}$$

where we observe that u is bounded in $U(\Gamma, d_0)$, in view of Lemma 1.

Then, taking the monotonicity of H and the relations (18), (84), (97), and (99) into account, we deduce that the inequality (92) is fulfilled.

Furthermore, choose $\alpha > |D\psi|_\Omega + |D\varphi^*|_{U(\Gamma_1, d_0)}$, $M = \max(m, \sup_{U(\Gamma_1, d_0)} |\varphi^*|)$, and h as above. This yields that

$$\delta_\varepsilon^+(x) = \varphi^*(x) + h(d(x)) \tag{100}$$

satisfies (94),

$$\delta_\varepsilon^+ \geqq \psi \quad \text{in } U(\Gamma, r), \tag{101}$$

$$\delta_\varepsilon^+ \geqq \varphi \quad \text{on } \Gamma, \tag{102}$$

and

$$\delta_\varepsilon^+ \geqq u \quad \text{on } \partial U(\Gamma, r) - \Gamma. \tag{103}$$

To define δ_ε^-, we choose φ^* such that

$$\varphi^* \leqq \varphi_\varepsilon^- \quad \text{on } \Gamma, \tag{104}$$

$$\varphi^* = \varphi_\varepsilon^- \quad \text{on } \Gamma', \tag{105}$$

and

$$\varphi^* \leqq -m \quad \text{on } [\partial U(\Gamma, r) \cup \Gamma_1] - \Gamma. \tag{106}$$

Then (95) is fulfilled. Defining h similarly as before, we deduce that

$$\delta_\varepsilon^-(x) = \varphi^*(x) - h(d(x)) \tag{107}$$

satisfies (96),

$$\delta_\varepsilon^- \leqq \varphi \quad \text{on } \Gamma, \tag{108}$$

and

$$\delta_\varepsilon^- \leqq u \quad \text{on } \partial U(\Gamma, r) - \Gamma. \tag{109}$$

If Γ_0 is only of class C^2, we assume without loss of generality that Γ_0 is the graph of a C^2 function. Then, by mollification we approximate Γ_0 by a sequence of smooth surfaces Γ_0^k, and we construct sequences $\delta_{\varepsilon,k}^+$, $\delta_{\varepsilon,k}^-$ the elements of which satisfy the relations above and have uniformly bounded second derivatives in $U(\Gamma, r)$ (cf. [48; Thm. 14.3]). Thus there are subsequences converging to some functions $\delta_\varepsilon^+, \delta_\varepsilon^- \in H^{2,\infty}(U(\Gamma, r))$ that still satisfy the above conditions.

We shall show that δ_ε^+ resp. δ_ε^- are super-resp. subsolutions in a variational sense. For brevity set $U = U(\Gamma, r)$ and define $I(v; w)$ by

$$I(v; w) = \int_U (1 + |Dv|^2)^{\frac{1}{2}} \, dx + \int_U \int_0^v H(x, t) \, dt \, dx + \int_{\partial U} |v - w| \, d\mathscr{H}_{n-1}. \tag{110}$$

Then, δ_ε^+ resp. δ_ε^- satisfy the relations

$$I(\delta_\varepsilon^+;\delta_\varepsilon^+)\leqq I(v;\delta_\varepsilon^+) \qquad \forall v\in BV(U)\cap\{v\geqq\delta_\varepsilon^+\} \tag{111}$$

resp.

$$I(\delta_\varepsilon^-;\delta_\varepsilon^-)\leqq I(v;\delta_\varepsilon^-) \qquad \forall v\in BV(U)\cap\{v\leqq\delta_\varepsilon^-\}. \tag{112}$$

We shall only prove (111). Consider the function

$$g(\tau)=I(\tau v+(1-\tau)\delta_\varepsilon^+;\delta_\varepsilon^+), \qquad \tau\in\mathbb{R}, \tag{113}$$

where v is smooth, $v\geqq\delta_\varepsilon^+$, and $v|_{\partial U}=\delta_\varepsilon^+|_{\partial U}$.

Obviously g is convex ($g''\geqq 0$) and

$$g'(0)=\langle A\delta_\varepsilon^+ +H(x,\delta_\varepsilon^+), v-\delta_\varepsilon^+\rangle\geqq 0. \tag{114}$$

Hence, we conclude

$$g(0)\leqq g(1). \tag{115}$$

The more general inequality (111) follows by approximation, cf. [11; Appendix III] and [14; Lemma A1 and Lemma A2].

Now, the estimate (89) follows easily.

Lemma 3. *Let u be a solution to* (15), *and let δ^+, $\delta^-\in H^{1,1}(U)$ be super-resp. subsolutions in $U=U_\Gamma$ satisfying*

$$\delta^+\geqq\psi \qquad \text{in } U, \tag{116}$$

$$\delta^-\leqq u\leqq\delta^+ \qquad \text{on } \Gamma^*=\partial U-\Gamma, \tag{117}$$

and

$$\delta^-\leqq\varphi\leqq\delta^+ \qquad \text{on } \Gamma. \tag{118}$$

Then u is estimated by

$$\delta^-\leqq u\leqq\delta^+ \qquad \text{in } U. \tag{119}$$

Proof. We shall only prove the second inequality in (119), since the first one can be proved in the same manner.

First of all, let us observe that u satisfies

$$\begin{aligned}&\int_U(1+|Du|^2)^{\frac12}\,dx+\int_U\int_0^u H(x,t)\,dt\,dx+\int_\Gamma|u-\varphi|\,d\mathcal{H}_{n-1}\\ &\quad\leqq\int_U(1+|Dv|^2)^{\frac12}\,dx+\int_U\int_0^v H(x,t)\,dt\,dx+\int_\Gamma|v-\varphi|\,d\mathcal{H}_{n-1}\\ &\qquad+\int_{\Gamma^*}|u-v|\,d\mathcal{H}_{n-1} \qquad \forall v\in BV(U)\cap\{v\geqq\psi\}.\end{aligned} \tag{120}$$

To verify this inequality, let $v\in BV(U)\cap\{v\geqq\psi\}$ and define

$$\tilde v=\begin{cases}v & \text{in } U\\ u & \text{in } \Omega-U.\end{cases} \tag{121}$$

Then we have in view of (15)

$$J(u)\leqq J(\tilde v). \tag{122}$$

Thus, our assertion follows by simple calculations, since

$$\int_{\Gamma^*} (1+|D\tilde{v}|^2)^{\frac{1}{2}}\, dx = \int_{\Gamma^*} |u-v|\, d\mathscr{H}_{n-1}. \tag{123}$$

Choosing in (111) $v=\max(u,\delta^+)$ and in (120) $v=\min(u,\delta^+)$ we deduce by combining the resulting inequalities and taking the estimate (117) into account that equality must hold in both inequalities provided that

$$\int_{\Gamma} |u-\varphi|\, d\mathscr{H}_{n-1} = \int_{\Gamma} \{|\min(u,\delta^+)-\varphi| + |\max(u,\delta^+)-\delta^+|\}\, d\mathscr{H}_{n-1} \tag{124}$$

which follows from the identity

$$|u-\varphi| = |\min(u,\delta^+)-\varphi| + |\max(u,\delta^+)-\delta^+| \qquad \mathscr{H}_{n-1}\text{-a.e. on } \Gamma. \tag{125}$$

The preceding equation might be easily checked by distinguishing the cases $\delta^+ < u$ and $\delta^+ > u$ in view of (118).

Thus, equality must hold in (120) where $v=\min(u,\delta^+)$. But the derivation of the relation (120) shows, that then equality must hold in (122). Hence, we deduce (since $\tilde{v}$ belongs to $H^{1,1}(\Omega)$)

$$u-\tilde{v} = \text{const} \quad \text{in } \Omega. \tag{126}$$

On the other hand, it follows from (117) that

$$\tilde{v} = u \qquad \mathscr{H}_{n-1}\text{-a.e. on } \Gamma^*. \tag{127}$$

Therefore, the constant in (126) is equal to zero, and our assertion is proved.

5. Proof of Theorem 2

If φ itself belongs to $C^2(\Gamma_0)$, then we may choose $\varphi_\varepsilon^- = \varphi_\varepsilon^+ = \varphi$ in (84). Thus, we can choose the functions φ^* in (100) resp. (107) such, that they coincide in a suitable neighbourhood $U_{\hat{\Gamma}} \subset U(\Gamma, r)$, $\hat{\Gamma} \subset \Gamma'$. Let us denote this common function by φ, too. Then, we deduce from the preceding results that u satisfies an estimate of the form

$$|u(x)-\varphi(x)| \leqq K_1 \cdot d(x) \qquad \forall x \in U_{\hat{\Gamma}}, \tag{128}$$

where K_1 only depends on the C^2-norm of φ, $|D\psi|_\Omega$, $\hat{\Gamma}$ and known quantities, and where φ has uniformly bounded gradient in $U_{\hat{\Gamma}}$.

To obtain a gradient bound for u, let us first assume that ψ belongs to $C^2(\bar{\Omega})$ and satisfies

$$\psi|_{\partial\Omega} \leqq \varphi - \varepsilon, \tag{129}$$

where ε is some positive constant. Since u is continuous in $U(\Gamma, r)$, we conclude that u is strictly greater than ψ near Γ'. Thus, let us suppose that we had chosen $U_{\hat{\Gamma}}$ in such a way that

$$u > \psi \quad \text{in } U_{\hat{\Gamma}} \tag{130}$$

and hence

$$Au + H(x,u) = 0 \quad \text{in } U_{\hat{\Gamma}}. \tag{131}$$

Now, we easily get a bound for $|Du|_{\hat{\Gamma}}$ using an idea of Giaquinta [19]. Let $x' \in U_{\hat{\Gamma}}$ be such that the ball $B=B(x', d(x'))$ of radius $d(x')$ and centered in x' is contained in $U_{\hat{\Gamma}}$. Then, applying the a priori estimates of Trudinger [58; Theorem 2] we deduce

$$|Du(x')| \leqq C_1 \cdot \exp\{C_2 \cdot \sup_B (u-u(x'))/d(x')\}, \tag{132}$$

where C_1 and C_2 are constants depending on n and $\sup_B \left\{ d\,|H| + d^2 \left| \frac{\partial}{\partial x} H(x, u(x)) \right| \right\}$.
Though Trudinger has only considered the case $H = H(x)$, the estimate remains unchanged in the more general case where H depends on t, provided that $\frac{\partial H}{\partial t} \geqq 0$ (cf [58; (44)]).

On the other hand, taking the inequalities

$$d(x) \leqq d(x') + |x - x'| \leqq 2 \cdot d(x') \qquad \forall x \in B \tag{133}$$

and (128) into account, we deduce from (132)

$$|Du(x')| \leqq L_1 = C_1 \cdot \exp\{C_2 \cdot (3K_1 + K_2)\}, \tag{134}$$

where we have set $K_2 = |D\varphi|_{U_{\hat{\Gamma}}}$. Hence, we obtain

$$|Du|_{\hat{\Gamma}} \leqq L_1 . \tag{135}$$

Now, let $U_{\hat{\Gamma}} \subset \Omega$ be any open set with $\bar{U}_{\hat{\Gamma}} \cap \partial\Omega = \hat{\Gamma}$. Then, we conclude that u is Lipschitz in $U_{\hat{\Gamma}}$ and is a solution of the following variational inequality

$$\begin{gathered} \langle Au + H(x, u), v - u \rangle \geqq 0 \qquad \forall v \in K, \\ K = \{v \in H^{1,\infty}(U_{\hat{\Gamma}}) \colon v \geqq \psi,\ v|_{\partial U_{\hat{\Gamma}}} = u\}, \end{gathered} \tag{136}$$

as we easily derive from the relation (120). Moreover, since we assumed ψ to be of class C^2, we obtain

$$u \in H^{2,p}(U_{\hat{\Gamma}}) \tag{137}$$

for any p, $1 \leqq p < \infty$ (cf. Section 6).

Thus, we conclude in view of (135) that $|Du|$ can be estimated by

$$|Du|_{U_{\tilde{\Gamma}}} \leqq L_2 = L_2(L_1, U_{\tilde{\Gamma}}) \tag{138}$$

where $U_{\tilde{\Gamma}}$ is any open set such that $\tilde{\Gamma} \subset \operatorname{int}(\hat{\Gamma})$ and $\bar{U}_{\tilde{\Gamma}} \cap \partial U_{\hat{\Gamma}} = \tilde{\Gamma}$. For a proof of this gradient estimate we refer the reader to the Appendix.

Finally, to remove the restrictions on ψ, let $\psi \in H^{1,\infty}(\Omega)$ with $\psi|_{\partial\Omega} \leqq \varphi$ be given. Then, by mollification, we can easily find smooth functions ψ_ε satisfying (129), which converge in $H^{1,2}(\Omega) \cap C^0(\bar{\Omega})$ towards ψ. Let u_ε be the solutions of the variational problem (15) with respect to the obstacles ψ_ε. Then, the sequence u_ε converges in $BV(\Omega)$ to u, since the solution is unique, and each u_ε satisfies an estimate of the form (138), independently of ε. Hence, this estimate is satisfied by u.

6. Proof of Theorem 3

In view of the Lipschitz regularity of solutions to the variational problem (15) which we developed in the Theorems 2 and 6, we may consider a solution $u \in \mathscr{K}$ of the variational inequality

$$\langle Au + H, v - u \rangle \geqq 0 \qquad \forall v \in \mathscr{K}, \tag{139}$$

where $\mathscr{K}$ is the convex set

$$\mathscr{K} = \{v \in H^{1,2}(\Omega) \colon v \geqq \psi,\ v|_{\partial\Omega} = f\}, \tag{140}$$

Ω is a bounded domain in $\mathbb{R}^n$, $n \geqq 2$, with Lipschitz boundary, ψ is a Lipschitz obstacle with $\psi|_{\partial\Omega} \leqq f$, f is the trace of a function $f \in H^{1,2}(\Omega)$, and where finally H is a given function in $L^2(\Omega)$ and

$$A = -D^i(a_i(x, p)) \tag{141}$$

is a uniformly elliptic operator whose coefficients satisfy

$$a_i \in C^1(\bar{\Omega} \times \mathbb{R}^n) \tag{142}$$

and

$$v_1 \cdot |\xi|^2 \leqq \frac{\partial a_i}{\partial p^j} \xi^i \xi^j \leqq v_2 \cdot |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n, \tag{143}$$

where v_1 and v_2 are positive constants.

Then the following lemma is valid

Lemma 4. *Let $f \in H^{1,2}(\Omega)$, $H \in L^p(\Omega)$, and $\psi \in H^{2,p}(\Omega)$, $2 \leqq p \leqq \infty$, be given functions, and let u be a solution of the variational inequality* (139). *Then we have the estimate*

$$\|Au\|_p \leqq \|A\psi\|_p + 2 \cdot \|H\|_p. \tag{144}$$

Proof. Let β be the following maximal monotone graph in $\mathbb{R} \times \mathbb{R}$

$$\beta(t) = \begin{cases} 0, & \text{if } t > 0 \\ [-1, 0], & \text{if } t = 0 \\ -1, & \text{if } t < 0. \end{cases} \tag{145}$$

Furthermore, let $\mu \in L^p(\Omega)$ be any function such that

$$\max\{A\psi(x) + H(x), 0\} \leqq \mu(x) \quad \text{a.e. in } \Omega. \tag{146}$$

Then, we consider the Dirichlet problem

$$\begin{aligned} Au^* + H + \mu \cdot \beta(u^* - \psi) &\ni 0 \\ u^*|_{\partial\Omega} &= f. \end{aligned} \tag{147}$$

It is well-known that (147) has a solution $u^* \in H^{1,2}(\Omega)$ (cf. [13; Appendix]). We are going to show that u^* belongs to $\mathscr{K}$; hence, it will be the unique solution of the variational inequality (139), since u^* satisfies

$$Au^* + H = \begin{cases} \text{nonnegative} & \text{a.e. in } \Omega \\ 0 & \text{in } \{u^* > \psi\}. \end{cases} \tag{148}$$

The estimate (144) then follows immediately.

In order to prove $u^* \geqq \psi$, let $\varepsilon > 0$ be given and set $\psi_\varepsilon = \psi - \varepsilon$. Taking (146) into account we conclude

$$A\psi_\varepsilon + H + \mu \cdot \beta(\psi_\varepsilon - \psi) = A\psi + H - \mu \leqq 0. \tag{149}$$

Hence (cf. Lemma 5 below)

$$\psi_\varepsilon \leqq u^* \tag{150}$$

by which the assertion is proved.

It remains to prove the following *comparison lemma.*

Lemma 5. *Let u resp. u' be super-resp. subsolutions in the sense that u, $u' \in H^{1,2}(\Omega)$ and the inequalities*

$$Au + H + \beta(u - \psi) \geqq 0 \tag{151}$$

and

$$Au' + H + \beta(u' - \psi') \leqq 0 \tag{152}$$

are valid, where $\psi, \psi' \in H^{1,\infty}(\Omega)$, and $H \in L^2(\Omega)$ are given such that

$$\psi \leqq u \quad \text{and} \quad \psi' \leqq u', \tag{153}$$

where, furthermore, β is a maximal monotone (multivalued) graph in $\mathbb{R} \times \mathbb{R}$, and where the inequalities have an obvious meaning. Then

$$u' - u \leqq \max\{\sup_{\Omega} |\psi - \psi'|, \sup_{\partial\Omega} |u - u'|\}. \tag{154}$$

Proof. Denote the right side of (154) by c, and set $\eta = \max\{u' - u, c\} - c$. Then, we have $0 \leqq \eta \in H_0^{1,2}(\Omega)$, and the inequalities (151) and (152) yield

$$\langle Au - Au' + \beta(u - \psi) - \beta(u' - \psi'), \eta \rangle \geqq 0. \tag{155}$$

The assertion $\eta = 0$ then follows from the fact that β is monotone and A elliptic. We leave the details to the reader.

The claims of Theorem 3 are now easily deduced from the estimate (144) in view of the results of [6] and [12].

7. Proof of Theorem 4

The proof of Theorem 4 is almost identical to that of Theorem 1. Let $x_0 \in \Gamma$ be a point of continuity of φ, and let $\alpha \in C^2(V_r)$, $V_r = \{x' \in \mathbb{R}^{n-1} : |x'| < r\}$, be a local boundary representation such that $(0, \alpha(0)) = x_0$. Moreover, set $\hat{\varphi}(x') = \varphi(x', \alpha(x'))$ and let $\hat{\varphi}_\varepsilon$ be a mollification of $\hat{\varphi}$.

Let u be a solution of the variational problem (15). Then u is bounded in a neighbourhood U_r with $x_0 \in \Gamma$, since φ is continuous at x_0 and hence bounded near x_0. Let m be an upper bound for u in U_r, and let ρ be an arbitrary positive number. Then, there exists a number $r' < r$ such that the inequality

$$\hat{\varphi}_\varepsilon + \rho \geqq \hat{\varphi} \qquad \text{a.e. in } V_{r'}, \tag{156}$$

is valid for sufficiently small ε. Furthermore, we can find a smooth function $\varphi_\varepsilon^+ \in C^2(\Gamma_0)$ such that

$$\varphi_\varepsilon^+(x) = \hat{\varphi}_\varepsilon(x') + \rho \qquad \forall x' \in V_{r'}, \tag{157}$$

where $x = (x', \alpha(x'))$.

Thus, from the proof of Theorem 1 we deduce that we can construct an upper barrier δ_ε^+ such that

$$\delta_\varepsilon^+ = \varphi_\varepsilon^+ \qquad \text{on } \Gamma' = \operatorname{graph} \alpha|_{V_{r'}}, \tag{158}$$

since the inequality

$$-H(x, \varphi_\varepsilon^+(x)) \leqq (n-1) \cdot H_{n-1}(x) \qquad \forall x \in \Gamma' \tag{159}$$

is valid in view of (23) and (156). Hence, we derive

$$\limsup_{\substack{x\in\Omega\\ x\to x_0}} u(x) \leqq \lim_{\substack{x\in\Omega\\ x\to x_0}} \delta_\varepsilon^+(x) = \varphi_\varepsilon^+(x_0), \tag{160}$$

where $\varphi_\varepsilon^+(x_0)$ tends to $\varphi(x_0)+\rho$ if ε goes to zero, from which we conclude

$$\limsup_{\substack{x\in\Omega\\ x\to x_0}} u(x) \leqq \varphi(x_0). \tag{161}$$

The lower estimate for u can be obtained by similar considerations, hence the result.

Remark. We suppose that the variational problem (15) has a unique solution u which coincides a.e. on Γ_0 with φ provided that (23) is satisfied. But, unfortunately, we could not prove this without assuming φ to be continuous.

A first step in this direction would be the following lemma.

Lemma 6. *Let u, u' be solutions of the variational problem* (15) *with respect to the data ψ, φ and ψ', φ' where we assume that*

$$\psi \leqq \psi' \quad \textit{and} \quad \varphi \leqq \varphi'. \tag{162}$$

Furthermore, we suppose that at least one solution is unique. Then we have

$$u \leqq u'. \tag{163}$$

Proof. Lemma 6 follows from a more general result which has been proved in [15; Lemma 3.3 and Remark 3.1].

8. A Counterexample

We shall show that the uniqueness of the solution to the variational problem fails to be true in general, even when there is a solution taking on the prescribed boundary values.

Let B be the unit ball in $\mathbb{R}^n$. Then, the upper hemisphere $u(x)=(1-|x|^2)^{\frac{1}{2}}$ satisfies the equation

$$Au - n = 0 \quad \text{in } B. \tag{164}$$

Moreover, let B_r, $0<r<1$, be the ball of radius r centered in the origin. We easily derive from (164) that u is the unique solution of the variational problem

$$J_r(v) \to \min \quad \text{in } H^{1,1}(B_r), \tag{165}$$

where

$$J_r(v) = \int_{B_r} (1+|Dv|^2)^{\frac{1}{2}}\, dx - n \int_{B_r} v\, dx + \int_{\partial B_r} |v-u|\, d\mathscr{H}_{n-1}. \tag{166}$$

Thus, u minimizes $J=J_1$ in $H^{1,1}(B)$. But, now, u is not the unique solution, since

$$J(u+c) = J(u) \qquad \forall c>0, \tag{167}$$

as we deduce from the identity

$$n\cdot|B| = \mathscr{H}_{n-1}(\partial B). \tag{168}$$

This result is not in contrast to Theorem 1, since the condition (18) is not satisfied.

Appendix

In the proof of the Theorems 2 and 6 we used a priori estimates for the gradient of the solution u of the variational inequality

$$\begin{aligned}&\langle Au+H(x,u), v-u\rangle \geqq 0 \qquad \forall v\in\mathscr{K},\\ &\mathscr{K}=\{v\in H^{1,\infty}(\Omega)\colon v\geqq\psi,\ v|_{\partial\Omega}=u\},\end{aligned} \tag{A1}$$

where A is the minimal surface operator, H is locally Lipschitz in $\mathbb{R}^n\times\mathbb{R}$ satisfying

$$\frac{\partial H}{\partial t}\geqq 0. \tag{A2}$$

We assume Ω to be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and ψ to be of class C^2 in $\bar{\Omega}$.

Then, the following theorem is valid

Theorem A1. *Under the preceding assumptions the gradient of the solution u of the variational inequality* (A1) *can be a priori estimated by*

$$|Du|_{\Omega'}\leqq \mathrm{const}(\Omega', |D\psi|_\Omega, |u|_\Omega, |DH(x,u(x))|_\Omega, n) \qquad \forall\Omega'\subset\subset\Omega. \tag{A3}$$

Moreover, let $\Gamma\subset\partial\Omega$ be relatively open and of class C^2. If we assume that

$$|Du|_\Gamma\leqq L_1, \tag{A4}$$

then we obtain

$$|Du|_{U_{\Gamma'}}\leqq L_2 \qquad \forall\Gamma'\subset\subset\Gamma \tag{A5}$$

where $U_{\Gamma'}$ is one of the open sets we described in the Definition 1, *and where L_2 depends on L_1, $U_{\Gamma'}$, and on the quantities in the estimate* (A3).

Proof. In the case $H=0$ the theorem has already been proved by Giusti [25]. For the generalization we need some techniques and results of [25] and [33].

We have to introduce some definitions. We denote by $\mathscr{S}$ the graph of u over Ω. The outward normal vector ν at a point $(x,u(x))$ is then equal to

$$\nu=(\nu_1,\ldots,\nu_{n+1})=(1+|Du|^2)^{-\frac{1}{2}}\cdot(-D^1u,\ldots,-D^nu,1). \tag{A6}$$

Furthermore, we define the differential operators $\delta=(\delta_1,\ldots,\delta_{n+1})$ and $\mathscr{D}$ by

$$\delta_i=D^i-\nu_i\cdot\nu_k\cdot D^k, \qquad i=1,\ldots,n+1, \tag{A7}$$

and

$$\mathscr{D}=\delta_i\delta_i, \tag{A8}$$

where now and in the following we sum over repeated indices from 1 to $n+1$.

Since u satisfies the equation

$$Au+H(x,u)=0 \quad \text{in } E=\{x\in\Omega\colon u(x)>\psi(x)\} \tag{A9}$$

we deduce from [2] that $w=-\log\nu_{n+1}$ satisfies in $\mathscr{S}_E=\mathrm{graph}\,u|_E$ the inequality

$$\mathscr{D}w\geqq|\delta w|^2+\frac{1}{\nu_{n+1}}\cdot\delta_{n+1}H. \tag{A10}$$

Thus, we obtain

$$\mathcal{D}w \geqq |\delta w|^2 - \sum_{i=1}^{n} v_i D^i H \geqq |\delta w|^2 + (1+|Du|^2)^{-\frac{1}{2}} \cdot \sum_{i=1}^{n} D^i u \cdot \frac{\partial H}{\partial x^i} \tag{A 11}$$

taking the monotonicity of $H(x, .)$ and the definition of v_i into account. Setting

$$c_1 = \sup_{\Omega} \left| \frac{\partial}{\partial x} H(x, u(x)) \right| \tag{A 12}$$

this yields

$$\mathcal{D}w \geqq |\delta w|^2 - c_1 \quad \text{in } \mathcal{S}_E. \tag{A 13}$$

Now, let η be a smooth function such that

$$0 \leqq \eta \leqq 1 \quad \text{and} \quad \operatorname{supp}\eta \cap \partial\Omega \subset \Gamma, \tag{A 14}$$

and define $z = \max(w\eta^2 - k, 0)$ where k is a real number satisfying

$$k > \max(L_1, |D\psi|_\Omega). \tag{A 15}$$

Then,

$$\operatorname{supp} z \subset E, \tag{A 16}$$

since we have $Du(x) = D\psi(x)$ for $x \in \{y \in \Omega : u(y) = \psi(y)\}$. Multiplying (A 13) with z and integrating over $\mathcal{S}$ we obtain

$$\int_{\mathcal{S}} |\delta w|^2 \cdot z \, d\mathcal{H}_n \leqq \int_{\mathcal{S}} \{\mathcal{D} w + c_1\} z \, d\mathcal{H}_n, \tag{A 17}$$

where $\mathcal{H}_n$ denotes the n-dimensional Hausdorff measure, i.e.

$$\int_{\mathcal{S}} h \, d\mathcal{H}_n = \int_{\Omega} h(x, u(x)) \cdot (1+|Du|^2)^{\frac{1}{2}} \, dx \qquad \forall h \in C^0(\mathbb{R}^{n+1}). \tag{A 18}$$

Moreover, observing that

$$\int_{\mathcal{S}} \delta_i h \, d\mathcal{H}_n = \int_{\mathcal{S}} h v_i H \, d\mathcal{H}_n \tag{A 19}$$

which is valid for all test functions h such that

$$\operatorname{supp} h \cap \mathcal{S} \subset\subset \mathcal{S}_E, \tag{A 20}$$

and using the identity

$$\delta_i w \cdot \delta_i z = -\mathcal{D} w \cdot z + \delta_i \{\delta_i w \cdot z\}, \tag{A 21}$$

we conclude from (A 17)

$$\begin{aligned} \int_{A(k,\eta)} |\delta w|^2 \cdot \eta^2 \, d\mathcal{H}_n \leqq c_1 \cdot \int_{\mathcal{S}} z \, d\mathcal{H}_n + \int_{\mathcal{S}} |\delta w| \cdot z \cdot |H| \, d\mathcal{H}_n \\ - \int_{\mathcal{S}} |\delta w|^2 \cdot z \, d\mathcal{H}_n + 2 \cdot \int_{A(k,\eta)} |\delta w| \cdot |\delta \eta| \cdot \eta \cdot w \, d\mathcal{H}_n, \end{aligned} \tag{A 22}$$

where $A(k, \eta)$ denotes the set: graph $u|_{\{x \in \Omega : z(x) > 0\}}$.

Thus, using

$$a \cdot b \leqq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2} b^2$$

and Schwarz's inequality we obtain

$$\int_{A(k,\eta)} |\delta w|^2 \cdot \eta^2 \, d\mathcal{H}_n \leqq c_2 \cdot \{ \int_{A(k,\eta)} [z + |\delta \eta|^2 \cdot |w|^2] \, d\mathcal{H}_n \} \tag{A23}$$

with a suitable constant c_2, $c_2 \geqq 1$.

Finally, taking the estimate

$$\begin{aligned} |\delta(w\eta^2)|^2 &\leqq 2 \cdot |\delta w|^2 \cdot \eta^4 + 8 \cdot |w|^2 \cdot \eta^2 \cdot |\delta \eta|^2 \\ &\leqq 2 \cdot |\delta w|^2 \cdot \eta^2 + 8 \cdot |w|^2 \cdot |\delta \eta|^2 \end{aligned} \tag{A24}$$

into account, we deduce

$$\int_{\mathscr{S}} |\delta z|^2 \, d\mathcal{H}_n \leqq 2c_2 \cdot \{ \int_{A(k,\eta)} [z + 5 \cdot |\delta \eta|^2 \cdot |w|^2] \, d\mathcal{H}_n \}, \tag{A25}$$

or finally

$$\int_{\mathscr{S}} |\delta z|^2 \, d\mathcal{H}_n \leqq c_3 \cdot \{ |A(k,\eta)| + \int_{A(k,\eta)} |w|^2 \, d\mathcal{H}_n \} \tag{A26}$$

where $|A(k,\eta)| = \mathcal{H}_n(A(k,\eta))$, c_3 depends on $|\delta \eta|$, and where we used the estimate $|z| \leqq |w|$.

On the other hand, the following imbedding theorem is valid (cf. [38])

$$\{ \int_{\mathscr{S}} |z|^{n/n-1} \, d\mathcal{H}_n \}^{(n-1)/n} \leqq \frac{4^{n+1}}{\omega_n^{1/n}} \cdot \int_{\mathscr{S}} [|\delta z| + z \cdot |H|] \, d\mathcal{H}_n, \tag{A27}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Using the Hölder inequality

$$\int_{\mathscr{S}} z \, d\mathcal{H}_n \leqq [\mathcal{H}_n(\mathscr{S} \cap \operatorname{supp} \eta)]^{1/n} \cdot \{ \int_{\mathscr{S}} |z|^{n/n-1} \, d\mathcal{H}_n \}^{(n-1)/n} \tag{A28}$$

and the estimate

$$\begin{aligned} \mathcal{H}_n(\mathscr{S} \cap \operatorname{supp} \eta) &\leqq \int_G (1 + |Du|^2)^{\frac{1}{2}} \, dx \leqq \int_G (1 + |D\psi|^2)^{\frac{1}{2}} \, dx \\ &\quad + \int_G \int_u^{\psi} H(x,t) \, dt \, dx + \int_{\partial G} |u - \psi| \, d\mathcal{H}_{n-1} \end{aligned} \tag{A29}$$

where $G \subset \Omega$ is an open set with finite perimeter containing $\operatorname{supp} \eta$ and where we suppose $\operatorname{supp} \eta$ to be both a subset of Ω and of $\mathscr{S}$ without changing the notation, we conclude

$$\{ \int_{\mathscr{S}} |z|^{n/n-1} \, d\mathcal{H}_n \}^{\frac{n-1}{n}} \leqq c_4 \cdot \int_{\mathscr{S}} |\delta z| \, d\mathcal{H}_n \tag{A30}$$

provided that $\operatorname{supp} \eta$ is small enough.

Hence, the preceding inequality yields

$$\int_{\mathscr{S}} |z|^2 \, d\mathcal{H}_n \leqq c_5 \cdot |A(k,\eta)|^{2/n} \cdot \int_{\mathscr{S}} |\delta z|^2 \, d\mathcal{H}_n \tag{A31}$$

in view of a well-known argument.

Combining the estimates (A26) and (A31) we thus obtain

$$\int_{\mathscr{S}} |z|^2 \, d\mathcal{H}_n \leqq c_6 \cdot \{ |A(k,\eta)|^{1+\frac{2}{n}} + |A(k,\eta)|^{\frac{2}{n}} \cdot \int_{A(k,\eta)} |w|^2 \, d\mathcal{H}_n \}, \tag{A32}$$

or finally,

$$\int_{\mathscr{S}} |z|\, d\mathscr{H}_n \leqq c_7 \cdot \{|A(k,\eta)|^{1+\frac{1}{n}} + |A(k,\eta)|^{\frac{1}{2}+\frac{1}{n}} \cdot [\int_{A(k,\eta)} |w|^2\, d\mathscr{H}_n]^{\frac{1}{2}}\} \tag{A33}$$

and

$$|h-k| \cdot |A(h,\eta)| \leqq c_7 \cdot \{|A(k,\eta)|^{1+\frac{1}{n}} + |A(k,\eta)|^{\frac{1}{2}+\frac{1}{n}} \cdot [\int_{A(k,\eta)} |w|^2\, d\mathscr{H}_n]^{\frac{1}{2}}\} \tag{A34}$$

for $h > k \geqq k_0 = \max\{|D\psi|_\Omega, L_1\} + 1$.

We are now in the same situation as in the proof of Lemma 1 (cf. the inequality (43)), and we can complete the proof of the a priori estimate for $|Du|$ following the same pattern as in Section 1. Thus, we have to show that

$$\int_{A(k,\eta)} |w|^2\, d\mathscr{H}_n \tag{A35}$$

is bounded in terms of η and known quantities independently of k.

In order to estimate this integral, let us first estimate

$$\int_{A(k_0)} |\delta w|^2 \cdot \eta^2\, d\mathscr{H}_n, \tag{A36}$$

where $A(k_0) = \{(x, u(x)) \in \mathscr{S} : w(x) > k_0\}$ and η is one of the test functions we considered above.

Using an idea of Giusti [25], we multiply the inequality (A13) with $\eta_\varepsilon^2 = \eta^2 \cdot \dfrac{z_0}{z_0+\varepsilon}$, where $z_0 = \max(w - k_0, 0)$ and ε is an arbitrary positive number. Hence, taking (A19) and (A21) into account we derive

$$\begin{aligned}\int_{\mathscr{S}} |\delta w|^2 \cdot \eta_\varepsilon^2\, d\mathscr{H}_n \leqq c_1 \cdot \int_{\mathscr{S}} \eta_\varepsilon^2\, d\mathscr{H}_n + 2 \cdot \int_{\mathscr{S}} |\delta w| \cdot |\delta\eta| \cdot \eta \cdot \frac{z_0}{z_0+\varepsilon}\, d\mathscr{H}_n \\ + \int_{\mathscr{S}} |\delta w| \cdot \eta_\varepsilon^2 \cdot |H|\, d\mathscr{H}_n,\end{aligned} \tag{A37}$$

observing that

$$\delta_i \left(\frac{z_0}{z_0+\varepsilon}\right) = \frac{\delta_i w \cdot \varepsilon}{(z_0+\varepsilon)^2} \quad \text{in } A(k_0). \tag{A38}$$

Moreover, since

$$\frac{z_0}{z_0+\varepsilon} \to \chi_{A(k_0)} \tag{A39}$$

we obtain

$$\int_{A(k_0)} |\delta w|^2 \cdot \eta^2\, d\mathscr{H}_n \leqq c_8 \cdot \{\int_{A(k_0)} [\eta^2 + |\delta\eta|^2]\, d\mathscr{H}_n\}. \tag{A40}$$

Finally, to estimate

$$\int_{A(k_0)} |w|^2 \cdot \eta^2\, d\mathscr{H}_n \tag{A41}$$

we shall follow the lines of [33] and multiply equation (A9) with $u \cdot w_{k_0}^2 \cdot \eta^2$, where $w_{k_0} = \max(w - k_0, 0)$, integrate over Ω and transform the equation so obtained in the following way

$$\begin{aligned}\int_\Omega a_i\{D^i u \cdot w_{k_0}^2 \cdot \eta^2 + 2 \cdot u \cdot D^i w_{k_0} \cdot w_{k_0} \cdot \eta^2 + 2 \cdot u \cdot w_{k_0}^2 \cdot D^i\eta \cdot \eta\}\, dx \\ \leqq \int_\Omega |H| \cdot u \cdot w_{k_0}^2 \cdot \eta^2\, dx,\end{aligned} \tag{A42}$$

where

$$a_i = D^i u \cdot (1+|Du|^2)^{-\frac{1}{2}}. \tag{A43}$$

Hence, we have, after some calculations which are identical to those in [33; p. 700] and in view of the estimate (A 40)

$$\int_\Omega w_{k_0}^2 \cdot \eta^2 \cdot (1+|Du|^2)^{\frac{1}{2}}\, dx \leqq c_9 \cdot \int_\Omega \{w_{k_0}^2 + (1+|Du|^2)^{\frac{1}{2}}\} \cdot \{\eta^2 + |D\eta|^2\}\, dx. \tag{A44}$$

Thus, observing that

$$w^2 = \tfrac{1}{4}\,|\log(1+|Du|^2)|^2 \leqq \alpha \cdot (1+|Du|^2)^{\frac{1}{2}}, \tag{A45}$$

where α is some suitable constant, we deduce

$$\int_{A(k_0)} w^2 \cdot \eta^2\, d\mathscr{H}_n \leqq c_{10} \cdot \int_{\mathscr{S}} k_0^2 \cdot \{\eta^2 + |D\eta|^2\}\, d\mathscr{H}_n, \tag{A46}$$

which is an estimate of the required form.

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Dr. Claus Gerhardt
FB Mathematik der Universität
D-6500 Mainz
Saarstr. 21
Federal Republic of Germany

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