Hypersurfaces of Prescribed Curvature in Lorentzian Manifolds

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ABSTRACT. The existence of closed hypersurfaces of prescribed curvature in globally hyperbolic Lorentzian manifolds is proved, provided there are barriers.

0. INTRODUCTION

Consider the problem of finding a closed hypersurface of prescribed curvature F in a complete (n+1)-dimensional manifold N. To be more precise, let Ω be a connected open subset of N, $f \in C^{2,\alpha}(\bar{\Omega})$, F a smooth, symmetric function defined in an open cone $\Gamma \subset \mathbb{R}^n$, then we look for a hypersurface $M \subset \Omega$ such that

$$F_{|_M} = f(x) \quad \text{for all } x \in M, \tag{0.1}$$

where $F_{|_M}$ means that F is evaluated at the vector $(\kappa_i(x))$ the components of which are the principal curvatures of M. The prescribed function f should satisfy natural structural conditions, e.g., if Γ is the positive cone and the hypersurface M is supposed to be convex, then f should be positive, but no further, merely technical, conditions should be imposed.

If N is a Riemannian manifold, then the problem has been solved in the case when $F = H$, the *mean curvature*, where in addition n had to be small, and N conformally flat, cf. [7], and for curvature functions F of class (K), no restrictions on n, cf. [4, 6]. We also refer to [5], where more special situations are considered, and the bibliography therein.

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In Lorentzian manifolds, existence results for space-like hypersurfaces of prescribed curvature have only been proved in the case $F = H$ so far, cf. [3], and [1, 2], where the results are better than in the Riemannian case, since no restrictions on n have to be imposed, and rather general ambient spaces can be considered. Thus, one would hope that for curvature functions of class (K) the existence results are at least as good as in the Riemannian case, and maybe the proof a little bit less demanding.

Unfortunately, the Lorentzian structure is only of advantage as far as the C^1-estimates are concerned, while the proof of the C^2-estimates is more difficult, if not impossible, for arbitrary functions of class (K). The complications derive from the Gauß equations, where the term stemming from the second fundamental form of the hypersurface has the opposite sign in the Lorentzian case as compared to the Riemannian case, which in turn leads to an unfavourable sign in the equation for the second fundamental form used for the a priori estimates.

We were only able to overcome these difficulties for curvature functions belonging to a fairly large subclass of (K), called (K^*), which will be defined in Section 1, and which includes the Gaussian curvature.

To give a precise statement of the existence result, we need a few definitions and assumptions. First, we assume that N is a smooth, connected, *globally hyperbolic* manifold with a compact *Cauchy hypersurface*, or equivalently, that N is topologically a product, $N = \mathbb{R} \times S_0$, where S_0 is a compact, n-dimensional Riemannian manifold, and there exists a Gaussian coordinate system $(x^\alpha)_{0\le\alpha\le n}$ such that x^0 represents the time, the $(x^i)_{1\le i\le n}$ are local coordinates for S_0, where we may assume that S_0 is equal to the level hypersurface $\{x^0 = 0\}$—we don't distinguish between S_0 and $\{0\} \times S_0$—, and such that the Lorentzian metric takes the form

$$d\bar{s}_N^2 = e^{2\psi}\{-dx^{0^2} + \sigma_{ij}(x^0, x)dx^i dx^j\}, \tag{0.2}$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the space-like components (x^i), see [9], [11, p. 212], [10, p. 252], and [3, Section 6].

In N we consider an open, precompact, connected set Ω that is bounded by two *achronal*, connected, space-like hypersurfaces M_1 and M_2, where M_1 is supposed to lie in the *past* of M_2.

Let F be of class (K^*), and $0 < f \in C^{2,\alpha}(\bar{\Omega})$. Then, we assume that the boundary components M_i act as barriers for (F, f).

Definition 0.1. M_2 is an *upper barrier* for (F, f), if M_2 is strictly convex and satisfies

$$F_{|M_2} \geq f, \tag{0.3}$$

and M_1 is a *lower barrier* for (F, f), if at the points $\Sigma \subset M_1$, where M_1 is strictly convex, there holds

$$F_{|\Sigma} \leq f. \tag{0.4}$$

Σ may be empty.

We shall clarify in Section 2 what *convexity* means for space-like hypersurfaces. Then, we can prove the following result:

Theorem 0.2. *Let M_1 be a lower and M_2 an upper barrier for (F, f). Then, the problem*

$$F_{|M} = f \tag{0.5}$$

has a strictly convex solution $M \subset \bar{\Omega}$ of class $C^{4,\alpha}$ that can be written as a graph over S_0, provided there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$.

Remark 0.3. As we shall show in Section 2, the existence of a strictly convex function χ is guaranteed by the assumption that the level hypersurfaces $\{x^0 = \text{const}\}$ are strictly convex in $\bar{\Omega}$.

The paper is organized as follows: In Section 1 we define the curvature functions of class (K^*) and examine their properties. In Section 2 we introduce the notations and common definitions we rely on, state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces in *pseudo-riemannian* manifolds, and analyze achronal hypersurfaces in some detail. In Section 3 we look at the curvature flow associated with our problem, and the corresponding evolution equations for the basic geometrical quantities of the flow hypersurfaces. In Section 4 we prove lower order estimates for the evolution problem, while a priori estimates in the C^2-norm are derived in Section 5. Finally, in Section 6, we demonstrate that the evolutionary solution converges to a stationary solution.

1. Curvature functions

Let $\Gamma_+ \subset \mathbb{R}^n$ be the open positive cone and $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0; \tag{1.1}$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices S_+, for, let $(h_{ij}) \in S_+$ with eigenvalues κ_i, $1 \le i \le n$, then define F on S_+ by

$$F(h_{ij}) = F(\kappa_i). \tag{1.2}$$

If we define

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{1.3}$$

and

$$F^{ij,k\ell} = \frac{\partial^2 F}{\partial h_{ij}\,\partial h_{k\ell}} \tag{1.4}$$

then,

$$F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}|\xi^i|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \tag{1.5}$$

$$F^{ij} \quad \text{is diagonal if } h_{ij} \text{ is diagonal}, \tag{1.6}$$

and

$$F^{ij,k\ell}\eta_{ij}\eta_{k\ell} = \frac{\partial^2 F}{\partial \kappa_i\,\partial \kappa_j}\eta_{ii}\eta_{jj} + \sum_{i \ne j} \frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2, \tag{1.7}$$

for any $(\eta_{ij}) \in S$, where S is the space of all symmetric matrices. The second term on the right-hand side of (1.7) is non-positive if F is concave, and non-negative if F is convex, and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

In [6] we defined the class (K) as follows:

Definition 1.1. A symmetric function $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ positively homogeneous of degree 1 is said to be of class (K) if

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+, \tag{1.8}$$

$$F \text{ is concave}, \tag{1.9}$$

$$F_{|\partial\Gamma_+} = 0, \tag{1.10}$$

and there exists a constant $c = c(F)$ such that

$$F^{ij,k\ell}\eta_{ij}\eta_{k\ell} \le cF^{-1}(F^{ij}\eta_{ij})^2 - F^{ik}\tilde{h}^{j\ell}\eta_{ij}\eta_{k\ell} \quad \text{for all } \eta \in S, \tag{1.11}$$

where F is evaluated at $(h_{ij}) \in S_+$ and $(\tilde{h}^{ij}) = (h_{ij})^{-1}$.

As we only recently became aware of, inequality (1.11) is valid with constant $c = 1$ if it is valid for a larger c.

Lemma 1.2. *Let $F \in C^2(\Gamma_+)$ be a symmetric curvature function, positively homogeneous of degree $d_0 > 0$ that satisfies the relations* (1.8) *and* (1.11)*, then it fulfills* (1.11) *with constant $c = 1$, i.e.,*

$$F^{ij,k\ell}\eta_{ij}\eta_{k\ell} \le F^{-1}(F^{ij}\eta_{ij})^2 - F^{ik}\tilde{h}^{j\ell}\eta_{ij}\eta_{k\ell} \quad \text{for all } \eta \in S, \tag{1.12}$$

or, equivalently, if we set $\hat{F} = \log F$,

$$\hat{F}^{ij,k\ell}\eta_{ij}\eta_{k\ell} \le -\hat{F}^{ik}\tilde{h}^{j\ell}\eta_{ij}\eta_{k\ell} \quad \text{for all } \eta \in S. \tag{1.13}$$

Equality holds in (1.12) *and* (1.13) *for* $(\eta_{ij}) = (h_{ij})$.

Proof. As we have shown in [6, Lemma 1.3 and Remark 1.4], a symmetric curvature function $F \in C^2(\Gamma_+)$ satisfies inequality (1.11) if and only if

$$F_i\kappa_i \le F_j\kappa_j, \quad \text{for } \kappa_j \le \kappa_i, \tag{1.14}$$

and

$$F_{ij}\xi^i\xi^j \le cF^{-1}(F_i\xi^i)^2 - F_i\kappa_i^{-1}|\xi^i|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \tag{1.15}$$

where F_i, F_{ij} are ordinary partial derivatives of F in Γ_+. Thus, we have to show that (1.15) holds with $c = 1$ for the F's under consideration.

We note that $F > 0$, cf. the proof of Lemma 1.8 below. Let $\hat{F} = \log F$ and

$$f_{ij} = \hat{F}_{ij} + \hat{F}_i\kappa_i^{-1}\delta_{ij}, \tag{1.16}$$

then the relation (1.15) is equivalent to

$$f_{ij} - (c-1)\hat{F}_i\hat{F}_j \le 0. \tag{1.17}$$

We shall demonstrate that

$$f_{ij} \le 0. \tag{1.18}$$

Define Λ by

$$\Lambda = \{\, \lambda \in \mathbb{R}_+ : f_{ij} - \lambda\hat{F}_i\hat{F}_j \le 0 \,\}, \tag{1.19}$$

and let $\lambda_0 = \inf \Lambda$.

Λ is non-empty, so that the infimum is well defined and attained. If $\lambda_0 = 0$, then the main part of the Lemma is proved. Thus, assume that $\lambda_0 > 0$, and let μ be the largest eigenvalue of

$$f_{ij} - \lambda_0 \hat{F}_i \hat{F}_j \tag{1.20}$$

with eigenspace E. Evidently, μ must be zero.

Let (κ^i) be the argument of F. Then, in view of the homogeneity of F we conclude

$$f_{ij}\kappa^j = 0 \tag{1.21}$$

and

$$\hat{F}_i \kappa^i = d_0. \tag{1.22}$$

Now, let $\eta = (\eta^i) \in E$, then

$$f_{ij}\eta^j - \lambda_0 \hat{F}_i \hat{F}_j \eta^j = 0, \tag{1.23}$$

and, multiplying this equation with (κ^i), we obtain

$$\lambda_0 \hat{F}_i \eta^i = 0, \tag{1.24}$$

i.e., $D\hat{F}$ is orthogonal to E, and

$$f_{ij}\eta^j = 0. \tag{1.25}$$

For $0 < \varepsilon < \lambda_0$ set

$$g^{\varepsilon}_{ij} = f_{ij} - (\lambda_0 - \varepsilon)\hat{F}_i \hat{F}_j. \tag{1.26}$$

Then the largest eigenvalue of g^{ε}_{ij}, has to be positive because of the definition of λ_0. Let η_ε be a corresponding unit eigenvector, then, η_ε has to be orthogonal to E, for E is also an eigenspace of g^{ε}_{ij}; but this is impossible, since a subsequence of the η_ε's converges to a unit vector in E, if ε tends to zero.

Hence, we conclude that $\lambda_0 = 0$ and that inequality (1.13) is valid. Finally, equality holds in (1.13) if we choose $(\eta_{ij}) = (h_{ij})$ in view of (1.21). $\square$

Thus, it seems worth to redefine the class (K).

Definition 1.3. A symmetric curvature function $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ positively homogeneous of degree $d_0 > 0$ is said to be of class (K) if

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0 \quad \text{in } \Gamma_+, \tag{1.27}$$

$$F_{|\partial\Gamma_+} = 0, \tag{1.28}$$

and

$$F^{ij,k\ell}\eta_{ij}\eta_{k\ell} \le F^{-1}(F^{ij}\eta_{ij})^2 - F^{ik}\tilde{h}^{j\ell}\eta_{ij}\eta_{k\ell} \quad \text{for all } \eta \in S, \tag{1.29}$$

or, equivalently, if we set $\hat{F} = \log F$,

$$\hat{F}^{ij,k\ell}\eta_{ij}\eta_{k\ell} \le -\hat{F}^{ik}\tilde{h}^{j\ell}\eta_{ij}\eta_{k\ell} \quad \text{for all } \eta \in S, \tag{1.30}$$

where F is evaluated at (h_{ij}).

Remark 1.4.

(i) The main difference in the new definition is that we no longer assume F to be concave. Instead, we deduce from (1.30) that $\hat{F} = \log F$ is concave, which is sufficient to apply the higher regularity results once the C^2-estimates are established.

(ii) We conclude immediately that products of functions of class (K) stay in this class, as is the case for positive powers.

(iii) If one wants to prove that a particular function is of class (K) it might be helpful to verify the formally less restrictive inequality (1.11) instead of (1.29).

We immediately deduce from (1.29) the following result:

Lemma 1.5. *Let F be of class (K), let κ_r be the largest eigenvalue of $(h_{ij}) \in S_+$, then, for any $(\eta_{ij}) \in S$ we have*

$$F^{ij,k\ell}\eta_{ij}\eta_{k\ell} \le F^{-1}(F^{ij}\eta_{ij})^2 - \kappa_r^{-1}F^{ij}\eta_{ir}\eta_{jr}, \tag{1.31}$$

where F is evaluated at (h_{ij}).

Let H_k be the symmetric polynomial of order k

$$H_k(\kappa_i) = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \quad 1 \le k \le n, \tag{1.32}$$

$\sigma_k = (H_k)^{1/k}$ and $\tilde{\sigma}_k$ the inverses of σ_k

$$\tilde{\sigma}_k(\kappa_i) = \frac{1}{\sigma_k(\kappa_i^{-1})}, \tag{1.33}$$

then, we proved in [6, Lemma 1.5], see also [8], that the $\tilde{\sigma}_k$ are of class (K).

Unfortunately, the class (K) is too large to prove existence results in the Lorentzian case. Instead, we have to consider a subclass (K^*) which is defined by the additional technical assumption

Definition 1.6. A function $F \in (K)$ is said to be of class (K^*) if there exists $0 < \varepsilon_0 = \varepsilon_0(F)$ such that

$$\varepsilon_0 F H \le F^{ij} h_{ik} h^k_j \tag{1.34}$$

for any $(h_{ij}) \in S_+$, where F is evaluated at (h_{ij}). H represents the mean curvature, i.e., the trace of (h_{ij}).

Here, the index is raised with respect to the Euclidean metric.

Evidently, $F = \sigma_n = \tilde{\sigma}_n$ is of class (K^*) since

$$F^{ij} = \frac{1}{n} F \tilde{h}^{ij}, \tag{1.35}$$

where $(\tilde{h}^{ij}) = (h_{ij})^{-1}$.

On the other hand, the $\tilde{\sigma}_k$, $1 \le k < n$, do not seem to belong to (K^*) as is easily checked for $k = 1$, while their inverses, the σ_k, fulfill (1.34). However, we shall show in Proposition 1.9 below that functions of the form FK, where $F \in (K)$ and $K = \sigma_n$, belong to (K^*).

We should note that any symmetric $F \in C^1(\Gamma_+)$, positively homogeneous of degree d_0, with $F_i > 0$, satisfies the estimate

$$F^{ij} h_{ik} h^k_j \le d_0 F H \tag{1.36}$$

for any $(h_{ij}) \in S_+$.

Before we establish some properties of (K^*), we need the following definition.

Definition 1.7. A symmetric curvature function $F \in C^{2,\alpha}(\Gamma_+)$ positively homogeneous of degree $d_0 > 0$ is said to be of class (K_b), if it satisfies the conditions of a function in class (K) except the relation (1.28).

Lemma 1.8. *Any $F \in (K_b)$ is bounded on bounded subsets of Γ_+ and positive.*

Proof. First, we note that $F > 0$ because of the homogeneity and Euler's formula. Let $\hat{F} = \log F$ and consider $\kappa = (\kappa^i) \in \Gamma_+$; in view of the concavity of $\hat{F}$, we deduce

$$\hat{F}(\kappa) \le \hat{F}(1,\dots,1) + \hat{F}_i(1,\dots,1)(\kappa^i - 1), \tag{1.37}$$

i.e., $\hat{F}$ is locally bounded from above. □

Now, we can prove the following:

Proposition 1.9.

(i) *Let* $F \in (K^*)$ *and* $r > 0$, *then* $F^r \in (K^*)$.
(ii) *Let* $F \in (K_b)$ *and* $K \in (K^*)$, *then* $FK \in (K^*)$.
(iii) *The* $F \in (K)$ *satisfying*

$$F_i \kappa_i \ge \varepsilon_0 F \quad \text{for all } i, \tag{1.38}$$

with some positive $\varepsilon_0 = \varepsilon_0(F)$, *are of class* (K^*), *and they are precisely those, that can be written in the form*

$$F = GK^a, \quad a > 0, \tag{1.39}$$

where $G \in (K_b)$ *and* $K = \sigma_n$.
(iv) *If* $n = 2$, *any* $F \in (K^*)$ *satisfies* (1.38), *i.e., the functions in* (K^*) *are exactly those given in* (1.39).

Proof. The demonstration of the first two properties is straight-forward, since the product FK, where $F \in (K_b)$ and $K \in (K)$, can be extended as a continuous function to $\bar{\Gamma}_+$ vanishing on the boundary, so that $FK \in (K)$.

To prove (iii), we first note that any $F \in (K)$ satisfying (1.38) certainly belongs to (K^*), and for any F of the form (1.39) the preceding estimate is valid. Thus, let us assume that $F \in (K^*)$ is given for which (1.38) holds. Let $\varepsilon > 0$ and set $G = FK^{-\varepsilon}$. We shall show $G \in (K_b)$, if ε is small, completing the proof of (iii).

As before, indicate the logarithm of a function by a hat; then

$$\hat{G}_i = \hat{F}_i - \varepsilon \hat{K}_i \ge \left(\varepsilon_0 - \frac{\varepsilon}{n}\right) \kappa_i^{-1} > 0, \tag{1.40}$$

if $\varepsilon < n\varepsilon_0$, i.e., (1.27) is satisfied.

The inequality (1.30) is valid, because this inequality becomes an equality when evaluated with $\hat{F} = \hat{K}$.

Finally, let us derive property (iv). Assume $n = 2$, and let $F \in (K^*)$, which, without loss of generality, should be homogeneous of degree 1. Consider $\kappa = (\kappa^1, \kappa^2) \in \Gamma_+$ and suppose for simplicity that $\kappa^1 \le \kappa^2$, then

$$F_2\kappa^2 \le F_1\kappa^1, \tag{1.41}$$

cf. (1.14), and

$$F = F_1\kappa^1 + F_2\kappa^2. \tag{1.42}$$

Suppose that there is a sequence κ_ε, with $\kappa_\varepsilon^1 \le \kappa_\varepsilon^2$, such that $\hat{F}_2\kappa_\varepsilon^2$ tends to 0. In view of the homogeneity, we may assume that

$$H = \kappa_\varepsilon^1 + \kappa_\varepsilon^2 = 1, \tag{1.43}$$

so that we conclude from (1.34) and (1.42)

$$\varepsilon_0 \le (\hat{F}_1\kappa_\varepsilon^1)\kappa_\varepsilon^1 + (\hat{F}_2\kappa_\varepsilon^2)\kappa_\varepsilon^2 \le \kappa_\varepsilon^1 + \frac{\varepsilon_0}{2}, \tag{1.44}$$

for small ε, i.e., $\kappa_\varepsilon^1 \ge \varepsilon_0/2$, contradicting the assumption that $\hat{F}_2(\kappa_\varepsilon)$ should tend to zero, which is only possible if $\kappa_\varepsilon^1 \to 0$. $\square$

The preceding considerations are also applicable if the κ_i are the principal curvatures of a hypersurface M with metric (g_{ij}). F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) with eigenvalues κ_i with respect to the metric.

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{1.45}$$

is then a contravariant tensor of second order. Sometimes it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$h_j^i = g^{ik}h_{kj}. \tag{1.46}$$

Then,

$$F_i^j = \frac{\partial F}{\partial h_j^i} \tag{1.47}$$

is also a mixed tensor with contravariant index j and covariant index i.

2. NOTATIONS AND PRELIMINARY RESULTS

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for hypersurfaces. In view of the subtle but important difference that is to be seen in the *Gauß equation* depending on the nature of the ambient space—Riemannian or Lorentzian—, which we already mentioned in the introduction, we shall formulate the governing equations of a hypersurface M in a pseudo-riemannian (n+1)-dimensional space N, which is either Riemannian or Lorentzian. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), $(R_{ijk\ell})$, etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e., for a function u in N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon \tag{2.1}$$

with obvious generalizations to other quantities.

Let M be a *space-like* hypersurface, i.e., the induced metric is Riemannian, with a differentiable normal ν. We define the signature of ν, $\sigma = \sigma(\nu)$, by

$$\sigma = \bar{g}_{\alpha\beta}\nu^\alpha\nu^\beta = \langle \nu, \nu \rangle . \tag{2.2}$$

In case N is Lorentzian, $\sigma = -1$, and ν is time-like.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the space-like hypersurface M are connected through the following equations

$$x_{ij}^\alpha = -\sigma h_{ij}\nu^\alpha \tag{2.3}$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.,

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma . \tag{2.4}$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to $-\sigma\nu$.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha , \tag{2.5}$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta \tag{2.6}$$

and the *Gauß equation*

$$R_{ijk\ell} = \sigma\{h_{ik}h_{j\ell} - h_{i\ell}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta}x_i^\alpha x_j^\beta x_k^\gamma x_\ell^\delta. \tag{2.7}$$

Here, the signature of ν comes into play.

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. As we have already pointed out in the introduction, N is then a topological product $\mathbb{R} \times S_0$, where S_0 is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that the metric in N has the form (0.2). We also assume that the coordinate system is *future oriented*, i.e., the time coordinate x^0 increases on future directed curves. Hence, the *contravariant* time-like vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \operatorname{graph} u_{|S_0}$ be a space-like hypersurface

$$M = \{\, (x^0, x) \colon x^0 = u(x),\ x \in S_0 \,\}, \tag{2.8}$$

then the induced metric has the form

$$g_{ij} = e^{2\psi}\{-u_i u_j + \sigma_{ij}\} \tag{2.9}$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$g^{ij} = e^{-2\psi}\left\{\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\right\}, \tag{2.10}$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \sigma^{ij}u_j, \\ v^2 &= 1 - \sigma^{ij}u_i u_j \equiv 1 - |Du|^2. \end{aligned} \tag{2.11}$$

Hence, $\operatorname{graph} u$ is space-like if and only if $|Du| < 1$.

The covariant form of a normal vector of a graph looks like

$$(\nu_\alpha) = \pm v^{-1}e^{\psi}(1, -u_i), \tag{2.12}$$

and the contravariant version is

$$(\nu^\alpha) = \mp v^{-1} e^{-\psi}(1, u^i). \tag{2.13}$$

Thus, we state the following:

Remark 2.1. Let M be space-like graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

$$(\nu^\alpha) = v^{-1} e^{-\psi}(1, u^i), \tag{2.14}$$

and the past directed

$$(\nu^\alpha) = -v^{-1} e^{-\psi}(1, u^i). \tag{2.15}$$

In the Gauß formula (2.3) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal for reasons that will be apparent in a moment.

Look at the component $\alpha = 0$ in (2.3) and obtain in view of (2.15)

$$e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar\Gamma^0_{00} u_i u_j - \bar\Gamma^0_{0i} u_j - \bar\Gamma^0_{0j} u_i - \bar\Gamma^0_{ij}. \tag{2.16}$$

Here, the covariant derivatives is taken with respect to the induced metric of M, and

$$-\bar\Gamma^0_{ij} = e^{-\psi} \bar h_{ij}, \tag{2.17}$$

where $(\bar h_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$\bar h_{ij} e^{-\psi} = -\frac{1}{2}\dot\sigma_{ij} - \dot\psi \sigma_{ij}, \tag{2.18}$$

where the dot indicates differentiation with respect to x^0.

Let us assume for the moment that the Gaussian coordinate system is *normal*, i.e., $\psi \equiv 0$, then

$$\bar h_{ij} = -\frac{1}{2}\dot\sigma_{ij}, \tag{2.19}$$

and the mean curvature of the level hypersurfaces, $\bar H = \sigma^{ij}\bar h_{ij}$, satisfies the equation

$$\dot{\bar H} = \bar R_{\alpha\beta}\nu^\alpha\nu^\beta + \bar h_{ij}\bar h^{ij}, \tag{2.20}$$

as one can easily check. If we assume now that the *time-like convergence* condition holds in N, i.e.,

$$\bar{R}_{\alpha\beta}\xi^\alpha\xi^\beta \ge 0 \tag{2.21}$$

for all time-like (ξ^α), then we deduce that $\bar{H}$ is monotone increasing in time.

Thus, we see that our intuitive understanding, namely, that lower barriers, as defined in Definition 0.1, should lie in the past of upper barriers is generically in accordance with Lorentzian geometry if we evaluate the second fundamental form with respect to the past directed normal.

Definition 2.2. A closed, space-like hypersurface M is said to be *convex* (*strictly convex*) if its second fundamental form evaluated with respect to the past directed normal is *positive* semi-definite (definite).

Remark 2.3. If in a particular setting the second fundamental forms of the barriers involved are negative semi-definite, when evaluated with respect to the past directed normal, then, changing the roles of the future and past directed light cones will establish the preferred situation, where convexity means non-negative principal curvatures.

Next, let us analyze under which condition a space-like hypersurface M can be written as a graph over the Cauchy hypersurface S_0.

We first need the following definition:

Definition 2.4. Let M be a closed, space-like hypersurface in N. Then,

(i) M is said to be *achronal*, if no two points in M can be connected by a future directed time-like curve.

(ii) M is said to *separate* N, if $N\backslash M$ is disconnected.

We can now prove the following result:

Proposition 2.5. *Let N be connected and globally hyperbolic, $S_0 \subset N$ a compact Cauchy hypersurface, and $M \subset N$ a compact, connected space-like hypersurface of class C^m, $m \ge 1$. Then, $M = \operatorname{graph} u_{|S_0}$ with $u \in C^m(S_0)$ iff M is achronal.*

Proof. (i) We first show that an achronal M is a graph over S_0. Let (x^α) be the special coordinate system associated with S_0 such that $S_0 = \{\, p \in N \colon x^0(p) = 0 \,\}$, and let $p \in M$ be arbitrary, $p = (x^0(p), x(p))$. Since M is achronal, the time-like curve $\{\gamma_p\} = \{\, (x^0, x(p)) \colon x^0 \in \mathbb{R} \,\}$ through $(0, x(p)) \in S_0$ intersects M exactly once, and we conclude that $M = \operatorname{graph} u_{|G}$ with $u \in C^0(G)$, where $G \subset S_0$ is closed. But G is also open, and hence $G = S_0$, for otherwise, there would be

$q \in M$ such that $\dot{\gamma}_q \in T_q(M)$, which is impossible since M has a continuous time-like normal.

Furthermore, there exists a neighbourhood $\mathcal{U} = \mathcal{U}(p)$ in N and a function $\Phi \in C^m(\mathcal{U})$ with time-like gradient such that

$$\mathcal{U} \cap M = \{\, (x^0, x) \colon \Phi(x^0, x) = 0 \,\}. \tag{2.22}$$

M is connected with a continuous time-like normal. Thus, we obtain

$$\frac{\partial \Phi}{\partial x^0} = \left\langle D\Phi, \frac{\partial}{\partial x^0} \right\rangle \neq 0, \tag{2.23}$$

and we deduce from the implicit function theorem, that there is a neighbourhood $\mathcal{V}$ of $x(p)$ in S_0 and a possibly smaller neighbourhood $\widetilde{\mathcal{U}}$ of p such that

$$\widetilde{\mathcal{U}} \cap M = \operatorname{graph} \varphi_{|\mathcal{V}}, \quad \varphi \in C^m(\mathcal{V}). \tag{2.24}$$

Hence, $\varphi = u_{|\mathcal{V}}$ and u is of class C^m.

(ii) To demonstrate the reverse implication, we use the fact that M is achronal if M separates N, cf. [13, p. 427], and observe that any graph over S_0 separates N. $\square$

In [13, p. 427] it is also proved that a closed, connected, space-like hypersurface M is achronal if N is simply connected. Hence, we infer the following:

Remark 2.6. Assume that the Cauchy hypersurface S_0 is homeomorphic to S^n, $n \geq 2$, then any closed, connected space-like hypersurface M is a graph over S_0.

One of the assumptions in Theorem 0.2 is that there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$. We shall state sufficient geometric conditions guaranteeing the existence of such a function.

Lemma 2.7. *Let N be globally hyperbolic, S_0 a Cauchy hypersurface, (x^α) a special coordinate system associated with S_0, and $\bar{\Omega} \subset N$ be compact. Then, there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$ provided the level hypersurfaces $\{x^0 = \text{const}\}$ that intersect $\bar{\Omega}$ are strictly convex.*

Proof. For greater clarity set $t = x^0$, i.e., t is a globally defined time function. Let $x = x(\xi)$ be a local representation for $\{t = \text{const}\}$, and t_i, t_{ij} be the covariant derivatives of t with respect to the induced metric, and t_α, $t_{\alpha\beta}$ be the covariant derivatives in N, then

$$0 = t_{ij} = t_{\alpha\beta} x_i^\alpha x_j^\beta + t_\alpha x_{ij}^\alpha, \tag{2.25}$$

and therefore,

$$t_{\alpha\beta}x_i^\alpha x_j^\beta = -t_\alpha x_{ij}^\alpha = -\bar{h}_{ij}t_\alpha \nu^\alpha. \tag{2.26}$$

Here, (ν^α) is past directed, i.e., the right-hand side in (2.26) is positive definite in $\bar{\Omega}$, since (t_α) is also past directed.

Choose $\lambda > 0$ and define $\chi = e^{\lambda t}$, so that

$$\chi_{\alpha\beta} = \lambda^2 e^{\lambda t} t_\alpha t_\beta + \lambda e^{\lambda t} t_{\alpha\beta}. \tag{2.27}$$

Let $p \in \Omega$ be arbitrary, $S = \{t = t(p)\}$ be the level hypersurface through p, and $(\eta^\alpha) \in T_p(N)$. Then, we conclude

$$e^{-\lambda t}\chi_{\alpha\beta}\eta^\alpha\eta^\beta = \lambda^2|\eta^0|^2 + \lambda t_{ij}\eta^i\eta^j + 2\lambda t_{0j}\eta^0\eta^i, \tag{2.28}$$

where t_{ij} now represents the left-hand side in (2.26), and we infer further

$$\begin{aligned} e^{-\lambda t}\chi_{\alpha\beta}\eta^\alpha\eta^\beta &\geq \frac{1}{2}\lambda^2|\eta^0|^2 + [\lambda\varepsilon - c_\varepsilon]\sigma_{ij}\eta^i\eta^j \\ &\geq \frac{\varepsilon}{2}\lambda\{-|\eta^0|^2 + \sigma_{ij}\eta^i\eta^j\} \end{aligned} \tag{2.29}$$

for some $\varepsilon > 0$, and where λ is supposed to be large. Therefore, we have in $\bar{\Omega}$

$$\chi_{\alpha\beta} \geq c\bar{g}_{\alpha\beta} \quad c > 0, \tag{2.30}$$

i.e., χ is strictly convex. $\square$

Sometimes, we need a Riemannian reference metric, e.g., if we want to estimate tensors. Since the Lorentzian metric can be expressed as

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{2\psi}\{-dx^{0^2} + \sigma_{ij}dx^i dx^j\}, \tag{2.31}$$

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

$$\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{2\psi}\{dx^{0^2} + \sigma_{ij}dx^i dx^j\}, \tag{2.32}$$

and we abbreviate the corresponding norm of a vectorfield η by

$$|||\eta||| = (\tilde{g}_{\alpha\beta}\eta^\alpha\eta^\beta)^{1/2}, \tag{2.33}$$

with similar notations for higher order tensors.

3. The evolution problem

Solving the problem (0.1) consists of two steps: first, one has to prove a priori estimates, and secondly, one has to find a procedure which, with the help of the priori estimates, leads to a solution of the problem.

When we first considered the problem for $F \in (K)$ in the Riemannian case, we used an evolutionary approach, which was rather aesthetic but had the shortcoming that for technical reasons the sectional curvatures of the ambient space had to be non-positive, cf. [4]. We were able to overcome this technical obstruction in [6], where we used the method of successive approximation to prove existence. An important ingredient of that proof was the property of the class (K) to be closed under *elliptic regularization*, see [6, Section 1] for details. However, the subclass (K^*) is not closed under elliptic regularization, so that this method of proof fails in the Lorentzian case. But, fortunately, we can apply the evolutionary approach without making any sacrifices with respect to the sectional curvatures of the ambient space, since the unfavourable sign condition that forces us to consider the class (K^*) instead of (K) eliminates that particular technical obstruction.

For greater transparency, we look at the problem in a pseudo-riemannian space N, where, as already stated in Section 2, we, really, only have the Riemannian and the Lorentzian case in mind. Properties like space-like, achronal, etc., however, only make sense, when N is Lorentzian and should be ignored otherwise.

We want to prove that the equation

$$F = f \tag{3.1}$$

has a solution. For technical reasons, it is convenient to solve instead the equivalent equation

$$\Phi(F) = \Phi(f), \tag{3.2}$$

where Φ is a real function defined on $\mathbb{R}_+$ such that

$$\dot{\Phi} > 0 \quad \text{and} \quad \ddot{\Phi} \le 0. \tag{3.3}$$

For notational reasons, let us abbreviate

$$\tilde{f} = \Phi(f). \tag{3.4}$$

We also point out that we may—and shall—assume without loss of generality that F is homogeneous of degree 1.

To solve (3.2) we look at the evolution problem

$$\begin{aligned} \dot{x} &= -\sigma(\Phi - \tilde{f})\nu, \\ x(0) &= x_0, \end{aligned} \tag{3.5}$$

where x_0 is an embedding of an initial strictly convex, compact, space-like hypersurface M_0, $\Phi = \Phi(F)$, and F is evaluated at the principal curvatures of the flow hypersurfaces $M(t)$, or, equivalently, we may assume that F depends on the second fundamental form (h_{ij}) and the metric (g_{ij}) of $M(t)$; $x(t)$ is the embedding of $M(t)$ and σ the signature of the (past directed) normal $\nu = \nu(t)$.

This is a parabolic problem, so short-time existence is guaranteed—the proof in the Lorentzian case is identical to that in the Riemannian case, cf. [4, p. 622]—and under suitable assumptions, we shall be able to prove that the solution exists for all time and converges to a stationary solution if t goes to infinity.

There is a slight ambiguity in the notation, since we also call the evolution parameter *time*, but this lapse shouldn't cause any misunderstandings.

Next, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives. The proofs are identical to those of the corresponding results in a Riemannian setting, cf. [4, Section 3], and will be omitted.

Lemma 3.1 (Evolution of the metric)**.** *The metric g_{ij} of $M(t)$ satisfies the evolution equation*

$$\dot{g}_{ij} = -2\sigma(\Phi - \tilde{f})h_{ij}. \tag{3.6}$$

Lemma 3.2 (Evolution of the normal)**.** *The normal vector evolves according to*

$$\dot{\nu} = \nabla_M(\Phi - \tilde{f}) = g^{ij}(\Phi - \tilde{f})_i x_j. \tag{3.7}$$

Lemma 3.3 (Evolution of the second fundamental form)**.** *The second fundamental form evolves according to*

$$\dot{h}_i^j = (\Phi - \tilde{f})_i^j + \sigma(\Phi - \tilde{f})h_i^k h_k^j + \sigma(\Phi - \tilde{f})\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \tag{3.8}$$

and

$$\dot{h}_{ij} = (\Phi - \tilde{f})_{ij} - \sigma(\Phi - \tilde{f})h_i^k h_{kj} + \sigma(\Phi - \tilde{f})\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{3.9}$$

Lemma 3.4 (Evolution of $(\Phi - \tilde{f})$). *The term $(\Phi - \tilde{f})$ evolves according to the equation*

$$(3.10)\quad (\Phi - \tilde{f})' - \dot{\Phi} F^{ij}(\Phi - \tilde{f})_{ij} = \sigma\dot{\Phi} F^{ij} h_{ik} h_j^k (\Phi - \tilde{f}) + \sigma \tilde{f}_\alpha \nu^\alpha (\Phi - \tilde{f}) + \sigma \dot{\Phi} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta (\Phi - \tilde{f}),$$

where

$$(3.11)\qquad (\Phi - \tilde{f})' = \frac{d}{dt}(\Phi - \tilde{f}),$$

and

$$(3.12)\qquad \dot{\Phi} = \frac{d}{dr}\Phi(r).$$

From (3.8) we deduce with the help of the Ricci identities a parabolic equation for the second fundamental form.

Lemma 3.5. *The mixed tensor h_i^j satisfies the parabolic equation*

$$\begin{aligned}(3.13)\quad \dot{h}_i^j - \dot{\Phi} F^{k\ell} h_{i;k\ell}^j &= \sigma \dot{\Phi} F^{k\ell} h_{rk} h_\ell^r h_i^j - \sigma \dot{\Phi} F h_{ri} h^{rj} + \sigma(\Phi - \tilde{f}) h_i^k h_k^j \\ &\quad - \tilde{f}_{\alpha\beta} x_i^\alpha x_k^\beta g^{kj} + \sigma \tilde{f}_\alpha \nu^\alpha h_i^j + \dot{\Phi} F^{k\ell,rs} h_{k\ell;i} h_{rs;}{}^j \\ &\quad + \ddot{\Phi} F_i F^j + 2\dot{\Phi} F^{k\ell} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_\ell^m g^{rj} \\ &\quad - \dot{\Phi} F^{k\ell} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_r^\gamma x_\ell^\delta h_i^m g^{rj} - \dot{\Phi} F^{k\ell} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_k^\beta x_i^\gamma x_\ell^\delta h^{mj} \\ &\quad + \sigma \dot{\Phi} F^{k\ell} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_k^\beta \nu^\gamma x_\ell^\delta h_i^j - \sigma \dot{\Phi} F \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\ &\quad + \sigma(\Phi - \tilde{f}) \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\ &\quad + \dot{\Phi} F^{k\ell} \bar{R}_{\alpha\beta\gamma\delta;\varepsilon} \{\nu^\alpha x_k^\beta x_\ell^\gamma x_i^\delta x_m^\varepsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_\ell^\varepsilon g^{mj}\}.\end{aligned}$$

The proof is identical to that of the corresponding result in the Riemannian case, cf. [4, Lemma 7.1 and Lemma 7.2]; we only have to keep track of the signature of the normal in the more general pseudo-riemannian setting.

If we had assumed F to be homogeneous of degree d_0 instead of 1, then, we would have to replace the explicit term F—occurring twice in the preceding lemma—by $d_0 F$.

We also point out that the technical differences we encounter, due to the nature of the ambient space—Riemannian or Lorentzian—, stem from the alternating sign of σ in (3.13).

Remark 3.6. In view of the maximum principle, we immediately deduce from (3.10) that the term $(\Phi - \tilde{f})$ has a sign during the evolution if it has one at the beginning, e.g., if the starting hypersurface M_0 is the upper barrier M_2, then $(\Phi - \tilde{f})$ is non-negative, or equivalently,

$$F \geq f. \tag{3.14}$$

4. LOWER ORDER ESTIMATES

From now on, we stick to our original assumption that the ambient space is globally hyperbolic with a compact Cauchy hypersurface S_0. The barriers M_i are then graphs over S_0, $M_i = \operatorname{graph} u_i$, because they are achronal, cf. Proposition 2.5, and we have

$$u_1 \leq u_2, \tag{4.1}$$

for M_1 should lie in the past of M_2, and the enclosed domain is supposed to be connected. Moreover, in view of the Harnack inequality, the strict inequality is valid in (4.1) unless the barriers coincide and are a solution to our problem, cf. the proof of Lemma 4.1.

Let us look at the evolution equation (3.5) with initial hypersurface M_0 equal to M_2. Then, because of the short-time existence, the evolution will exist on a maximal time interval $I = [0, T^*)$, $T^* \leq \infty$, as long as the evolving hypersurfaces are space-like, strictly convex and smooth.

Furthermore, since the initial hypersurface is a graph over S_0, we can write

$$M(t) = \operatorname{graph} u(t)_{|S_0} \quad \text{for all } t \in I, \tag{4.2}$$

where u is defined in the cylinder $Q_{T^*} = I \times S_0$. We then deduce from (3.5), looking at the component $\alpha = 0$, that u satisfies a parabolic equation of the form

$$\dot{u} = -e^{-\psi} v^{-1} (\Phi - \tilde{f}), \tag{4.3}$$

where we use the notations in Section 2, and where we emphasize that the time derivative is a total derivative, i.e.,

$$\dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i. \tag{4.4}$$

Since the past directed normal can be expressed as

$$(\nu^\alpha) = -e^{-\psi} v^{-1} (1, u^i), \tag{4.5}$$

we conclude from (3.5), (4.3), and (4.4)

$$\frac{\partial u}{\partial t} = -e^{-\Psi} v (\Phi - \tilde{f}). \tag{4.6}$$

Thus, $\partial u/\partial t$ is non-positive in view of Remark 3.6.

Next, let us state our first a priori estimate:

Lemma 4.1. *During the evolution the flow hypersurfaces stay in $\bar{\Omega}$.*

Proof. Since $\partial u/\partial t$ is non-positive, we only have to consider the case that the flow reaches the boundary component M_1. Suppose that the flow hypersurfaces would touch M_1 for the first time at time $t = t_0$ in $x_0 \in M_1$, then we deduce, from the equation (2.16) and the maximum principle, that $x_0 \in \Sigma$ and conclude further that, in view of the relation (3.14), the Harnack inequality can be applied to $(u - u_1)$ to yield $M(t_0) = M_1$, and hence, that M_1 is already a solution to our problem; the flow would become stationary for $t \geq t_0$. □

Remark 4.2. It is important to allow non-convex lower barriers, because the *big bang* and *big crunch* hypotheses of the standard cosmological model assert that there are sequences $M_{1,k}$ and $M_{2,k}$ of closed, achronal, space-like hypersurfaces such that, in our setting, $M_{i,k} = \operatorname{graph} u_{i,k|S_0}$, for $i = 1, 2$,

$$\lim_{k\to\infty} \sup_{S_0} u_{1,k} = -\infty, \quad \lim_{k\to\infty} \inf_{S_0} u_{2,k} = \infty, \tag{4.7}$$

and the principal curvatures with respect to the past directed normal of $M_{1,k}$ tend to $-\infty$, while those of $M_{2,k}$ tend to ∞.

Thus, the $M_{2,k}$ could serve as upper barriers for our purposes, but the $M_{1,k}$ would fail to be lower barriers, if we would only consider convex hypersurfaces.

As a consequence of Lemma 4.1 we obtain

$$\inf_{S_0} u_1 \leq u \leq \sup_{S_0} u_2 \quad \text{for all } t \in I. \tag{4.8}$$

We are now able to derive the C^1-estimates, i.e., we shall show that the hypersurfaces remain uniformly space-like, or equivalently, that the term

$$\tilde{v} = v^{-1} = \frac{1}{\sqrt{1 - |Du|^2}} \tag{4.9}$$

is uniformly bounded.

In the Riemannian case, C^1-estimates for closed, convex hypersurfaces can only be derived if they are graphs in a *normal* Gaussian coordinate system, in the Lorentzian case the Gaussian coordinate system no longer needs to be normal, and also, the convexity assumption can be relaxed to a unilateral bound for the second fundamental form.

Lemma 4.3. *Let* $M = \operatorname{graph} u_{|S_0}$ *be a compact, space-like hypersurface represented in a Gaussian coordinate system with unilateral bounded principal curvatures, e.g.,*

$$\kappa_i \ge \kappa_0 \quad \text{for all } i. \tag{4.10}$$

Then, the quantity $\tilde v = 1/\sqrt{1-|Du|^2}$ *can be estimated by*

$$\tilde v \le c(|u|, S_0, \sigma_{ij}, \psi, \kappa_0), \tag{4.11}$$

where we used the notation in (0.2)*, i.e., in the Gaussian coordinate system the ambient metric has the form*

$$d\bar s_N^2 = e^{2\psi}\{-{dx^0}^2 + \sigma_{ij}(x^0,x)dx^i dx^j\}. \tag{4.12}$$

Proof. We suppose that the Gaussian coordinate system is future oriented, and that the second fundamental form is evaluated with respect to the past-directed normal. From formulas (2.10) and (2.11) we get

$$\|Du\|^2 = g^{ij}u_iu_j = e^{-2\psi}\frac{|Du|^2}{v^2}, \tag{4.13}$$

hence, it is equivalent to find an a priori estimate for $\|Du\|$.

Let λ be a real parameter to be specified later, and set

$$w = \frac{1}{2}\log\|Du\|^2 + \lambda u. \tag{4.14}$$

We may regard w as being defined on S_0; thus, there is $x_0 \in S_0$ such that

$$w(x_0) = \sup_{S_0} w, \tag{4.15}$$

and we conclude

$$0 = w_i = \frac{1}{\|Du\|^2}u_{ij}u^j + \lambda u_i \tag{4.16}$$

in x_0, where the covariant derivatives are taken with respect to the induced metric g_{ij}, and the indices are also raised with respect to that metric.

In view of (2.16) we deduce further

$$\begin{aligned}\text{(4.17)}\quad \lambda\|Du\|^4 = -u_{ij}u^iu^j &= e^{-\psi}\tilde{v}h_{ij}u^iu^j + \bar{\Gamma}^0_{00}\|Du\|^4 \\ &\quad + 2\bar{\Gamma}^0_{0j}u^j\|Du\|^2 + \bar{\Gamma}^0_{ij}u^iu^j.\end{aligned}$$

Now, there holds

$$\text{(4.18)}\qquad u^i = g^{ij}u_j = e^{-2\psi}\sigma^{ij}u_jv^{-2},$$

and by assumption,

$$\text{(4.19)}\qquad h_{ij}u^iu^j \geq \kappa_0\|Du\|^2,$$

i.e., the critical terms on the right-hand side of (4.17) are of fourth order in $\|Du\|$ with bounded coefficients, and we conclude that $\|Du\|$ can't be too large in x_0 if we choose λ such that

$$\text{(4.20)}\qquad \lambda \leq -c|||\bar{\Gamma}^0_{\alpha\beta}||| - 1,$$

with a suitable constant c; w, or equivalently, $\|Du\|$ is therefore uniformly bounded from above. $\square$

For convex graphs over S_0 the term $\tilde{v}$ is uniformly bounded as long as they stay in a compact set. Moreover, we shall see, that $\tilde{v}$ satisfies a useful parabolic equation that we shall exploit to estimate the principal curvatures of the hypersurfaces $M(t)$ from above.

Lemma 4.4 (Evolution of $\tilde{v}$). *Consider the flow* (3.5) *in the distinguished coordinate system associated with S_0. Then, $\tilde{v}$ satisfies the evolution equation*

$$\begin{aligned}\text{(4.21)}\quad \dot{\tilde{v}} - \dot{\Phi}F^{ij}\tilde{v}_{ij} = &-\dot{\Phi}F^{ij}h_{ik}h^k_j\tilde{v} + [(\Phi - \tilde{f}) - \dot{\Phi}F]\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &- 2\dot{\Phi}F^{ij}h^k_jx^\alpha_ix^\beta_k\eta_{\alpha\beta} - \dot{\Phi}F^{ij}\eta_{\alpha\beta\gamma}x^\beta_ix^\gamma_j\nu^\alpha \\ &- \dot{\Phi}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_ix^\gamma_kx^\delta_j\eta_\varepsilon x^\varepsilon_\ell g^{k\ell} - \tilde{f}_\beta x^\beta_ix^\alpha_k\eta_\alpha g^{ik},\end{aligned}$$

where η is the covariant vector field $(\eta_\alpha) = e^\psi(-1,0,\dots,0)$.

Proof. We have $\tilde{v} = \langle \eta, \nu \rangle$. Let (ξ^i) be local coordinates for $M(t)$. Differentiating $\tilde{v}$ covariantly we deduce

$$\tilde{v}_i = \eta_{\alpha\beta} x_i^\beta \nu^\alpha + \eta_\alpha \nu_i^\alpha, \tag{4.22}$$

$$\begin{aligned} \tilde{v}_{ij} = {} & \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha + \eta_{\alpha\beta} x_{ij}^\beta \nu^\alpha \\ & + \eta_{\alpha\beta} x_i^\beta \nu_j^\alpha + \eta_{\alpha\beta} x_j^\beta \nu_i^\alpha + \eta_\alpha \nu_{ij}^\alpha. \end{aligned} \tag{4.23}$$

The time derivative of $\tilde{v}$ can be expressed as

$$\begin{aligned} \dot{\tilde{v}} &= \eta_{\alpha\beta} \dot{x}^\beta \nu^\alpha + \eta_\alpha \dot{\nu}^\alpha \\ &= \eta_{\alpha\beta} \nu^\alpha \nu^\beta (\Phi - \tilde{f}) + (\Phi - \tilde{f})^k x_k^\alpha \eta_\alpha \\ &= \eta_{\alpha\beta} \nu^\alpha \nu^\beta (\Phi - \tilde{f}) + \dot{\Phi} F^k x_k^\alpha \eta_\alpha - \tilde{f}_\beta x_i^\beta x_k^\alpha g^{ik} \eta_\alpha, \end{aligned} \tag{4.24}$$

where we have used (3.7).

Substituting (4.23) and (4.24) in (4.21), and simplifying the resulting equation with the help of the Weingarten and Codazzi equations, we arrive at the desired conclusion. □

5. A PRIORI ESTIMATES IN THE C^2-NORM

Let $M(t)$ be a solution of the evolution problem (3.5) with initial hypersurface $M_0 = M_2$, defined on a maximal time interval $I = [0, T^*)$. We assume that F is of class (K^*) according to Definition 1.6, homogeneous of degree 1, and we choose $\Phi(r) = \log r$; alternatively, we could use $\Phi(r) = -(1/m) r^{-m}$, $m \geq 1$, but with the logarithm the proof of the C^2-estimates is a bit simpler. Furthermore, we suppose that there exists a strictly convex function $\chi \in C^2(\bar{\Omega})$, i.e., there holds

$$\chi_{\alpha\beta} \geq c_0 \bar{g}_{\alpha\beta}, \tag{5.1}$$

with a positive constant c_0.

We observe that

$$\begin{aligned} \dot{\chi} - \dot{\Phi} F^{ij} \chi_{ij} &= [(\Phi - \tilde{f}) - \dot{\Phi} F] \chi_\alpha \nu^\alpha - \dot{\Phi} F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta \\ &\leq [(\Phi - \tilde{f}) - \dot{\Phi} F] \chi_\alpha \nu^\alpha - c_0 \dot{\Phi} F^{ij} g_{ij}, \end{aligned} \tag{5.2}$$

where we used the homogeneity of F.

From Remark 3.6 we infer

$$\Phi \geq \tilde{f} \quad \text{or} \quad F \geq f, \tag{5.3}$$

and from the results in Section 4 that the flow stays in the compact set $\bar{\Omega}$.

Furthermore, due to (5.3) and the fact that M_0 is strictly convex, the $M(t)$ remain strictly convex during the evolution; hence, $\tilde{v}$ is uniformly bounded.

We are now able to prove this result:

Lemma 5.1. *Let F be of class (K^*). Then, the principal curvatures of the evolution hypersurfaces $M(t)$ are uniformly bounded.*

Proof. Let φ and w be defined respectively by

$$\varphi = \sup\{\, h_{ij}\eta^i\eta^j : \|\eta\| = 1 \,\}, \tag{5.4}$$

$$w = \log\varphi + \lambda\tilde{v} + \mu\chi, \tag{5.5}$$

where λ, μ are large positive parameters to be specified later. We claim that w is bounded for a suitable choice of λ, μ.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\, \sup_{M(t)} w : 0 < t \le T \,\} = w(x_0). \tag{5.6}$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h^n_n. \tag{5.7}$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$\tilde{\eta} = (0, \dots, 0, 1), \tag{5.8}$$

and set

$$\tilde{\varphi} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}. \tag{5.9}$$

$\tilde{\varphi}$ is well defined in neighbourhood of (t_0, ξ_0).

Now, define $\tilde{w}$ by replacing φ by $\tilde{\varphi}$ in (5.5); then, $\tilde{w}$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$\dot{\tilde{\varphi}} = \dot{h}^n_n, \tag{5.10}$$

and the spatial derivatives do also coincide; in short, at (t_0,ξ_0) $\tilde\varphi$ satisfies the same differential equation (3.13) as h^n_n. For the sake of greater clarity, let us therefore treat h^n_n like a scalar and pretend that w is defined by

$$w=\log h^n_n+\lambda\tilde v+\mu\chi. \tag{5.11}$$

At (t_0,ξ_0) we have $\dot w\ge 0$, and, in view of the maximum principle, we deduce from (1.34), (3.13), (4.21), and (5.2)

$$\begin{aligned}0\le{}&\dot\Phi Fh^n_n-(\Phi-\tilde f)h^n_n+\lambda c_1-\lambda\varepsilon_0\dot\Phi FH\tilde v+\lambda c_1[(\Phi-\tilde f)+\dot\Phi F]\\&+\lambda c_1\dot\Phi F^{ij}g_{ij}+\mu c_1[(\Phi-\tilde f)+\dot\Phi F]-\mu c_0\dot\Phi F^{ij}g_{ij}\\&+\dot\Phi F^{ij}(\log h^n_n)_i(\log h^n_n)_j+\{\ddot\Phi F_nF^n+\dot\Phi F^{k\ell,rs}h_{k\ell;n}h_{rs;}{}^n\}(h^n_n)^{-1},\end{aligned}\tag{5.12}$$

where we have estimated bounded terms by a constant c_1, assumed that h^n_n, λ, and μ are larger than 1, and used (5.3) as well as the simple observation

$$|F^{ij}h^k_j\eta_k|\le\|\eta\|F \tag{5.13}$$

for any vector field (η_k), cf. [4, Lemma 7.4].

Now, the last term in (5.12) is estimated from above by

$$\{\ddot\Phi F_nF^n+\dot\Phi F^{-1}F_nF^n\}(h^n_n)^{-1}-\dot\Phi F^{ij}h_{in;n}h_{jn;}{}^n(h^n_n)^{-2}, \tag{5.14}$$

cf. (1.31), where the sum in the braces vanishes, due to the choice of Φ. Moreover, because of the Codazzi equation, we have

$$h_{in;n}=h_{nn;i}+\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_n x^\gamma_i x^\delta_n, \tag{5.15}$$

and hence, using the abbreviation $\bar R_i$ for the curvature term, we conclude that (5.14) is bounded from above by

$$-(h^n_n)^{-2}\dot\Phi F^{ij}(h^n_{n;i}+\bar R_i)(h^n_{n;j}+\bar R_j). \tag{5.16}$$

Thus, the terms in (5.12) containing the derivatives of h^n_n are estimated from above by

$$-2\dot\Phi F^{ij}(\log h^n_n)_i\bar R_j(h^n_n)^{-1}. \tag{5.17}$$

Moreover, Dw vanishes at ξ_0, i.e.,

$$D\log h^n_n=-\lambda D\tilde v-\mu D\chi, \tag{5.18}$$

where only $D\tilde v$ deserves further consideration.

Replacing then $D\tilde{v}$ by the right-hand side of (4.22), and using the Weingarten equation and (5.13), we finally conclude from (5.12)

$$\begin{aligned}(5.19)\quad 0 \le \dot{\Phi} F h_n^n - (\Phi - \tilde{f}) h_n^n + \lambda c_1 - \lambda \varepsilon_0 \dot{\Phi} F H \tilde{v} + (\lambda + \mu) c_1 [(\Phi - \tilde{f}) + \dot{\Phi} F] \\ + \lambda c_1 \dot{\Phi} F^{ij} g_{ij} - \mu [c_0 - c_1 (h_n^n)^{-1}] \dot{\Phi} F^{ij} g_{ij}.\end{aligned}$$

Then, if we suppose h_n^n to be so large that

$$c_1 \le \frac{1}{2} c_0 h_n^n, \tag{5.20}$$

and if we choose λ, μ such that

$$2 \le \lambda \varepsilon_0 \tag{5.21}$$

and

$$4 \lambda c_1 \le \mu c_0, \tag{5.22}$$

we derive

$$0 \le -\frac{1}{2} \lambda \varepsilon_0 \dot{\Phi} F H \tilde{v} - (\Phi - \tilde{f}) h_n^n + (\lambda + \mu) c_1 [(\Phi - \tilde{f}) + \dot{\Phi} F] + \lambda c_1. \tag{5.23}$$

We now observe that $\dot{\Phi} F = 1$, and deduce in view of (5.3) that h_n^n is a priori bounded at (t_0, ξ_0). ◻

The result of Lemma 5.1 can be restated as a uniform estimate for the functions $u(t) \in C^2(S_0)$. Since, moreover, the principal curvatures of the flow hypersurfaces are not only bounded, but also uniformly bounded away from zero, in view of (5.3) and the assumption that F vanishes on $\partial \Gamma_+$, we conclude that F is uniformly elliptic on $M(t)$.

6. Convergence to a stationary solution

We are now ready to prove Theorem 0.2. Let $M(t)$ be the flow with initial hypersurface $M_0 = M_2$. Let us look at the scalar version of the flow (3.5)

$$\frac{\partial u}{\partial t} = -e^{-\psi} v (\Phi - \tilde{f}). \tag{6.1}$$

This is a scalar parabolic differential equation defined on the cylinder

$$Q_{T^*} = [0, T^*) \times S_0, \tag{6.2}$$

with initial value $u(0) = u_2 \in C^{4,\alpha}(S_0)$. In view of the a priori estimates, which we have established in the preceding sections, we know that

$$|u|_{2,0,S_0} \le c, \tag{6.3}$$

and

$$\Phi(F) \text{ is uniformly elliptic in } u, \tag{6.4}$$

independent of t. Moreover, $\Phi(F)$ is concave, and thus, we can apply the regularity results of [12, Chapter 5.5] to conclude that uniform $C^{2,\alpha}$-estimates are valid, leading further to uniform $C^{4,\alpha}$-estimates due to the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e., $T^* = \infty$.

Now, integrating (6.1) with respect to t, and observing that the right-hand side is non-positive, yields

$$\begin{aligned} u(0,x) - u(t,x) &= \int_0^t e^{-\psi} v(\Phi - \tilde{f}) \\ &\ge c \int_0^t (\Phi - \tilde{f}), \end{aligned} \tag{6.5}$$

i.e.,

$$\int_0^\infty |\Phi - \tilde{f}| < \infty \quad \text{for all } x \in S_0. \tag{6.6}$$

Hence, for any $x \in S_0$ there is a sequence $t_k \to \infty$ such that $(\Phi - \tilde{f}) \to 0$.

On the other hand, $u(\cdot, x)$ is monotone decreasing and therefore

$$\lim_{t\to\infty} u(t,x) = \tilde{u}(x) \tag{6.7}$$

exists and is of class $C^{4,\alpha}(S_0)$ in view of the a priori estimates. We, finally, conclude that $\tilde{u}$ is a stationary solution of our problem, and that

$$\lim_{t\to\infty} (\Phi - \tilde{f}) = 0. \tag{6.8}$$

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