ON THE FOLIATION OF SPACE-TIME BY CONSTANT MEAN CURVATURE HYPERSURFACES

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ABSTRACT. We prove that the mean curvature τ of the slices given by a constant mean curvature foliation can be used as a time function, i.e. τ is smooth with non-vanishing gradient.

CONTENTS

0. INTRODUCTION

In [2] it is proved that a globally hyperbolic Lorentzian manifold N with a compact Cauchy hypersurface can be foliated by constant mean curvature hypersurfaces if the *big bang* and *big crunch* hypotheses are valid, and if the *time-like convergence* condition holds, i.e. if

$$(0.1) \qquad \bar{R}_{\alpha\beta}\eta^{\alpha}\eta^{\beta} \geq 0$$

for all time-like vectorfields (η^{α}).

If we assume for simplicity that the mean curvature of the barriers provided by the big bang resp. big crunch hypotheses tend to $-\infty$ resp. $+\infty$, the foliation can be described as follows: Indicate by M_{τ} a closed hypersurface of mean curvature τ, then, the foliation consists of the uniquely determined M_{τ}, $0 \neq \tau \in \mathbb{R}$, and of the set $\mathcal{C}_0$ of *maximal* slices. If there is more than one maximal slice, then $\mathcal{C}_0$ comprises a whole continuum of maximal slices, which are all totally geodesic and the ambient metric is static in $\mathcal{C}_0$.

The mean curvature of the slices of the foliation can be looked at as a function τ on N, which is continuous as one easily checks. However, in order to use τ as a new time function, τ has to be smooth with non-vanishing gradient.

We prove that this is indeed the case in $\{\tau \neq 0\}$ and also globally, if there is a maximal slice that is not totally geodesic or if the strict inequality is valid in (0.1). Evidently, if we have two (and more) maximal slices, then, $D\tau$ vanishes in the interior of $\mathcal{C}_0$, and we shall also give an example of a foliation with exactly one totally geodesic slice, where $D\tau$ vanishes on that slice, i.e. the assumptions guaranteeing τ to be smooth and $D\tau \neq 0$ are also sharp.

The paper is organized as follows: In Section 1 we introduce the notations and common definitions we rely on.

The main theorem is proved in Section 2, while the counterexample is given in Section 3.

1. Notations and definitions

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces M in a (n+1)-dimensional Lorentzian manifold N. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijk)}$, etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function u in N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$(1.1) \qquad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a *space-like* hypersurface, i.e. the induced metric is Riemannian, with a differentiable normal ν which is time-like.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the space-like hypersurface M are connected through the following equations

$$(1.2) \qquad x_{ij}^\alpha = h_{ij}\nu^\alpha$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$(1.3) \qquad x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma.$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to ν.

The second equation is the *Weingarten equation*

$$(1.4) \qquad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$(1.5) \qquad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauß equation*

$$(1.6) \qquad R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta}x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. N is then a topological product $\mathbb{R} \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that the metric in N has the form

$$(1.7) \qquad d\bar{s}_N^2 = e^{2\psi}\{-{dx^0}^2 + \sigma_{ij}(x^0, x)dx^i dx^j\},$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the space-like components (x^i), see [4], [6, p. 212], [5, p. 252], and [2, Section 6]. We also assume that the coordinate system is *future oriented*, i.e. the time coordinate x^0 increases on future directed curves. Hence, the *contravariant*

time-like vector $(\xi^\alpha) = (1, 0, \ldots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \ldots, 0)$.

Let $M = \operatorname{graph} u_{|S_0}$ be a space-like hypersurface

$$(1.8) \qquad M = \{\, (x^0, x) \colon x^0 = u(x),\ x \in \mathcal{S}_0 \,\},$$

then the induced metric has the form

$$(1.9) \qquad g_{ij} = e^{2\psi}\{-u_i u_j + \sigma_{ij}\}$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$(1.10) \qquad g^{ij} = e^{-2\psi}\{\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\},$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$(1.11) \qquad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2. \end{aligned}$$

Hence, $\operatorname{graph} u$ is space-like if and only if $|Du| < 1$.

The covariant form of a normal vector of a graph looks like

$$(1.12) \qquad (\nu_\alpha) = \pm v^{-1} e^{\psi}(1, -u_i).$$

and the contravariant version is

$$(1.13) \qquad (\nu^\alpha) = \mp v^{-1} e^{-\psi}(1, u^i).$$

Thus, we have

Remark 1.1. Let M be space-like graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

$$(1.14) \qquad (\nu^\alpha) = v^{-1} e^{-\psi}(1, u^i)$$

and the past directed

$$(1.15) \qquad (\nu^\alpha) = -v^{-1} e^{-\psi}(1, u^i).$$

In the Gauß formula (1.2) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal for reasons that we have explained in [3, Section 2].

Look at the component $\alpha = 0$ in (1.2) and obtain in view of (1.15)

$$(1.16) \qquad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{ij}.$$

Here, the covariant derivatives are taken with respect to the induced metric of M, and

$$(1.17) \qquad -\bar{\Gamma}^0_{ij} = e^{-\psi}\bar{h}_{ij},$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$(1.18) \qquad \bar{h}_{ij} e^{-\psi} = -\tfrac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij},$$

where the dot indicates differentiation with respect to x^0.

2. τ IS A TIME FUNCTION

Let $M_0 = M_{\tau_0}$ be a hypersurface of constant mean curvature τ_0, and let (x^α) be a future oriented, normal Gaussian coordinate system relative to M_0, i.e.

$$(2.1) \qquad d\bar{s}_N^2 = -dx^{0^2} + \sigma_{ij}(x^0, x)dx^i dx^j$$

in a tubular neighbourhood $\mathcal{U}$ of M_0; here, x is an abbreviation for the spatial components (x^i).

Since we have a continuous foliation of N, the constant mean curvature slices contained in $\mathcal{U}$ can be written as graphs over M_0

$$(2.2) \qquad M_\tau = \{ (x^0, x^i) \colon x^0 = u(\tau, x^i) \}$$

with a continuous function $u \in C^0(I \times M_0)$, I an open interval containing τ_0.

Let us assume for the moment that u is smooth—we shall see later that this is indeed the case, at least for $\tau_0 \neq 0$—, and let us define the transformation

$$(2.3) \qquad \Phi(\tau, x^i) = (u(\tau, x^i), x^i).$$

Then, we have

$$(2.4) \qquad \det D\Phi = \frac{\partial u}{\partial \tau} \equiv \dot{u}.$$

In view of the monotonicity of the foliation, cf. [2, Lemma 7.2], we know that $\dot{u} \geq 0$. Thus, if we could show that u is smooth and $\dot{u}$ strictly positive, we would obtain that Φ is a diffeomorphism, and hence, that τ is smooth with non-vanishing gradient.

Let us first show that u is smooth.

Lemma 2.1. *Let $\tau_0 \in \mathbb{R}$ be such that M_{τ_0} is not totally geodesic or assume that the strict inequality is valid in (0.1). Consider a tubular neighbourhood $\mathcal{U}$ of $M_0 = M_{\tau_0}$, and choose normal Gaussian coordinates as above, such that the $M_\tau \subset \mathcal{U}$ are graphs*

$$(2.5) \qquad M_\tau = \operatorname{graph} u(\tau)_{|_{M_0}}.$$

Then, u is smooth in $I \times M_0$, where $I = (\tau_0 - \epsilon, \tau_0 + \epsilon)$ for small $\epsilon = \epsilon(\tau_0) > 0$.

Proof. Let $u_0 = u(\tau_0)$, which is incidentally identical zero due to our choice of coordinates, and define the operator G

$$(2.6) \qquad G(\tau, \varphi) = H(\varphi) - \tau,$$

where $H(\varphi)$ is an abbreviation for the mean curvature of $\operatorname{graph} \varphi_{|_{M_0}}$.

It is well-known, cf. [1, Section 5], that $D_2 G(\tau_0, u_0)\varphi$ equals

$$(2.7) \qquad -\Delta\varphi + \varphi\{\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta\},$$

where $\|A\|^2 = h_{ij}h^{ij}$, and the Laplacian, the second fundamental form and ν are evaluated with respect to M_0.

Hence, $D_2 G$ is invertible at (τ_0, u_0) in view of our assumptions, and thus, the *implicit function theorem* is applicable to yield the existence of a smooth function $u = u(\tau, x)$ satisfying

$$(2.8) \qquad G(\tau, u(\tau, x)) = 0$$

for $|\tau - \tau_0| < \epsilon$, ϵ small; of course, the new smooth function coincides with the original continuous function. $\qquad\square$

It remains to demonstrate that $\dot{u}$ is strictly positive.

We use the same setting as above, i.e. we work in a tubular neighbourhood $\mathcal{U}$ of a slice M_{τ_0} with corresponding normal Gaussian coordinates, and look at the $M_\tau \subset \mathcal{U}$, which are then graphs over M_{τ_0}.

Lemma 2.2. *Let $M_\tau = \operatorname{graph} u(\tau)$ and $M_{\bar{\tau}} = \operatorname{graph} u(\bar{\tau})$, then, there exists a constant $c = c(\mathcal{U})$ such that*

$$(2.9) \qquad |\tau - \bar{\tau}| \le c \inf_{M_{\tau_0}} |u - \bar{u}|,$$

where $\bar{u} = u(\bar{\tau})$.

Proof. Assume for simplicity $\bar{\tau} < \tau$. We want to employ the relation (1.16)—now with $\psi \equiv 0$—, but the covariant derivatives of u should be expressed with respect to the metric $(\sigma_{ij}(u, x))$; recall that

$$(2.10) \qquad g_{ij} = -u_i u_j + \sigma(u, x).$$

An easy calculation reveals that

$$(2.11) \qquad u_{ij} = v^{-2} u_{;ij},$$

where the semicolon indicates the covariant derivatives with respect to the metric $(\sigma_{ij}(u, x))$. Let $\tilde{\Gamma}_{ij}^k$ be the corresponding Christoffel symbols and indicate by a comma ordinary partial derivatives, then, we obtain

$$(2.12) \qquad h_{ij} = -v^{-1} u_{,ij} + v^{-1} \tilde{\Gamma}_{ij}^k u_k - v \bar{\Gamma}_{ij}^0$$

with an analogous relation valid for $\bar{u}$; note that the other Christoffel symbols in (1.16) vanish, since $\psi \equiv 0$.

Let $x_0 \in M_{\tau_0}$ be such that

$$(2.13) \qquad (u - \bar{u})(x_0) = \inf_{M_{\tau_0}} (u - \bar{u}).$$

We now observe that in x_0

$$(2.14) \qquad Du = D\bar{u} \quad \text{and} \quad D^2 u \ge D^2 \bar{u},$$

and deduce from (2.12), and the corresponding relation for $M_{\bar{\tau}}$,

$$(2.15) \qquad \tau - \bar{\tau} \le f(x_0, u(x_0)) - f(x_0, \bar{u}(x_0))$$

with a smooth function f, i.e.

$$(2.16) \qquad \tau - \bar{\tau} \le c[u(x_0) - \bar{u}(x_0)].$$

$\square$

Remark 2.3. An estimate of the form (2.9) is valid in *any* Gaussian coordinate system in which the slices of the foliation are represented as graphs. The constant c depends on the compact set containing the slices under consideration.

Thus, we have proved

Theorem 2.4. *The function τ is smooth with non-vanishing gradient in $\{\tau \neq 0\}$. If there is a maximal slice that is not totally geodesic or if the strict inequality is valid in (0.1), then τ is a globally defined time function.*

We note that by assumption there are maximal slices in the foliation.

3. A COUNTEREXAMPLE

Let S^n be the standard unit sphere with metric σ_{ij}, and for $t \in I = (-\epsilon, \epsilon)$ let

$$(3.1) \qquad f = f(t) = -\int_0^t \frac{s^3}{\epsilon^2 - s^2} \, ds.$$

Then, we define $N = I \times S^n$ with metric

$$(3.2) \qquad d\bar{s}_N^2 = -dt^2 + e^{2f} \sigma_{ij} dx^i dx^j,$$

where the coordinates (t, x^i) are supposed to be future oriented.

The level hypersurfaces $\{t = \text{const}\}$ are all totally umbilical, and their mean curvature is equal to

$$(3.3) \qquad \tau = H = -n\dot{f} = n\frac{t^3}{\epsilon^2 - t^2}.$$

Thus, we see that the big bang and big crunch hypotheses are satisfied and that $D\tau$ vanishes on the unique totally geodesic maximal slice.

It remains to verify the time-like convergence condition.

We first note, that for $\epsilon \leq 1$

$$(3.4) \qquad \ddot{f} + \dot{f}^2 = -\frac{3t^2}{\epsilon^2 - t^2} - \frac{2t^4}{(\epsilon^2 - t^2)^2} + \frac{t^6}{(\epsilon^2 - t^2)^2} \leq 0,$$

which will be the core inequality to estimate the Ricci tensor.

Secondly, let us write the metric as a conformal metric

$$(3.5) \qquad \bar{g}_{\alpha\beta} = e^{2\psi} g_{\alpha\beta}$$

in coordinates (x^0, x^i), where

$$(3.6) \qquad g_{\alpha\beta} dx^\alpha dx^\beta = -dx^{0^2} + \sigma_{ij} dx^i dx^j,$$

$$(3.7) \qquad \frac{dx^0}{dt} = e^{-f(t)},$$

and $\psi(x^0) = f(t)$.

The metric $(g_{\alpha\beta})$ is a product metric and, therefore, the only non-zero components of its Ricci tensor $(R_{\alpha\beta})$ are of the form R_{ij} and coincide with the components of the Ricci tensor of S^n.

The Ricci tensor $(\bar{R}_{\alpha\beta})$ of the metric $(\bar{g}_{\alpha\beta})$ is connected to $(R_{\alpha\beta})$ through the relation

$$(3.8) \qquad \begin{aligned} \bar{R}_{\alpha\beta} = R_{\alpha\beta} &- (n-1)[\psi_{\alpha\beta} - \psi_\alpha \psi_\beta] \\ &- g_{\alpha\beta}[\Delta\psi + (n-1)|D\psi|^2], \end{aligned}$$

where covariant derivatives and the norm are calculated with respect to $(g_{\alpha\beta})$; note that $|D\psi|^2$ can be negative.

Hence, we conclude

$$(3.9) \qquad \begin{aligned} \bar{R}_{00} &= -n\ddot{\psi} = -ne^{2f}[\ddot{f} + \dot{f}^2], \\ \bar{R}_{0i} &= 0, \end{aligned}$$

and

$$(3.10) \qquad \begin{aligned} \bar{R}_{ij} &= R_{ij} + \sigma_{ij}[\ddot{\psi} + (n-1)\dot{\psi}^2], \\ &= R_{ij} + \sigma_{ij}[\ddot{f} + n\dot{f}^2]e^{2f}. \end{aligned}$$

Let (η^α) be a time-like vector with component $\eta^0 = 1$, so that

$$(3.11) \qquad \sigma_{ij}\eta^i \eta^j < 1.$$

Then, taking into account that $\ddot{f} \leq 0$, we derive

$$
\begin{aligned}
\bar{R}_{\alpha\beta}\eta^{\alpha}\eta^{\beta} = {} & -ne^{2f}[\ddot{f} + \dot{f}^2] + R_{ij}\eta^i\eta^j \\
& + \sigma_{ij}\eta^i\eta^j[\ddot{f} + n\dot{f}^2]e^{2f} \\
\geq {} & ne^{2f}[\ddot{f} + \dot{f}^2][-1 + \sigma_{ij}\eta^i\eta^j],
\end{aligned}
$$

(3.12)

and we conclude that the time-like convergence condition is satisfied in view of (3.4).

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