Übungen zur Funktionalanalysis

Blatt 4

1 Prove that the direct sum $\bigoplus_{i\in\mathbb{N}} H_i$ of a sequence of Hilbert spaces is again a Hilbert space. $\boxed{2}$

2 Let H be a scalar product space and $0 \neq x \in H$, then

(i) $\langle x, y\rangle = \|x\|\,\|y\| \iff y = \lambda x,\ \lambda \geq 0.$ $\boxed{2}$

(ii) $\|x+y\| = \|x\| + \|y\| \iff y = \lambda x,\ \lambda \geq 0.$ $\boxed{2}$

(iii) $\|\cdot\|^2$ ist *strictly convex*, i.e.,

$$\|tx + (1-t)y\|^2 < t\|x\|^2 + (1-t)\|y\|^2 \qquad \forall\, x \neq y,\ \forall\, 0 < t < 1.$$

$\boxed{2}$

3 Let H be a Hilbert space, then

$$x_n \rightharpoonup x_0 \quad \wedge \quad \|x_n\| \to \|x_0\| \quad \Longrightarrow \quad x_n \to x_0.$$

$\boxed{8}$

4 Let H be Hilbert space and (x_n) a weakly convergent sequence, $x_n \rightharpoonup x_0$. Then there exists a subsequence (x_{n_k}), such that

$$y_k = \frac{1}{k}\sum_{i=1}^{k} x_{n_i} \to x_0.$$

$\boxed{6}$