Übungen zur Funktionalanalysis

Blatt 3

1 Show that the real logarithm is a concave function, and then use this fact to prove the so-called *Young's inequality*

$$xy \le \frac{1}{p}x^p + \frac{1}{p'}y^{p'} \qquad \forall\, x, y \in \mathbb{R}^+,$$

where, p, p', are so-called *conjugate exponents*, i.e.,

$$p, p' \in (1, \infty) \quad \text{such that} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

$\boxed{6}$

2 Use Young's inequality to prove

(i) For $p \in [1, \infty)$, define the so-called p-norm on $\mathbb{R}^n$

$$\|x\|_p = \Big(\sum_{i=1}^n |x^i|^p\Big)^{\frac{1}{p}} \qquad \forall\, x = (x^i) \in \mathbb{R}^n;$$

then, for $p \in (1, \infty)$, the so-called *Hölder's inequality*

$$|\langle x, y\rangle| \le \|x\|_p \, \|y\|_{p'} \qquad \forall\, x, y \in \mathbb{R}^n$$

is valid, where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. $\boxed{2}$

(ii) $\|x + y\|_p \le \|x\|_p + \|y\|_p \qquad \forall\, x, y \in \mathbb{R}^n$. $\boxed{4}$

(iii) Set $\|x\|_\infty = \max_i |x^i|$, then the inequalities in (i) and (ii) are also valid for the exponents $p = 1$ and $p' = \infty$. $\boxed{2}$

3 Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable subset, then the preceding results are also valid in the corresponding L^p-spaces:

(i)

$$\Big|\int_\Omega fg\Big| \le \|f\|_p \|g\|_{p'} \qquad \forall\, f \in L^p(\Omega),\ g \in L^{p'}(\Omega).$$

$\boxed{2}$

(ii)

$$\|f + g\|_p \le \|f\|_p + \|g\|_p \qquad \forall\, f, g \in L^p(\Omega).$$

$\boxed{2}$

(iii) The estimates in (i) and (ii) are also valid for $p = 1$ and $p' = \infty$, or vice versa.

$\boxed{2}$