Notice that we only consider hypersurfaces where the induced metric is really a non-singular metric.

Since M has codimension 1, its normal space $N_x(M)$ is spanned by a single vector ν in each point $x \in M$. In the present situation we can always define a continuous normal vector field $\nu \in C^{m-1}(\Omega, T^{1,0}(N))$ by requiring that

$$\det(x_1, \ldots, x_n, \nu) > 0 \qquad \forall \xi \in \Omega, \tag{12.2.3}$$

cf. exercise 1 of Exercises 12.2.13; notice that we assume to work in an arbitrary but fixed coordinate system (x^α) of N.

Unless otherwise stated a normal vector ν is always supposed to be normalized to

$$\langle \nu, \nu \rangle = \sigma = \pm 1. \tag{12.2.4}$$

σ is called the *signature* of ν.

In each point $x \in M$ the tangent vectors x_i span $T_x(M) \subset T_x(N)$, and

$$\langle x_i, \nu \rangle = 0 \qquad \forall 1 \le i \le n. \tag{12.2.5}$$

Geometric quantities in the *ambient space* N are denoted by $\bar{g}_{\alpha\beta}$, $\bar{R}_{\alpha\beta\gamma\delta}$, etc., and those in M by g_{ij}, R_{ijkl}, etc. Greek indices range from 0 to n and Latin from 1 to n.

12.2.1. Remark. The tangent vectors $x_i = (x_i^\alpha)$ of M are full tensors and have to be regarded as tensor fields over (Ω, x), cf. (12.1.6).

We also recall that covariant derivatives are always supposed to be full tensors.

There are four basic equations governing the geometry of a hypersurface. The first is

12.2.2. Theorem (Gaussian formula). *Let $x = x(\xi)$ be the local representation of a hypersurface $M \subset N$ of class C^m, $2 \le m \le \infty$, where (x^α) are local coordinates in N. Then the second covariant derivatives of x, x_{ij}, with respect to the induced metric (g_{ij}), are a symmetric tensor with values in the normal space of M, $N(M)$, i.e., if $\nu = (\nu^\alpha)$ is a continuous normal vector field of M with signature σ, then we can express x_{ij} in the form*

$$x_{ij} = -\sigma h_{ij} \nu, \tag{12.2.6}$$

with a symmetric tensor $(h_{ij}) \in T^{0,2}(M)$, which is called the second fundamental form of M with respect to $-\sigma\nu$.

The preceding equation is called the Gaussian formula. Notice that (h_{ij}) changes sign, if we choose the opposite normal vector.

Proof. Differentiating $g_{ij} = \langle x_i, x_j \rangle$ covariantly with respect to ξ^k, we obtain

$$0 = D_k \langle x_i, x_j \rangle = \langle x_{ik}, x_j \rangle + \langle x_i, x_{jk} \rangle. \tag{12.2.7}$$

Now x_{ik} can be expressed in the form

$$x_{ik} = -\sigma h_{ik}\nu + a^m{}_{ik} x_m \tag{12.2.8}$$

with symmetric tensors in the indices (i,k) on the right-hand side.

Multiplying this equation with x_j we deduce in view of (12.2.7)

$$\langle x_{ik}, x_j\rangle = a^m{}_{ik} g_{mj} = -\langle x_{jk}, x_i\rangle = -a^m{}_{jk} g_{mi}, \tag{12.2.9}$$

from which we infer that the tensor

$$a_{jik} = g_{mj} a^m{}_{ik} \tag{12.2.10}$$

is antisymmetric in the first two indices and symmetric in the last two, i.e.,

$$\begin{aligned} a_{jik} &= -a_{ijk} = -a_{ikj} = a_{kij} \\ &= a_{kji} = -a_{jki} = -a_{jik}, \end{aligned} \tag{12.2.11}$$

and hence $a_{ijk} = 0$, and the Gaussian formula is proved. □

12.2.3. Remark. (i) Let $(h_{ij}) \in T^{0,2}(M)$ be symmetric. $\eta = (\eta^i) \in T^{1,0}(M)$ is said to be an *eigenvector* of (h_{ij}) with respect to the metric (g_{ij}), if here exists $\lambda \in \mathbb{R}$ such that

$$\lambda g_{ij}\eta^j \equiv \lambda \eta_i = h_{ij}\eta^j; \tag{12.2.12}$$

λ is then called an *eigenvalue* of (h_{ij}) with respect to the metric (g_{ij}).

(ii) λ is eigenvalue of (h_{ij}) with respect to (g_{ij}) if and only if

$$\det(h_{ij} - \lambda g_{ij}) = 0. \tag{12.2.13}$$

(iii) Let (h^i_j) be the mixed representation of (h_{ij}), then the eigenvectors and eigenvalues of (h^i_j) can be defined without any reference to g_{ij} by requiring

$$\lambda \eta^i = h^i_j \eta^j. \tag{12.2.14}$$

λ satisfies the preceding relation if and only if

$$\det(h^i_j - \lambda \delta^i_j) = 0. \tag{12.2.15}$$

(iv) The eigenvalues of the second fundamental form of a hypersurface M with respect to the induced metric, which is also referred to as the *first fundamental form*, are called *principal curvatures* of M. We usually denote the principal curvatures of a hypersurface M by κ_i, $1 \le i \le n$.

(v) In the choice of the normal ν in the Gaussian formula we are completely free, i.e., we could just as well have replaced ν by $-\nu$, then the principal curvatures κ_i would have been replaced by $-\kappa_i$.

However, if the ambient space is Riemannian, then we stipulate to always choose the *outer normal* to M, if such a choice is possible. This will be the case if M is compact and $\complement M$ consists of exactly two components one of which is bounded and the other unbounded. The outer normal is then the normal which points to the unbounded component.

If the ambient space is Lorentzian and M spacelike, then we always choose ν to be *past directed*, and in this case the coordinate system (x^α) is supposed to be *future directed*, where x^0 represents the time function, i.e., x^0 should increase on future directed curves, or equivalently,

$$dx^0 \in C_-, \tag{12.2.16}$$

for let $\gamma = (\gamma^\alpha(t))$ be a future directed curve, i.e., $\dot\gamma \in C_+$, and let $\varphi(t) = f \circ \gamma(t)$, where $f = x^0$, then

$$\dot\varphi = f_\alpha \dot\gamma^\alpha = \langle Df, \dot\gamma\rangle \ge 0, \tag{12.2.17}$$

which is only possible, if Df is past directed.

If N is Riemannian, then we shall always choose a coordinate system (x^α) such that

$$\langle \frac{\partial}{\partial x^0}, \nu\rangle > 0. \tag{12.2.18}$$

Notice that with the preceding stipulations the relation

$$\langle \frac{\partial}{\partial x^0}, \nu\rangle > 0 \tag{12.2.19}$$

will be valid in Riemannian spaces N as well as in Lorentzian manifolds N, if, in the latter case, spacelike hypersurfaces will be considered, since $\langle dx^0, \frac{\partial}{\partial x^0}\rangle = 1$, i.e., $\frac{\partial}{\partial x^0}$ is future directed.

12.2.4. Theorem (Weingarten equation)**.** *For a hypersurface M of N there holds*

$$\nu_i = h_i^k x_k, \tag{12.2.20}$$

where the covariant derivative is a full tensor, and the index of the second fundamental form is raised with respect to the induced metric. This equation is known as the Weingarten equation.

Proof. Differentiate the equation

$$0 = \langle x_k, \nu\rangle \tag{12.2.21}$$

covariantly with respect to ξ^i to get

$$0 = \langle x_{ki}, \nu\rangle + \langle x_k, \nu_i\rangle, \tag{12.2.22}$$

hence

$$\langle x_k, \nu_i\rangle = h_{ik}, \tag{12.2.23}$$

since $\sigma^2 = 1$.

On the other hand, because of $\sigma = \langle \nu, \nu\rangle$, we deduce

$$0 = \langle \nu_i, \nu\rangle, \tag{12.2.24}$$

i.e., for fixed i, ν_i is a tangential vector

$$\nu_i = a_i^m x_m. \tag{12.2.25}$$

Multiplying this equation with x_k we infer

$$\langle \nu_i, x_k\rangle = a_i^m g_{mk} = h_{ik}, \tag{12.2.26}$$

12.2.13. Exercises.

1 Let $x \in C^m(\Omega, N)$ be the embedding of a hypersurface M such that Ω is connected and M covered by a coordinate system (x^α). Then there is a unique normal unit vector field $\nu \in C^{m-1}(\Omega), T^{1,0}(N))$ satisfying (12.2.3).

2 Let M be a hypersurface in N of class C^m, $m \ge 1$, and assume that (Ω, x) is a local embedding of M, $\Omega \subset \mathbb{R}^n$. Let $x_0 \in M$, then any tensor field A over (Ω, x) of class C^q, $0 \le q \le m-1$, and of order (k,l) in M and of order (r,s) in N can be extended as a C^q-tensor field of order $(r+k, s+l)$ in a small neighbourhood $V = V(x_0) \subset N$, i.e., $A \in C^q(V, T^{r+k,s+l}(N))$, where we used the same notation for A and its extension.

3 Prove that the Einstein tensor of a semi-Riemannian manifold N is divergence free, i.e.,

(12.2.59) $$G^\alpha_{\beta;\alpha} = 0.$$

4 Prove Proposition 12.2.11.

12.3. Submanifolds of higher codimension

In this section $(N, \bar g)$ is a semi-Riemannian manifold of dimension $(n+m)$, $1 \le m$, $M \subset N$ a semi-Riemannian submanifold of dimension n. Generic coordinates in N are denoted by (x^α), $1 \le \alpha \le n+m$, and those in M by (ξ^i), $1 \le i \le n$.

We are mainly interested in the case $m \ge 2$, though of course the subsequent results are also valid for $m = 1$, but are then already known.

Let us start with the higher codimensional version of the Gaussian formula, Theorem 12.2.2 on page 334.

12.3.1. Theorem. *Let M be a semi-Riemannian submanifold of N and $x = x(\xi)$, $\xi \in \Omega \subset \mathbb{R}^n$, a local embedding of M, then x_{ij} is symmetric in the indices (i,j) and, for fixed (i,j), a normal vector, i.e., for $p \in M$ we have*

(12.3.1) $$x^\alpha_{ij}(p) \in T^{0,2}_p(M) \otimes N_p(M),$$

where $N_p(M) \subset T_p(N)$ is the m-dimensional normal space of M in p,

(12.3.2) $$T_p(N) = T_p(M) \oplus N_p(M).$$

Proof. The proof is identical to the corresponding proof in the codimension 1 case. Differentiate $g_{ij} = \langle x_i, x_j \rangle$ covariantly with respect to ξ^k, we obtain

(12.3.3) $$0 = D_k \langle x_i, x_j \rangle = \langle x_{ik}, x_j \rangle + \langle x_i, x_{jk} \rangle.$$

Now, for fixed $p \in M$, x_{ik} can be expressed in the form

(12.3.4) $$x_{ik} = y_{ik} + a^m{}_{ik} x_m, \qquad y^\alpha_{ik}(p) \in T^{0,2}_p(M) \otimes N_p(M),$$

with symmetric tensors in the indices (i,k) on the right-hand side of the equation.