10.1.13. Theorem. *Let $(E, \mathcal{M}, \mu)$ be a measure space. Then the following relations are valid*

(i) *Let A_i, $1 \le i \le n$, be finitely many pairwise disjoint measurable sets, then*

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i). \tag{10.1.43}$$

(ii) *Let $A, B \in \mathcal{M}$, then*

$$A \subset B \implies \mu(A) \le \mu(B). \tag{10.1.44}$$

(iii) *Let $(A_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence of measurable sets, then*

$$A = \bigcup_{n\in\mathbb{N}} A_n \implies \mu(A) = \lim \mu(A_n). \tag{10.1.45}$$

(iv) *Let $(A_n)_{n\in\mathbb{N}}$ be a monotone decreasing sequence of measurable sets, then*

$$A = \bigcap_{n\in\mathbb{N}} A_n \ \wedge\ \mu(A_0) < \infty \implies \mu(A) = \lim \mu(A_n). \tag{10.1.46}$$

(v) *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets, then*

$$A = \bigcup_{n\in\mathbb{N}} A_n \implies \mu(A) \le \sum_{n\in\mathbb{N}} \mu(A_n). \tag{10.1.47}$$

Proof. „(i)“ Set $A_i = \emptyset$ for $i > n$, and then apply (10.1.39) and (10.1.40).

„(ii)“ Decompose B in the form

$$B = A \,\dot\cup\, (B \backslash A) \tag{10.1.48}$$

and deduce from (10.1.43)

$$\mu(B) = \mu(A) + \mu(B\backslash A) \ge \mu(A). \tag{10.1.49}$$

„(iii)“ Define $B_0 = A_0$ and

$$B_n = A_n \backslash \bigcup_{i=0}^{n-1} B_i, \quad n \ge 1, \tag{10.1.50}$$

then the B_n are pairwise disjoint and there holds

$$A = \bigcup_{n\in\mathbb{N}} A_n = \dot{\bigcup_{n\in\mathbb{N}}} B_n, \tag{10.1.51}$$

as well as

$$A_n = \bigcup_{i=0}^{n} B_i, \tag{10.1.52}$$

(iii) *μ is a measure in $\mathcal{M}$.*

Proof. (1) First we observe, that $\emptyset \in \mathcal{M}$ and $\complement A \in \mathcal{M}$, if $A \in \mathcal{M}$.

(2) Let $M \subset E$ be arbitrary and A, B measurable, then

$$\begin{aligned} \mu(M) &= \mu(M \cap A) + \mu(M \backslash A) \\ &= \mu(M \cap A) + \mu((M \backslash A) \cap B) + \mu((M \backslash A) \backslash B) \\ &\geq \mu(M \cap (A \cup B)) + \mu(M \backslash (A \cup B)), \end{aligned} \tag{10.1.67}$$

hence $A \cup B$ are measurable, in view of Remark 10.1.17.

(3) By induction we conclude that finite unions and intersections of measurable sets are measurable.

(4) If $(A_i)_{i \in \mathbb{N}}$ are disjoint μ measurable sets, then

$$\mu(\dot{\bigcup_{i \in \mathbb{N}}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i). \tag{10.1.68}$$

Since μ is an outer measure, we have

$$\mu(\bigcup_{i=1}^{n} A_i) \leq \mu(\bigcup_{i \in \mathbb{N}} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i). \tag{10.1.69}$$

Hence, (10.1.68) will be proved, if we can show that

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i). \tag{10.1.70}$$

Now, let $B_n = \bigcup_{i=1}^{n} A_i$, then

$$B_{n+1} \cap A_{n+1} = A_{n+1} \quad \wedge \quad B_{n+1} \backslash A_{n+1} = B_n \tag{10.1.71}$$

implies

$$\mu(B_{n+1}) = \mu(A_{n+1}) + \mu(B_n) \tag{10.1.72}$$

and the relation (10.1.70) follows by induction.

(5) Let (B_i) be an increasing sequence of measurable sets, then

$$\mu(\bigcup_{i} B_i) = \lim \mu(B_i). \tag{10.1.73}$$

This follows by applying (10.1.68) to the disjoint sets

$$A_1 = B_1, \quad A_i = B_i \backslash B_{i-1}, \tag{10.1.74}$$

which are measurable because of (1). Next let (A_i), $i \in \mathbb{N}$, be measurable and $M \subset E$, then

$$\begin{aligned} \mu(M) &\geq \mu(M \cap (\bigcup_{i=1}^{n} A_i)) + \mu(M \backslash (\bigcup_{i=1}^{n} A_i)) \\ &\geq \mu(M \cap (\bigcup_{i=1}^{n} A_i)) + \mu(M \backslash \bigcup_{i=1}^{\infty} A_i). \end{aligned} \tag{10.1.75}$$