then we have—omitting the arguments as usual

$$\operatorname{rg} D\psi = \operatorname{rg}(Df \circ D\varphi_n') = \operatorname{rg} Df = r, \tag{8.4.11}$$

since $D\varphi_n'(y) \in GL(n)$, and moreover

$$\psi(y) = (y^1, \dots, y^r, h^{r+1}(y), \dots, h^m(y)), \quad y \in W_\epsilon^n, \tag{8.4.12}$$

with C^k-functions h^j, $r+1 \le j \le m$.

Claim: The functions h^j only depend on $(y^1, \dots, y^r)$.

Let e_i resp. e_j' be the canonical basis vectors of $\mathbb{R}^n$ resp. $\mathbb{R}^m$ and $(a_i^j) = (a_i^j(y))$ the Jacobian of $D\psi(y)$. Then

$$D\psi(y)e_i = a_i^j e_j' \qquad \forall\, 1 \le i \le n, \tag{8.4.13}$$

from which we deduce with the help of (8.4.12)

$$D\psi(y)e_i = e_i' + \sum_{j=r+1}^m \frac{\partial h^j}{\partial y^i} e_j' \qquad \forall\, 1 \le i \le r \tag{8.4.14}$$

and

$$D\psi(y)e_i = \sum_{j=r+1}^m \frac{\partial h^j}{\partial y^i} e_j' \qquad \forall\, r+1 \le i \le n. \tag{8.4.15}$$

Because of (8.4.11) and (8.4.14) the vectors $D\psi(y)e_i$, $1 \le i \le r$, generate $R(D\psi(y))$; on the other hand the vectors on the right-hand side of (8.4.15) cannot belong to the span of $D\psi(y)e_i$, $1 \le i \le r$, unless they vanish, hence

$$\frac{\partial h^j}{\partial y^i}(y) = 0 \qquad \forall\, r+1 \le i \le n, \quad \forall\, r+1 \le j \le m \tag{8.4.16}$$

and for all $y \in W_\epsilon^n$.

Decomposing the cube W_ϵ^n, $W_\epsilon^n = W_\epsilon^r \times W_\epsilon^{n-r}$, with a corresponding decomposition for the elements $y \in W_\epsilon^n$, $y = (y_1, y_2)$, then (8.4.16) can be reformulated as

$$D_{y_2} h^j \equiv 0 \qquad \forall\, r+1 \le j \le m. \tag{8.4.17}$$

Since each subcube is connected, we deduce

$$h^j(y_1, y_2) = h^j(y_1) = h^j(y^1, \dots, y^r) \qquad \forall\, r+1 \le j \le m, \tag{8.4.18}$$

in view of Corollary 7.2.3 on page 70.

(iii) Let $h : W_\epsilon^r \times \mathbb{R}^{m-r} \to W_\epsilon^r \times \mathbb{R}^{m-r}$ be defined by

$$h = (0, \dots, 0, h^{r+1}, \dots, h^m), \tag{8.4.19}$$

where we observe that h only depends on the first r variables $(y^1, \dots, y^r) \in W_\epsilon^r$, and set $\alpha : W_\epsilon^r \times \mathbb{R}^{m-r} \to W_\epsilon^r \times \mathbb{R}^{m-r}$

$$\alpha(y) = y - h(y). \tag{8.4.20}$$

Decomposing the vectors $y \in W_\epsilon^r \times \mathbb{R}^{m-r}$ in the form $y = (y_1, y_2)$, then

$$\alpha(y_1, y_2) = (y_1, y_2) - h(y_1). \tag{8.4.21}$$