Übungen zu Analysis III

Blatt 7

1 Find a second proof of Theorem 8.5.2, by using in the arguments following equation (8.5.7) the coordinate's notation. As a starting point for the formulation of (8.5.8) in coordinates use the equation

$$0 = \frac{\partial \Phi^j}{\partial x^i} + \frac{\partial \Phi^j}{\partial y^k}\frac{\partial \varphi^k}{\partial x^i},$$

which is derived by differentiating $\Phi(x, \varphi(x)) = 0$ with respect to x^i.

Note: Set $(a^k_j) = (\frac{\partial \Phi^j}{\partial y^k})^{-1}$. $\boxed{4}$

2 Determine the distance of the origin from the set

$$M = \{\, (x, y) \in \mathbb{R}^2 \colon xy = 1 \,\}.$$

$\boxed{2}$

3 Determine the distance of the origin from the surface

$$M = \operatorname{graph} f, \qquad f(x, y) = \frac{1}{xy} \quad \forall\, x, y \neq 0.$$

$\boxed{2}$

4 Prove that the solution of the variational problem (8.5.1) is an eigenvector of A and that the corresponding eigenvalue is the smallest eigenvalue of A. $\boxed{4}$

5 Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be symmetric and $j(x) = \langle Ax, x\rangle$. Prove, by using exercise 4, that A can be diagonalized, where the diagonal elements are the eigenvalues λ_i and the basis vectors are orthonormal eigenvectors e_i, $1 \le i \le n$.
Assume that the eigenvalues are ordered such that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then

(i) $\lambda_1 = \min_{x \in S^{n-1}} j(x)$, $\lambda_1 = j(e_1)$ and

$$\begin{aligned}\lambda_k &= \min\{\, j(x) \colon x \in S^{n-1} \ \wedge\ \langle x, e_i\rangle = 0 \quad \forall\, 1 \le i \le k-1 \,\}\\ &= j(e_k).\end{aligned}$$

(ii) The λ_k can be characterized by a minimax principle

$\lambda_k = \max\{\, \min\{\, j(x) \colon x \in S^{n-1} \cap E \,\} \colon E \subset \mathbb{R}^n,\ \dim E \ge n - k + 1 \,\}$,

where E are subspaces of $\mathbb{R}^n$.

$\boxed{6}$