Übungen zu Analysis III

Blatt 3

1 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

and prove that the second partial derivatives of f exist in the origin, but that they are neither continuous nor symmetric there. $\boxed{4}$

2 Let $A \in L(E,F)$, then $A \in C^\infty(E,F)$ and $D^2 A \equiv 0$. $\boxed{4}$

3 Prove the relation in (7.5.86). $\boxed{4}$

4 Let $a \in L_k(E_1, \dots E_k; F)$, then $a \in C^\infty(\prod_{i=1}^k E_i, F)$ and $D^{k+1} a \equiv 0$.
Note: Give a simple and elegant proof. $\boxed{4}$

5 Consider in $\Omega \subset \mathbb{R}^n$ a symmetric, covariant tensor of second order g_{ij}, which is supposed to be invertible with inverse $(g^{ij}) = (g_{ij})^{-1}$. A covariant tensor of this kind is said to be a *metric*.
Prove that

(i) The inverse g^{ij} is a contravariant tensor of second order. $\boxed{3}$

(ii) Relative to a fixed metric g_{ij} define the *divergence* of a contravariant vector field $\xi = (\xi^i)$ by

$$\operatorname{div} \xi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|}\, \xi^i \right),$$

where $g = \det(g_{ij})$, and the *rotation* of a covariant vector field $\lambda = (\lambda_i)$ by

$$\operatorname{rot} \lambda = (\lambda_{i,j} - \lambda_{j,i}).$$

The comma indicates an ordinary partial derivative

$$\lambda_{i,j} = \frac{\partial \lambda_i}{\partial x^j}.$$

Prove that $\operatorname{div} \xi$ is a function and $\operatorname{rot} \lambda$ a covariant tensor of order two. $\boxed{6}$