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Global Regularity of the Solutions to the Capillarity Problem (*).

CLAUS GERHARDT (**)

0. – Introduction.

Let Ω be a bounded domain of $\mathbb{R}^n$, $n\geqslant 2$, with smooth boundary $\partial\Omega$, and let A be the minimal surface operator

$$(0.1) \qquad A = -D^i\big(a_i(p)\big)\,, \quad a_i = p^i\cdot(1+|p|^2)^{-\frac{1}{2}}\,(^1)\,.$$

Then, a (regular) solution of the capillarity problem can be looked at as a solution $u\in C^2(\bar{\Omega})$ of the following equation

$$(0.2) \qquad Au + H(x, u) = 0 \quad \text{in } \Omega$$

subject to the boundary conditions

$$(0.3) \qquad a_i\cdot\gamma_i = \beta \quad \text{on } \partial\Omega\,,$$

where H and β are given functions, and $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the exterior normal vector to $\partial\Omega$.

Recently, SPRUCK [12] and URAL'CEVA [15] solved this question partially: In the case $n=2$ Spruck could show the existence of a solution $u\in C^{2,\alpha}(\bar{\Omega})$ provided that $\partial\Omega$ is of class C^4, β belongs to $C^{1,\varepsilon}(\partial\Omega)$ such that $0<\varepsilon<1$ and $|\beta|<1$, and provided that H has the form $H(x,t)=\varkappa\cdot t$, $\varkappa>0$. Spruck's methods are completely two-dimensional.

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(^{1}) Here and in the following we sum over repeated indices from 1 to n.

A different approach has been made by Ural'ceva which will be valid for arbitrary dimension. She proved the existence of a solution $u \in C^{2,\alpha}(\bar{\Omega})$ under the assumptions

$$\partial\Omega \in C^{2,\lambda}, \quad H \in C^{1,\lambda}(\mathbb{R}^n \times \mathbb{R}), \quad |\beta| < 1 \tag{0.4}$$

where H satisfies

$$\frac{\partial H}{\partial t} \geqslant \varkappa > 0 \tag{0.5}$$

and where Ω is supposed to be convex and β is constant.

The last assumptions are rather restrictive, and it is the aim of this paper to exclude these restrictions by a suitable modification of Ural'ceva's proof.

In the second part of this article we shall apply this result to the capillarity problem with constant volume—which is an obstacle problem—and we shall show that this problem has a solution $u \in H^{2,p}(\Omega)$ for any $p > n$.

Since the paper of Ural'ceva is written in Russian we shall repeat many proofs of that paper almost literally for the convenience of the reader.

1. – A priori estimates for $|Du|$.

In this section we shall assume that $u \in C^2(\bar{\Omega})$ is a solution to the differential equation (0.2), (0.3). Furthermore, let us suppose that $\partial\Omega$ is of class C^2, and that the functions

$$H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}) \quad \text{and} \quad \beta \in C^{0,1}(\partial\Omega) \tag{1.1}$$

satisfy the conditions

$$\frac{\partial H}{\partial t} \geqslant 0 \tag{1.2}$$

and

$$|\beta| \leqslant 1 - a, \quad a > 0. \tag{1.3}$$

Then, the following theorem is valid.

THEOREM 1.1. *Under the assumptions stated above the modulus of the gradient of u can be estimated by a constant depending on $|u|_\Omega$, $|H(x, u(x))|_\Omega$, $|(\partial/\partial x) H(x, u(x))|_\Omega$, $\partial\Omega$, n, a, and on the Lipschitz constant of β.*

PROOF. First of all, let us extend β and γ into the whole domain Ω such that β belonging to $C^{0,1}(\bar{\Omega})$ still satisfies (1.3), and such that the vector field γ is uniformly Lipschitz continuous in Ω and absolutely bounded by 1. These extensions are possible in view of the smoothness of $\partial\Omega$.

Then, following Ural'ceva's ideas, we are going to prove that the function

$$v = (1 + |Du|^2)^{\frac{1}{2}} - \beta \cdot D^i u \cdot \gamma_i \tag{1.4}$$

is uniformly bounded in Ω by some constant which only depends on the quantities we have just mentioned in Theorem 1.1. Precisely, we shall show that v is bounded locally near $\partial\Omega$. The global estimate then follows from well-known interior gradient bounds.

In order to prove the main result we need some lemmata which will be derived in the following.

We denote by $\mathcal{S}$ the graph of u

$$\mathcal{S} = \{X = (x, x^{n+1}) : x \in \bar{\Omega},\, x^{n+1} = u(x)\} \tag{1.5}$$

and by $\delta = (\delta^1, \dots, \delta^{n+1})$ the usual differential operators on $\mathcal{S}$, i.e. for $g \in C^1(\bar{\Omega}^{n+1})$ we have

$$\delta^i g = D^i g - \nu_i \cdot \sum_{k=1}^{n+1} \nu_k \cdot D^k g\,, \qquad i = 1, \dots, n+1\,, \tag{1.6}$$

where $\nu = (\nu_1, \dots, \nu_{n+1})$ is the exterior normal vector to $\mathcal{S}$

$$\nu = (1 + |Du|^2)^{-\frac{1}{2}} \cdot (-D^1 u, \dots, -D^n u, 1)\,. \tag{1.7}$$

Then the following Sobolev Imbedding Lemma is valid:

LEMMA 1.1. *For any function $g \in C^1(\bar{\Omega})$ the inequality*

$$\Big(\int_{\mathcal{S}} |g|^{n/(n-1)} d\mathcal{H}_n\Big)^{(n-1)/n} \leq c_1 \cdot \Big\{ \int_{\mathcal{S}} (|\delta g| + |g|)\, d\mathcal{H}_n + \int_{\partial\Omega} |g| \cdot (1 + |Du|^2)^{\frac{1}{2}}\, d\mathcal{H}_{n-1} \Big\} \tag{1.8}$$

holds, where $\mathcal{H}_n$ is the n-dimensional Hausdorff measure, and where the constant depends on n and $|H(x, u(x))|_\Omega$.

PROOF OF LEMMA 1.1. This Sobolev inequality for functions equal to zero on all of $\partial\Omega$ was established in [9] for solutions of (0.2). Here, we shall not assume that g is equal to zero on $\partial\Omega$.

Denote by d, $d(x) = \operatorname{dist}(x, \partial\Omega)$, the distance function to $\partial\Omega$, and let

$$\eta_k = \min(1, kd) \tag{1.9}$$

for $k \in \mathbf{N}$.

Let $g \in C^1(\bar{\Omega})$ be given. Then

$$g_k = g \cdot \eta_k \tag{1.10}$$

has boundary values equal to zero, so that in view of the result in [9] inequality (1.8) is valid for g_k.

If k goes to infinity the integrals

$$\Big(\int_S |g_k|^{n/(n-1)}\, d\mathcal{H}_n\Big)^{(n-1)/n} \quad \text{and} \quad \int_S |g_k|\, d\mathcal{H}_n \tag{1.11}$$

tend to the respective integrals with g_k replaced by g, while

$$\int_S |\delta g_k|\, d\mathcal{H}_n \tag{1.12}$$

is estimated by

$$\int_S |\delta g| \cdot \eta_k\, d\mathcal{H}_n + \int_\Omega |g| \cdot |D\eta_k| \cdot (1 + |Du|^2)^{\frac{1}{2}}\, dx\,. \tag{1.13}$$

The last integral converges to

$$\int_{\partial\Omega} |g| \cdot (1 + |Du|^2)^{\frac{1}{2}}\, d\mathcal{H}_{n-1} \tag{1.14}$$

(cf. [5; Appendix III]), hence the result.

Next we need to technical lemmata. Let us denote by a_{ij}

$$a_{ij} = \frac{\partial a_i}{\partial p^j} \tag{1.15}$$

then we have

LEMMA 1.2. *On the boundary of Ω we have the following estimate*

$$|\gamma_i \cdot a_{ij}(D^j v + D^j(\beta \cdot \gamma_k) \cdot D^k u)| \leqslant c_2 \tag{1.16}$$

where the constant c_2 depends on $\partial\Omega$ and the Lipschitz constant of β.

PROOF OF LEMMA 1.2. Let x_0 be an arbitrary boundary point and let us introduce new coordinates $y = y(x)$ which are related to x by an orthogonal transformation such that the y^n-axis is directed along the exterior normal vector at x_0. Assume, furthermore, that in a neighbourhood of x_0 the surface $\partial\Omega$ is specified by

$$y^n = \omega(y^1, \dots, y^{n-1}) . \tag{1.17}$$

If we now differentiate the equation (0.3) with respect to the operator

$$(a_k - \beta\cdot\gamma_k)\cdot \sum_{s=1}^{n-1} \frac{\partial y^s}{\partial x^k}\cdot\frac{\partial}{\partial y^s}, \tag{1.18}$$

then we obtain at x_0

$$\left|(a_k - \beta\cdot\gamma_k)\cdot \sum_{s=1}^{n-1} \frac{\partial y^s}{\partial x^k}(\gamma_i\cdot a_{ij}\cdot D_x^j D_y^s u + a_i\cdot D_y^s\gamma_i)\right| \leqslant \text{const}. \tag{1.19}$$

Moreover, since $\partial\omega/\partial y_l = 0$ et x_0 for $s = 1, \dots, n-1$ we deduce

$$\gamma_k = \frac{\partial y^n}{\partial x^k}, \qquad k = 1, \dots, n . \tag{1.20}$$

Thus, we have in view of (0.3)

$$(a_k - \beta\cdot\gamma_k)\cdot\frac{\partial y^n}{\partial x^k} = 0 , \tag{1.21}$$

and hence the relation (1.19) is also true for $s = n$.

On the other hand, combining (1.19) and

$$D^j v = (a_k - \beta\cdot\gamma_k) D^k D^j u - D^j(\beta\cdot\gamma_k)\cdot D^k u \tag{1.22}$$

we derive that the left side of (1.16) is bounded at x_0 by

$$\text{const} + |(a_k - \beta\cdot\gamma_k)\cdot a_i\cdot D_x^k\gamma_i| \tag{1.23}$$

by which the assertion is proved.

LEMMA 1.3. *In the whole domain Ω the following pointwise estimate is valid*

$$|a_{ij} D^k D^j u[a_{kl} D^l D^i u - D^i(\beta\cdot\gamma_k)]| \leqslant c_3\cdot[|\delta v|\cdot(1 + |Du|^2)^{-\frac{1}{2}} + 1] , \tag{1.24}$$

where the constant depends on $|D\gamma|_\Omega$, $|D\beta|_\Omega$, and on a.

PROOF OF LEMMA 1.3. During the proof we have to work in the $(n+1)$-dimensional Euclidean space rather than in the n-dimensional one; therefore we regard a function $g = g(x^1, \ldots, x^n)$ as being defined in $\mathbb{R}^{n+1}$ via the mapping $x \to (x, 0)$. Let us introduce the notation Φ for the Euclidean norm in $\mathbb{R}^{n+1}$

$$\Phi(q) = |q|, \qquad q \in \mathbb{R}^{n+1}. \tag{1.25}$$

We shall consider the Hessian matrix of Φ, $(\Phi_{ij})_{i,j=1,\ldots,n+1}$, evaluated at $q_0 = \big(-Du(x_0), 1\big)$ where x_0 is an arbitrary but fixed point in Ω. Let $z^1, \ldots, z^{n+1}$ be the eigenvectors of that matrix and $\lambda_1, \ldots, \lambda_{n+1}$ be the corresponding eigenvalues. Evidently, $q_0/|q_0|$ is itself an eigenvector, which is just the exterior normal vector of the surface S at the point $\big(x_0, u(x_0)\big)$.

Assume, that the eigenvectors are numbered in such a way that $z^{n+1} = q_0/|q_0|$. Then, we have $\lambda_{n+1} = 0$. Furthermore, the eigenvectors z_i, $i < n+1$, are orthogonal to z^{n+1}, and we easily derive

$$\lambda_i = \big(1 + |Du|^2\big)^{-\frac{1}{2}}, \qquad i = 1, \ldots, n. \tag{1.26}$$

Finally, if we denote by $\cos(z^i, x^k)$ the scalar product of the corresponding vectors, where the indices run from 1 to $n+1$, then we obtain

$$\cos(z^{n+1}, x^k) = \begin{cases} -D^k u(1 + |Du|^2)^{-\frac{1}{2}}, & k = 1, \ldots, n, \\ (1 + |Du|^2)^{-\frac{1}{2}}, & k = n+1. \end{cases} \tag{1.27}$$

To estimate

$$A = a_{ij} \cdot D^k D^j u \cdot a_{kl} \cdot D^l D^i u = \Phi_{ij} \cdot D^k D^j u \cdot \Phi_{kl} \cdot D^l D^i u$$

at x_0, let us observe that we may sum from 1 to $n+1$ in this expression since u does not depend on x^{n+1}. Moreover, as A is the trace of a product of matrices it is invariant under orthogonal transformations of the coordinate system. Thus, we derive,

$$A = \lambda_s \lambda_t |D_z^s D_z^t u^2| \tag{1.28}$$

having in mind that $\lambda_{n+1} = 0$.

From (1.28) we conclude in view of (1.26)

$$A = \big(1 + |Du|^2\big)^{-1} \cdot |\delta^2 u|^2, \tag{1.29}$$

where

$$|\delta^2 u| = \Big(\sum_{s,t=1}^{n} |D_z^s D_z^t u|^2\Big)^{\frac{1}{2}}.$$

On the other hand, we have

$$(1.30)\qquad \begin{aligned} a_{ij}\cdot D^k D^j u\cdot D^i(\beta\gamma_k) &= \Phi_{ij}\cdot D^k D^j u\cdot D^i(\beta\gamma_k) = \\ &= \sum_{r,s,t=1}^{n+1}\Phi_{ij}\cdot\cos(z^s, x^j)\cdot\cos(z^t, x^i)\cdot\cos(z^r, x^k)\cdot D_z^r D_z^s u\cdot D_z^t(\beta\gamma_k) = \\ &= \sum_{r,s=1}^{n}\lambda_s\cdot\cos(z^r, x^k)\cdot D_z^r D_z^s u\cdot D_z^s(\beta\gamma_k) + \\ &\qquad + \sum_{s=1}^{n}\lambda_s\cdot\cos(z^{n+1}, x^k)\cdot D_z^{n+1}D_z^s u\cdot D_z^s(\beta\gamma_k)\,, \end{aligned}$$

where now we also sum over the repeated indices i, j, and k from 1 to $n+1$.

Furthermore, to estimate $D_z^{n+1}D_z^s u$ for $s\neq n+1$ we observe that for any C^1-function g which does not depend on x^{n+1} there holds

$$(1.31)\qquad D_z^{n+1}g = -D_x^k g\cdot D_x^k u\cdot\left(1+|Du|^2\right)^{-\frac12}$$

and

$$(1.32)\qquad |\delta g|^2 = \sum_{s=1}^{n}|D_z^s g|^2 = |D_x g|^2 - |D_z^{n+1}g|^2 \geqslant |D_x g|^2\cdot\left(1+|Du|^2\right)^{-1},$$

hence

$$(1.33)\qquad \sum_{s=1}^{n}|D_z^{n+1}D_z^s u|^2 \leqslant \left(1+|Du|^2\right)\cdot|\delta^2 u|^2\,.$$

Finally, if differentiate v with respect to z^s for $s\neq n+1$ we obtain

$$(1.34)\qquad \begin{aligned} D_z^s v = (a_k-\beta\cdot\gamma_k)\cdot\sum_{t=1}^{n+1}D_z^s D_z^t u\cdot\cos(z^t, x^k) - D_x^k u\cdot D_z^s(\beta\cdot\gamma_k) &\equiv \\ \equiv \sum_{t=1}^{n+1}\alpha_t\cdot D_z^s D_z^t u - D_x^k u\cdot D_z^s(\beta\cdot\gamma_k)\,&, \end{aligned}$$

where the α_t's satisfy

$$(1.35)\qquad |\alpha_t|\leqslant 2$$

and—in view of (1.27)—

$$(1.36)\qquad \alpha_{n+1} = -(a_k-\beta\cdot\gamma_k)\cdot D^k u\cdot\left(1+|Du|^2\right)^{-\frac12}.$$

Taking the estimate

$$(1.37)\qquad (a_k-\beta\cdot\gamma_k)\cdot D^k u \geqslant a\cdot\left(1+|Du|^2\right)^{\frac12} - c_4$$

with some suitable constant c_4 into account, we thus deduce from (1.33) and (1.36)

$$(1.38)\qquad |\alpha_{n+1}\cdot D_z^{n+1}D_z^s u| \geqslant a\cdot|D_z^{n+1}D_z^s u| - c_4\cdot|\delta^2 u|\,.$$

Combining the relations (1.34), (1.35), and (1.38) we then conclude

$$(1.39)\qquad |D_z^s D_z^{n+1}u| \leqslant a^{-1}\cdot\big[|D_z^s v| + (2\cdot n + c_4)\cdot|\delta^2 u| + |Du|\cdot|D(\beta\gamma)|\big]\,.$$

Hence, there exists a constant c_3 depending on a, $|D(\beta\gamma)|$, and known quantities such that in view of (1.29)

$$(1.40)\qquad |a_{ij}D^kD^ju\cdot D^i(\beta\gamma_k)| \leqslant \Lambda + c_3\cdot\big[|\delta v|\cdot(1+|Du|^2)^{-\frac12}+1\big]$$

from which the assertion (1.24) immediately follows.

As we are treating the case of a non-convex domain and a variable β we need the following estimate

LEMMA 1.4. *For any positive function $\eta \in H^{1,\infty}(\Omega)$ we have the estimate*

$$(1.41)\qquad \int_{\partial\Omega} v\eta\, d\mathcal{H}_{n-1} \leqslant c_5\cdot\int_{S}\big[|\delta\eta| + \eta\big]\, d\mathcal{H}_n\,,$$

where the constant depends on $|\delta\gamma|_\Omega$ and $|H(x, u(x))|_\Omega$.

PROOF OF LEMMA 1.4. Let $W = (1+|Du|^2)^{\frac12}$ and $\varphi \in H^{1,\infty}(\Omega)$. Then, for $i = 1, \dots, n$, we have

$$\begin{aligned}(1.42)\qquad \int_S \delta^i\varphi\, d\mathcal{H}_n &= \int_\Omega \delta^i\varphi\cdot W\, dx = \int_\Omega \{D^i\varphi - a_i\cdot a_k\cdot D^k\varphi\}\, W\, dx = \\ &= \int_\Omega \{D^i\varphi\cdot W + a_k\cdot D^kD^iu\cdot\varphi\}\, dx - \int_\Omega a_k\cdot D^k(\varphi\cdot D^iu)\, dx = \\ &= \int_\Omega D^i(\varphi\cdot W)\, dx - \int_\Omega a_k\cdot D^k(\varphi\cdot D^iu)\, dx = \\ &= \int_{\partial\Omega}\gamma_i\cdot\varphi\cdot W\, d\mathcal{H}_{n-1} - \int_{\partial\Omega} a_k\cdot\gamma_k\cdot D^iu\cdot\varphi\, d\mathcal{H}_{n-1} + \int_\Omega D^ka_k\cdot\varphi\cdot D^iu\, dx\,.\end{aligned}$$

Thus, we deduce the identity

$$(1.43)\qquad \int_S \delta^i\varphi\, d\mathcal{H}_n = \int_{\partial\Omega}\{\gamma_i\cdot W - a_k\cdot D^iu\cdot\gamma_k\}\cdot\varphi\, d\mathcal{H}_{n-1} - \int_S \varphi\cdot \mathrm{H}\cdot\nu_i\, d\mathcal{H}_n\,.$$

Inserting $\varphi = \eta \cdot \gamma_i$ in this equality and summing over i from 1 to n yields

$$\int_S \delta^i(\eta \cdot \gamma_i)\, d\mathcal{H}_n = \int_{\partial\Omega} v \cdot \eta\, d\mathcal{H}_{n-1} - \int_S H \cdot \nu_i \cdot \gamma_i \cdot \eta\, d\mathcal{H}_n\,, \tag{1.44}$$

hence the result.

Up to now we have only proved auxiliary propositions which we shall need for estimating certain expressions that will appear in the following calculations. As we mentioned at the beginning we are going to show that v, or better,

$$w = \log v \tag{1.45}$$

is uniformly bounded in Ω. To accomplish this, let us look at the integral identity

$$\int_\Omega D^k a_i \cdot D^i \varphi\, dx = -\int_\Omega D^k D^i a_i \cdot \varphi\, dx + \int_{\partial\Omega} \gamma_i \cdot D^k a_i \varphi\, d\mathcal{H}_{n-1}\,. \tag{1.46}$$

If we choose $\varphi = (a_k - \beta \cdot \gamma_k)\eta$, $0 \leqslant \eta \in H^{1,\infty}(\Omega)$ and $\operatorname{supp} \eta \subset \{w > h\}$, where h is large, then we obtain in view of (0.2) and (1.22)

$$\begin{aligned}\int_\Omega \{a_{ij}[D^j v + D^j(\beta\gamma_k) \cdot D^k u] D^i \eta + a_{ij} D^k D^j u [a_{kl} D^l D^i u - D^i(\beta\gamma_k)] \cdot \eta\}\, dx \\ + D^k H \cdot (a_k - \beta\gamma_k)\eta\, dx = \int_{\partial\Omega} \gamma_i \cdot a_{ij}[D^j v + D^j(\beta\gamma_k) \cdot D^k u] \cdot \eta\, d\mathcal{H}_{n-1}\,.\end{aligned} \tag{1.47}$$

Moreover, observing that

$$D^k H = \frac{\partial H}{\partial x^k} + \frac{\partial H}{\partial t} \cdot D^k u \tag{1.48}$$

we deduce from (1.47) in view of the Lemmata 1.2 and 1.3, and in view of the assumption (1.3)

$$\int_\Omega a_{ij}[D^j v + D^j(\beta\gamma_k) \cdot D^k u] D^i \eta\, dx \leqslant c_2 \cdot \int_{\partial\Omega} \eta\, d\mathcal{H}_{n-1} + c_6 \cdot \int_\Omega \left[\frac{|\delta v|}{W} + 1\right] \eta\, dx, \tag{1.49}$$

where c_6 is a suitable constant and η any positive C^1-function.

We shall use this relation with $\eta = v \cdot \max\{w\zeta^2 - h, 0\}$, where h is a large positive number and ζ, $0 \leqslant \zeta \leqslant 1$, a smooth function. Introducing the notations $z = \max\{w\zeta^2 - h, 0\}$, $A(h, \zeta) = \{X \in \mathcal{S} : w(x)\zeta^2(x) > h\}$, and $|A(h, \zeta)| = \mathcal{H}_n(A(h, \zeta))$ we then obtain in view of Lemma 1.4

$$(1.50) \qquad \int_{A(h,\zeta)} \{a_{ij} D^j v \cdot D^i v \cdot z + a_{ij} \cdot v^2 \cdot D^i w \cdot D^j w \cdot \zeta^2\} W^{-1} d\mathcal{H}_n \leqslant$$

$$\leqslant -\int_{A(h,\zeta)} a_{ij} D^j w \cdot v^2 \cdot w \cdot 2 \cdot \zeta \cdot D^i \zeta \cdot W^{-1} d\mathcal{H}_n + c_2 \cdot c_5 \cdot \int_{A(h,\zeta)} [|\delta w| \cdot \zeta^2 + 2w\zeta \cdot |\delta \zeta| + z]\, d\mathcal{H}_n -$$

$$-\int_{A(h,\zeta)} a_{ij} \cdot D^j(\beta\gamma_k) \cdot D^k u [D^i v \cdot z + v D^i w \zeta^2 + 2vw\zeta \cdot D^i \zeta] W^{-1} d\mathcal{H}_n +$$

$$+ c_6 \cdot \int_{A(h,\zeta)} \left[\frac{|\delta v|}{W} + 1\right] \cdot v \cdot z \cdot W^{-1} d\mathcal{H}_n .$$

Thus, taking the relations

$$(1.51) \qquad a_{ij} D^i g D^j g = W^{-1} \cdot |\delta g|^2 \qquad \forall g \in C^1(\bar{\Omega}),$$

$$(1.52) \qquad |a_{ij} D^i g \cdot D^j \phi| \leqslant W^{-1} \cdot |\delta g| \cdot |D\phi| \qquad \forall \phi \in C^1(\bar{\Omega}),$$

$$(1.53) \qquad a \cdot W \leqslant v \leqslant 2 \cdot W,$$

$$(1.54) \qquad a_{ij} p^i q^j \leqslant \frac{\varepsilon}{2} \cdot a_{ij} p^i p^j + \frac{1}{2\varepsilon} \cdot a_{ij} q^i q^j,$$

and

$$(1.55) \qquad |\delta(w\zeta^2)|^2 \leqslant 8 \cdot \{|\delta w|^2 \zeta^2 + w^2 |\delta \zeta|^2\}$$

into account, we derive

$$(1.56) \qquad \int_{\mathcal{S}} |\delta z|^2 d\mathcal{H}_n = \int_{A(h,\zeta)} |\delta(w\zeta^2)|^2 d\mathcal{H}_n \leqslant [c_7 + |\delta\zeta|^2_{\Omega}] \Big\{ |A(h, \zeta)| + \int_{A(h,\zeta)} w^2 d\mathcal{H}_n \Big\} .$$

Moreover, from the Lemmata 1.1 and 1.4, and from (1.53) we conclude

$$(1.57) \qquad \Big(\int_{\mathcal{S}} |z|^{n/(n-1)} d\mathcal{H}_n \Big)^{(n-1)/n} \leqslant c_1 \cdot \Big\{ \int_{\mathcal{S}} |\delta z| + |z|\, d\mathcal{H}_n + + a^{-1} \cdot c_5 \cdot \int_{\mathcal{S}} (|\delta z| + |z|)\, d\mathcal{H}_n \Big\} .$$

Thus, using the Hölder inequality

$$(1.58) \qquad \int_{\mathcal{S}} |z|\, d\mathcal{H}_n \leqslant |A(h, \zeta)|^{1/n} \cdot \Big(\int_{\mathcal{S}} |z|^{n/(n-1)} d\mathcal{H}_n \Big)^{(n-1)/n}$$

and choosing supp ζ small enough we deduce

$$\left(\int_{S}|z|^{n/(n-1)}\,d\mathcal{H}_n\right)^{(n-1)/n}\leqslant c_8\cdot\int_{S}|\delta z|\,d\mathcal{H}_n \tag{1.59}$$

from which we derive by a well-known argument

$$\int_{S}|z|^2\,d\mathcal{H}_n\leqslant c_9\cdot|A(h,\zeta)|^{2/n}\cdot\int_{S}|\delta z|^2\,d\mathcal{H}_n\,. \tag{1.60}$$

Hence, we conclude

$$\int_{S}|z|\,d\mathcal{H}_n\leqslant\left[c_{10}+|\delta\zeta|_{\Omega}\right]\cdot\left\{|A(h,\zeta)|^{1+1/n}+|A(h,\zeta)|^{1/2+1/n}\cdot\left[\int_{A(h,\mathfrak{J})}w^2\,d\mathcal{H}_n\right]^{1/2}\right\}. \tag{1.61}$$

Now, the boundedness of $w\cdot\zeta^2$ follows immediately provided that

$$\int_{B(h_0)}w^2\cdot W\cdot dx \tag{1.62}$$

is bounded, where $B(h_0)=\{x\in\Omega\colon v(x)>h_0\}$ and h_0 is sufficiently large (cf. [3; p. 195]).

As a first step we prove

LEMMA 1.5. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Then we have*

$$\int_{B(h_0)}\left[W^{-3}\cdot|Dv|^2+v\right]dx\leqslant c_{11}\,. \tag{1.63}$$

PROOF OF LEMMA 1.5. We insert $\eta=\max(v-h_0,0)$ in the inequality (1.49) and conclude with the help of the relations (1.51)-(1.54)

$$\int_{B(h_0)}W^{-3}\cdot|Dv|^2\,dx\leqslant\int_{B(h_0)}W^{-1}|\delta v|^2\,dx\leqslant c_{12}\cdot\int_{\Omega}W\,dx\,. \tag{1.64}$$

Thus it remains to prove that $\int_{\Omega}v\,dx$ or equivalently $\int_{\Omega}W\,dx$ is bounded.

To accomplish this, we consider the identity

$$\int_{\Omega}a_i\cdot D^i\eta\,dx+\int_{\Omega}H\cdot\eta\,dx-\int_{\partial\Omega}\beta\eta\,d\mathcal{H}_{n-1}=0\,,\qquad\forall\eta\in C^1(\bar{\Omega})\,. \tag{1.65}$$

Choosing $\eta = u$ the result follows from the inequality

$$\int_{\partial\Omega} |\beta\eta| d\mathcal{H}_{n-1} \leqslant (1-a)\cdot \int_{\Omega} |D\eta|\, dx + c_{13}\cdot \int_{\Omega} |\eta| dx \tag{1.66}$$

(cf. [6; Lemma 1]).

After having established the estimate (1.63) we use the relation (1.65) once more, this time with $\eta = u\cdot \max(w^2 - h, 0)$ and we obtain in view of the preceding inequality

$$\begin{aligned}(1.67)\quad & \int_{\{w^2>h\}} \{a_i\cdot D^i u\cdot (w^2 - h) + u\cdot a_i\cdot D^i v\cdot v^{-1}\cdot 2w + H\cdot u\cdot (w^2-h)\}\, dx \leqslant \\ & \leqslant (1-a)\cdot \int_{\{w^2>h\}} \{|Du|\cdot (w^2-h) + |u|\cdot 2w\cdot |Dv|\cdot v^{-1}\}\, dx + c_{13}\cdot \int_{\{w^2>h\}} |u|\cdot (w^2-h)\, dx\,.\end{aligned}$$

Hence, we deduce

$$\int_{\{w^2>h\}} |Du|\cdot (w^2-h)\, dx \leqslant c_{14}\cdot \int_{\{w^2>h\}} [w^2 + W^{-3}\cdot |Dv|^2]\, dx\,, \tag{1.68}$$

where we used the inequality

$$a\cdot b \leqslant \frac{\varepsilon}{2}\cdot a^2 + \frac{1}{2\varepsilon}\cdot b^2\,. \tag{1.69}$$

To complete the proof of the boundedness of the integral (1.62) we observe that

$$w^2 \leqslant \alpha\cdot W \tag{1.70}$$

for some suitable constant α.

Thus, we have proved that, given a suitable boundary neighbourhood U, $|Du|$ is bounded in U. Together with well-known interior gradient estimates (cf. [1, 8, 13]) this completes the proof of Theorem 1.1.

2. – Existence of a solution u.

In view of the a priori estimates which we have just established the existence of a solution will be proved by a continuity method.

THEOREM 2.1. *Suppose that the boundary of Ω is of class $C^{2,\lambda}$, and that H and β are $C^{1,\lambda}$-functions in their arguments. Furthermore, assume that H*

satisfies

$$\frac{\partial H}{\partial t} \geqslant \varkappa > 0 \,. \tag{2.1}$$

Then the boundary value problem (0.2), (0.3) *has a unique solution* $u \in C^{2,\alpha}(\bar{\Omega})$, *where the exponent* α, $0 < \alpha < 1$, *is determined by the above quantities.*

PROOF. Let τ be a real number with $0 \leqslant \tau \leqslant 1$, and consider the boundary value problems

$$Au_\tau + \tau \cdot H(x, u_\tau) = 0 \,, \tag{2.2.τ}$$

$$a_i \cdot \gamma_i = \tau \cdot \beta \,. \tag{2.3.τ}$$

Let T be the set

$$T = \{\tau\colon \text{ there exists a solution } u_\tau \in C^2(\bar{\Omega})\} \,. \tag{2.4}$$

T is obviously not empty for $u_0 = 0$ belongs to it, and we shall show that it is both open and closed.

In view of the assumption (2.1) we obtain an a priori bound of $|u_\tau|_\Omega$ for any $\tau \in T$ independent of τ (cf. [2]). Furthermore, let us remark that any solution $u_\tau \in C^2(\bar{\Omega})$ is of class $C^{2,\alpha}(\bar{\Omega})$ with some fixed α, $0 < \alpha < 1$, such that the norm of u_τ in $C^{2,\alpha}(\bar{\Omega})$ is bounded independently of τ.

To prove this, we first deduce from Theorem 1.1 that $|Du_\tau|_\Omega$ is uniformly bounded

$$|Du_\tau|_\Omega \leqslant K_1 \,. \tag{2.5}$$

Then, we choose a smooth vector field $\tilde{a}_i$ such that $\partial \tilde{a}_i / \partial p^j$ is uniformly elliptic, and such that

$$\tilde{a}_i(p) = a_i(p) \qquad \text{for } |p| \leqslant 3 \cdot K_1 \,. \tag{2.6}$$

From [7; Chapter 10, Theorem 2.2] we conclude that the problem

$$\tilde{A}\tilde{u}_\tau + \tau \cdot H(x, \tilde{u}_\tau) = 0 \qquad \text{in } \Omega \,, \tag{2.7.τ}$$

$$\tilde{a}_i \cdot \gamma_i = \tau \cdot \beta \qquad \text{on } \partial\Omega \,, \tag{2.8.τ}$$

has a solution $\tilde{u}_\tau \in C^{2,\alpha}(\bar{\Omega})$ for any τ. Moreover in view of (2.5) and (2.6) we derive

$$\tilde{A}u_\tau = Au_\tau \,. \tag{2.9}$$

Hence, we obtain from the uniqueness of the solution

$$\tilde{u}_\tau = u_\tau . \tag{2.10}$$

Thus, we can finally conclude that $|u_\tau|_{2,\alpha,\Omega}$ is uniformly bounded

$$|u_\tau|_{2,\alpha,\Omega} \leqslant K_2 \tag{2.11}$$

where the constant is determined by known quantities.

From the estimate (2.11) it follows immediately that T is closed.

On the other hand, let $\tau_0 \in T$. Then, we consider the boundary value problems (2.7.τ), (2.8.τ) as before. Since $|D\tilde{u}_\tau|_\Omega$ depends continuously on τ, it turns out that

$$|D\tilde{u}_\tau|_\Omega \leqslant 2\cdot K_1 \quad \text{for } |\tau - \tau_0| < \varepsilon . \tag{2.12}$$

But this yields $\tilde{u}_\tau = u_\tau$ for those τ's. Thus, the proof of Theorem 2.1 is completed.

For our considerations in the next section it will be necessary to bound the norm of u in the function space $H^{2,p}(\Omega)$, $p > n$, by a constant which only depends on $|Du|_\Omega$, $|u|_\Omega$, p, n, the C^2-norm of $\partial\Omega$, $|D\beta|_\Omega$, and on the L^p-norm of $H(x, u(x))$.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1 *the norm of* u *in* $H^{2,p}(\Omega)$, $n < p < \infty$, *is bounded by a constant being only determined by the quantities mentioned above.*

PROOF. Since in the interior this result follows from the well-known Calderon-Zygmund–Inequalities, we have only to prove it near the boundary.

Let Γ be a part of the boundary and suppose that an open subset Ω^* of Ω adjacent to Γ is transformed into some open subset G of the half-space $\{y \in \mathbb{R}^n : y^n > 0\}$ via a C^2-diffeomorphism $y = y(x)$ such that $y(\Gamma) \subset \{y \in \mathbb{R}^n : y^n = 0\}$.

In G the equation assumes the form

$$- D_y^k(a_i)\cdot D_x^i y^k + H = o \tag{2.13}$$

and the boundary condition (0.3) is transformed into

$$a_i \cdot D_x^i y^n = \beta\cdot |D_x y^n| \equiv \tilde{\beta} \qquad \text{on } y(\Gamma) , \tag{2.14}$$
$$|D_x y^n| = \left(\sum_{i=1}^n \left|\frac{\partial y^n}{\partial x^i}\right|^2\right)^{\frac{1}{2}} .$$

We are going to prove that the norm of $\tilde{u}(y) = u(y(x))$ in $H^{2,p}(G)$ can be estimated by the quantities mentioned in the theorem, where we assume that u — and hence $\tilde{u}$ — is of class C^2.

It will be sufficient to bound the L^p-norm of $D_y^k D_y^r \tilde{u}$ where k ranges from 1 to n and r from 1 to $n-1$. The estimate for $D_y^n D_y^n \tilde{u}$ then follows from the equation.

We already know from the results of [7; p. 468] that the norm of $\tilde{u}$ in $H^{2,2}(G) \cap C^{1,\alpha}(\bar{G})$ with some suitable α can be estimated appropriately.

Let ξ and η be arbitrary functions in $C^1(\bar{G})$ vanishing on $\partial G - y(\Gamma)$. Then we obtain from (2.13) and (2.14)

$$\int_G [a_i \cdot D_x^i y^k \cdot D_y^k \eta + a_i \cdot D_x^i D_y^k y^k \cdot \eta + H \cdot \eta]\, dy = \int_{\{y^n=0\}} \tilde{\beta} \cdot \eta \, dy^n . \tag{2.15}$$

Inserting $\eta = D_y^r \xi$, $r \neq n$, in this identity and integrating by parts yields

$$\int_G [a_{ij} \cdot D_x^i y^k \cdot D_x^j y^l \cdot D_y^l D_y^r \tilde{u} \cdot D_y^k \xi - H \cdot D_y^r \xi]\, dy = \int_{\{y^n=0\}} D_y^r \tilde{\beta} \cdot \xi \, dy^n . \tag{2.16}$$

Thus, we deduce

$$\int_G b_{kl} \cdot D_y^l D_y^r \tilde{u} \cdot D_y^k \xi \, dy \leqslant K_3 \cdot \|\xi\|_{1,q,G} \tag{2.17}$$

where we have set

$$b_{kl} = a_{ij} \cdot D_x^i y^k \cdot D_x^j y^l . \tag{2.18}$$

and where q is the conjugate exponent to p, and ξ is any function belonging to $H^{1,q}(G)$ vanishing on $\partial G - y(\Gamma)$. The constant K_3 depends on the quantities mentioned in the theorem.

If $\xi \in H^{1,q}(G)$ is arbitrary, choose $\phi \in C^1(\bar{G})$, $0 \leqslant \phi \leqslant 1$, which vanishes on $\partial G - y(\Gamma)$. Then the inequality (2.17) is satisfied for $\xi \cdot \phi$, so that we obtain

$$\int_G b_{kl} \cdot D_y^l (D_y^r \tilde{u} \cdot \phi) \cdot D_y^k \xi \, dy \leqslant K_4 \cdot \|\xi\|_{1,q,G} \tag{2.19}$$

with some constant K_4 depending on K_3, $|D\tilde{u}|_{G'}$, $|D\phi|_{G'}$ and on $|b_{kl}|_G$.

Thus, we finally deduce

$$\int_G [b_{kl} \cdot D_y^l (D_y^r \tilde{u} \cdot \phi) \cdot D_y^k \xi + D_y^r \tilde{u} \cdot \phi \cdot \xi]\, dy \leqslant K_5 \cdot \|\xi\|_{1,q\,G} . \tag{2.20}$$

Since the bilinear form

$$(u, v) = \int_G [b_{kl} \cdot D_y^l u \cdot D_y^k v + u \cdot v] \, dy \tag{2.21}$$

is coercive and non-degenerate, and since the coefficients b_{kl} are continuous we conclude from [10; Theorem 5.2] that $D_y^r \tilde{u} \cdot \phi$ belongs to $H^{1,p}(G)$ and that its norm can be estimated by a constant depending on K_5, the ellipticity constant of the b_{kl}'s, and on the modulus of continuity of the coefficients.

3. – The capillarity problem with constant volume.

In a former paper [4] we considered the variational problem

$$J(v) = \int_\Omega (1 + |Dv|^2) \, dx + \int_\Omega \int_0^v H(x, t) \, dt \, dx - \int_{\partial\Omega} \beta v \, d\mathcal{H}_{n-1} \to \min \tag{3.1}$$
$$\text{in } BV(\Omega) \cap \{v \geqslant \psi\} \cap \left\{ \int_\Omega (v - \psi) \, dx = V \right\}.$$

We could prove that this problem has a solution $u \in C^{0,1}(\Omega) \cap L^\infty(\Omega)$, and u also minimizes the functional

$$J_\lambda(v) = J(v) + \lambda \cdot \int_\Omega v \, dx \tag{3.2}$$

in the convex set $BV(\Omega) \cap \{v \geqslant \psi\}$, where λ is a suitable Lagrange multiplier. We had only to assume

$$\psi \in C^{0,1}(\bar{\Omega}), \qquad H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}), \tag{3.3}$$

and

$$\frac{\partial H}{\partial t} \geqslant 0 \qquad \text{and} \qquad |\beta| \leqslant 1 - a, \ a > 0. \tag{3.4}$$

Here, we shall give sufficient conditions which imply that the variational problem (3.1) has a (unique) solution $u \in H^{2,p}(\Omega)$ for any finite p. It will be important to remark that H need not be strictly monotone in t. However, we deduce from [4] that in the following we may consider the variational problem

$$J(v) \to \min \text{ in } BV(\Omega) \cap \{v \geqslant \psi\}, \tag{3.5}$$

where H is supposed to satisfy the inequality (2.1). Moreover, we shall obviously assume that the conditions of Theorem 1.1 are fulfilled, and that ψ belongs to $H^{2,\infty}(\Omega)$. But we still have to impose a further conditions on ψ which ensures, that a solution $u \in H^{2,p}(\Omega)$ of (3.5) satisfies the boundary condition (0.3) which is absolutely necessary in order to obtain a priori estimates for the gradient. Therefore, we suppose that the relation

$$a_i(D\psi)\cdot\gamma_i \leqslant \beta \tag{3.6}$$

is valid on $\partial\Omega$.

Then, we have the following result

THEOREM 3.1. *Under the above assumptions the variational problem* (3.5) *has a solution* $u \in H^{2,p}(\Omega)$ *for any finite* p, *which is uniquely determined in that function class.*

PROOF. Let θ be the maximal monotone graph

$$\theta(t) = \begin{cases} -1, & t < 0, \\ [-1, 0], & t = 0, \\ 0, & t > 0, \end{cases} \tag{3.7}$$

and let θ_ε be a sequence of smooth monotone grphs tending to θ in such a way that

$$\theta_\varepsilon(t) = \begin{cases} -1, & t \leqslant -\varepsilon, \\ 0, & t \geqslant 0. \end{cases} \tag{3.8}$$

Furthermore, let μ be a positive constant such that

$$A\psi + H(x, \psi) \leqslant \mu \quad \text{in } \Omega. \tag{3.9}$$

Then, we consider the approximating boundary value problems

$$Au_\varepsilon + H(x, u_\varepsilon) + \mu\cdot\theta_\varepsilon(u_\varepsilon - \psi) = 0 \quad \text{in } \Omega, \qquad a_i\gamma_i = \beta \quad \text{on } \partial\Omega. \tag{3.10}$$

We shall show that the a priori estimates of Section 1 are still valid in this case, where the estimate depends on $|D\psi|_\Omega$, μ, $|u_\varepsilon|_\Omega$, and on known quantities.

1) The Lemmata 1.1-1.5 are still valid, but the constants might depend on μ.

2) The only difficulty arises in the estimate (1.49). But, since we apply this estimate with $\eta = v \cdot \max\{w\zeta^2 - h, 0\}$, we deduce that for $h \geqslant h_0$ the critical term

$$\mu \cdot \int_\Omega \theta_\varepsilon' \cdot D^k(u_\varepsilon - \psi) \cdot [a_k - \beta \cdot \gamma_k] \cdot \eta \, dx \tag{3.12}$$

in (1.47) is positive and can therefore be neglected, where h_0 depends on $|D\psi|_\Omega$ and a.

Hence, we obtain

$$|Du_\varepsilon|_\Omega \leqslant K_2 , \tag{3.13}$$

where the constant is independent of ε, provided that $|u_\varepsilon|_\Omega$ is uniformly bounded. But this follows from the strict monotonicity of H.

Thus, we deduce from Theorem 2.3 that for any $p \|u_\varepsilon\|_{2,p}$ is uniformly bounded, where we may assume without loss of generality that $\partial\Omega$, H, and β satisfy the further smoothness conditions of Theorem 2.2.

To complete the proof of Theorem 2.1, we shall show that the relation

$$\psi - \varepsilon \leqslant u_\varepsilon \tag{3.14}$$

is valid in Ω.

But this estimate is an immediate consequence of the assumptions (3.6), (3.8), and (3.9). Indeed, set $\psi_\varepsilon = \psi - \varepsilon$ and $\eta = \min(u_\varepsilon - \psi_\varepsilon, 0)$. Then, we deduce from

$$A\psi_\varepsilon + H(x, \psi_\varepsilon) + \mu \cdot \theta_\varepsilon(\psi_\varepsilon - \psi) = A\psi_\varepsilon + H(x, \psi_\varepsilon) - \mu \leqslant 0 , \tag{3.15}$$

$$\begin{aligned} &\int_\Omega \{a_i(Du_\varepsilon) - a_i(D\psi_\varepsilon)\} D^i\eta + \\ &\quad + \{H(x, u_\varepsilon) + \mu \cdot \theta_\varepsilon(u_\varepsilon - \psi) - H(x, \psi_\varepsilon) - \mu \cdot \theta_\varepsilon(\psi_\varepsilon - \psi)\} \eta \, dx + \\ &\qquad + \int_{\partial\Omega} \{a_i(D\psi_\varepsilon) \cdot \gamma_i - \beta\} \eta \, d\mathcal{H}_{n-1} \leqslant 0 . \end{aligned} \tag{3.16}$$

Hence, we obtain $\eta \equiv 0$ in view of the strict monotonicity of H.

In the limit case a subsequence of the u_ε's converges uniformly to some function $u \in H^{2,p}(\Omega)$ satisfying

$$u \geqslant \psi \tag{3.17}$$

$$Au + H(x, u) + \mu \cdot \theta(u - \psi) \ni 0 \quad \text{in } \Omega \tag{3.18}$$

and

$$a_i \cdot \gamma_i = \beta \quad \text{on } \partial\Omega . \tag{3.19}$$

But this is an equivalent formulation of the variational problem (3.5), if we restrict the variations to the convex set $H^{2,p}(\Omega) \cap \{v \geq \psi\}$, as one easily checks.

After having finished the present article the author became acquainted with a paper of Simon and Spruck [11] who proved similar results.

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