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BOUNDARY VALUE PROBLEMS FOR SURFACES OF PRESCRIBED MEAN CURVATURE

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0. Introduction

We are interested in boundary value problems for non-parametric surfaces of prescribed mean curvature, i. e. we are interested in solutions of the equation

$$(0.1)\qquad \mathrm{A}\,u+\mathrm{H}(x, u)=0$$

in a domain $\Omega \subset \mathbf{R}^n$, $n \geqq 2$, where A is the minimal surface operator in divergence form

$$(0.2)\qquad \mathrm{A}\,u=-\mathrm{D}^i(a_i(\mathrm{D}\,u)),\qquad a_i(p)=p^i(1+|p|^2)^{-1/2},$$

and where $n^{-1}\,\mathrm{H}(x, u)$ is the mean curvature of the surface $\mathscr{S}=\{(x, u(x)) : x\in\Omega\}$. The equation (0.1) has been intensively studied with two entirely different types of boundary conditions, namely, with Dirichlet boundary conditions

$$(0.3)\qquad u=\varphi \quad \text{on } \partial\Omega$$

and with Neumann boundary conditions

$$(0.4)\qquad -a_i\,\rho_i=\beta \quad \text{on } \partial\Omega,$$

where $\rho=(\rho_1, \ldots, \rho_n)$ is the outward unit normal to $\partial\Omega$, and where φ and β are given functions on the boundary. These problems are well-known under the names *Plateau's problem* and *capillarity problem*, respectively.

As the equation (0.1) is the Euler equation of the functional

$$\mathrm{I}(v)=\int_\Omega (1+|\mathrm{D}\,v|^2)^{1/2}\,dx+\int_\Omega\int_0^v \mathrm{H}(x, t)\,dt\,dx,$$

there correspond the following variational problems to Plateau's problem and to the capillarity problem

$$(0.5)\qquad \mathrm{I}(v)\to \min,\qquad \forall\, v\in \mathrm{H}^{1,1}(\Omega)\cap\{v|_{\partial\Omega}=\varphi\}$$

and

$$I(v)+\int_{\partial\Omega}\beta v\, d\mathscr{H}_{n-1}\to\min, \qquad \forall v\in H^{1,1}(\Omega), \tag{0.6}$$

where we use the standard notation for Sobolev spaces and where $\mathscr{H}_k$ is the k-dimensional Hausdorff measure.

Many mathematicians investigated these problems, and nowadays we know rather well the *sufficient* (and in some sense also *necessary*) conditions guaranteeing the existence of solutions to the variational problems (0.5) and (0.6), namely: restricting us to the case of a bounded domain Ω with C^2-boundary $\partial\Omega$ and to Lipschitz continuous mean curvature functions $H=H(x, t)$, we have to assume for the Plateau problem $\varphi\in C^0(\partial\Omega)$ and

$$|H(x, \varphi(x))|\leqq(n-1)H_{n-1}(x), \qquad \forall x\in\partial\Omega, \tag{0.7}$$

$$\frac{\partial H}{\partial t}\geqq 0 \tag{0.8}$$

and

$$\left|\int_G H_0\, dx\right|\leqq(1-\varepsilon_0)\mathbf{P}(G), \tag{0.9}$$

for any subset $G\subset\Omega$, where $H_0=H(\,.\,,0)$, ε_0 is a small positive number independent of G, and where $\mathbf{P}(G)$ denotes the *perimeter* of G in the sense of de Giorgi [5]. H_{n-1} denotes the mean curvature of $\partial\Omega$.

For the capillarity problem we need the conditions $\beta\in L^\infty(\partial\Omega)$:

$$|\beta|\leqq 1 \tag{0.10}$$

and

$$\frac{\partial H}{\partial t}\geqq\varkappa>0. \tag{0.11}$$

Under these assumptions it is known that both problems have uniquely determined solutions of class $C^{2,\alpha}$ in the interior of Ω, which are uniformly continuous resp. bounded.

Moreover, assuming $\beta\in C^{0,1}(\partial\Omega)$ and

$$|\beta|<1, \tag{0.12}$$

we may conclude that the solution of the capillarity problem is of class $H^{2,p}(\Omega)$ for any finite $p\geqq 1$ (*cf.* [7] and [25], [26] also). In both cases the regularity (up to the boundary) of the solutions increases with the regularity of the data.

The condition (0.7) relating the mean curvature of the surface with the mean curvature of the boundary of Ω is absolutely necessary for the solvability of the Plateau problem. This has already been recognized by Rado [21] for the classical Plateau problem and by Serrin [22] for the more general problem of finding a surface of prescribed mean curvature having given boundary values. Nevertheless, starting with works of de Giorgi, Miranda, and Giusti the variational problem (0.5), which only makes sense for domains satisfying the mean curvature inequality, has been generalized to the following version of Plateau's problem, namely, to find a solution $u \in \mathrm{H}^{1,1}(\Omega)$ of the variational problem

$$(0.13) \qquad \mathrm{I}(v)+\int_{\partial\Omega}|v-\varphi|\,d\mathscr{H}_{n-1}\to \min, \qquad \forall v\in \mathrm{H}^{1,1}(\Omega).$$

It is not yet known if a solution of this problem is always uniquely determined, except in the case when the condition (0.7) is valid locally on $\partial\Omega$. Then we may also conclude

$$(0.14) \qquad u(x)=\varphi(x),$$

for those points $x\in\partial\Omega$ (*cf.* [8], [20]).

To solve the variational problems (0.6) and (0.13) quite different techniques are used, and one aim of this paper is to indicate a common approach to these problems studying the corresponding *boundary value problems*.

To give the idea, let u be a solution of the problem (0.13) and let $\eta \ni \mathrm{C}^1(\Omega)$ be arbitrary. Calculating formally the first variation of the functional in (0.13) we obtain

$$(0.15) \qquad \int_{\Omega} a_i(\mathrm{D}u)\,\mathrm{D}^i\eta\,dx+\int_{\Omega}\mathrm{H}(x,u)\,\eta\,dx+\int_{\partial\Omega}\beta(u-\varphi)\,\eta\,d\mathscr{H}_{n-1}\ni 0,$$

where β is the *subdifferential* of the convex function $t\to|t|$, i. e:

$$(0.16) \qquad \beta(t)=\begin{cases} -1, & t<0,\\ [-1,1], & t=0,\\ 1, & t>0.\end{cases}$$

The "equation" (0.15) says that the *multivalued* left-hand sight—representing a set—contains zero. Thus, we are led to the following boundary value problem

$$(0.17) \qquad \begin{cases} \mathrm{A}u+\mathrm{H}(x,u)=0 & \text{in } \Omega,\\ -a_i\,\rho_i\in\beta(u-\varphi) & \text{on } \partial\Omega,\end{cases}$$

where β is a *maximal monotone graph* (1).

This is a boundary value problem of capillary type, and we shall try to solve it using the techniques appropriate to those problems. The crucial step is to prove *a priori* estimates for the gradient of smooth solutions u_ε of approximating problems

$$(0.18) \qquad \begin{cases} \mathrm{A}u_\varepsilon+\mathrm{H}(x,u_\varepsilon)=0 & \text{in } \Omega,\\ -a_i\,\rho_i=\beta_\varepsilon(u_\varepsilon-\varphi) & \text{on } \partial\Omega,\end{cases}$$

corresponding to smooth monotone approximations β_ε of β.

(1) For uniformly elliptic (linear) operators A boundary value problems of this type have already been studied in [2].

Unfortunately, we cannot apply the methods of [7] since the estimates are not to depend on $|\partial\beta_\varepsilon/\partial t|$; instead we shall use a version of Simon's and Spruck's proof for the gradient estimate of capillary surface [25]. The disadvantage of that proof is, that Simon and Spruck have to assume H to satisfy the strict inequality (0.11) and not only the less rigorous condition (0.7). Thus, the minimal surface case is excluded in their setting.

Our methods will also be applicable to variational problems of the form

$$(0.19) \qquad I(v) \to \min, \qquad \forall v \in H^{1,1}(\Omega) \cap \{\varphi_1 \leqq v|_{\partial\Omega} \leqq \varphi_2\},$$

where constraints are given on the boundary. We shall prove the existence of uniformly Lipschitz continuous solutions in this case under suitable assumptions on the data. In the physical interesting case $n=2$ we are able to consider general mean curvature functions H satisfying (0.7) instead of (0.11). The *parametric* analogue of (0.19) for minimal surfaces has been considered by Hildebrandt and Nitsche [15].

In Section 6 we also prove *local* regularity results for solutions of *mixed boundary value problems* which have been studied by Giusti [14].

Finally, let us mention the striking result that in the case of strictly increasing mean curvature H, i. e. in the case when (0.11) is valid, the solutions of the generalized Plateau problem (0.13) are uniformly Hölder continuous provided φ is Lipschitz and $\partial\Omega$ of classe C^4. This result also holds locally near the boundary if the assumptions are only locally fulfilled.

At this place we should like to thank Leon Simon for his interest in this paper and for some stimulating discussions.

1. Notations and preliminaries

In the next section we shall be interested in *a priori* estimates for the gradient of smooth solutions of the boundary value problem

$$(1.1) \qquad \begin{cases} Au + H(x, u) = 0 & \text{in } \Omega, \\ -a_i \rho_i = \beta(x, u-\varphi) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbf{R}^n$, $n \geqq 2$, with boundary $\partial\Omega$ of class C^4. $H = H(x, t)$, $\beta = \beta(x, t)$, and $\varphi = \varphi(x)$ are given Lipschitz continuous functions satisfying the conditions

$$(1.2) \qquad \frac{\partial H}{\partial t} \geqq \varkappa > 0,$$

$$(1.3) \qquad \frac{\partial \beta}{\partial t} \geqq 0,$$

and

$$(1.4) \qquad |\beta| \leqq 1 - a, \qquad a > 0.$$

d will denote the distance function, $d(x)=\operatorname{dist}(x, \partial\Omega)$. We shall derive *local* estimates near the boundary, i. e. we shall work in a neighbourhood $\Omega_\delta = B_\delta(x_0) \cap \Omega$ of a point $x_0 \in \partial\Omega$. δ will be assumed small enough to ensure that d is of class C^4 in Ω_δ.

Let L, M, and N be constants such that

$$\sup_{\Omega_\delta} \left| \frac{\partial}{\partial x} \beta(x, u-\varphi) \right| \leqq L, \tag{1.5}$$

$$\sup_{\Omega_\delta} \left\{ |u| + |H(x, u)| + \left| \frac{\partial}{\partial x} H(x, u) \right| \right\} \leqq M, \tag{1.6}$$

and

$$\sup_{\partial\Omega \cap \partial\Omega_\delta} \left| \frac{\partial}{\partial x} \varphi \right| \leqq N, \tag{1.7}$$

The definitions that will follow are almost identical to the corresponding ones in [25]. We repeat them for the convenience of the reader.

We define $(Du)_T(x)$ to be the tangential derivative of u relative to the hypersurface $\{\xi \in \Omega_\delta : d(x)=d(\xi)\}$, i. e.:

$$(Du)_T(x) = Du(x) - [Du(x) . Dd(x)]\, Dd(x),$$

v_T is defined on Ω_δ by

$$v_T = (1 + |(Du)_T|^2)^{1/2}.$$

δ is assumed to be small enough to ensure that we can introduce local coordinates $y=y(x)$ in Ω_δ which "flattens" $\partial\Omega$ near x_0. We may choose $y=(y^1, \ldots, y^n)$ to be a diffeomorphism from Ω_δ into $\mathbf{R}^n$ such that

$$y^i \in C^3(\Omega_\delta), \qquad i=1, \ldots, n-1,$$
$$y^n \equiv d \quad \text{on } \Omega_\delta,$$

and such that the transposed Jacobian matrix J [i. e. the matrix with ith row $(\partial y/\partial x^i)(x)$] satisfies

$$J^t(x)\, J(x) = (e^{ij}(y)), \qquad x \in \Omega_\delta,$$

where

$$e^{in}=0, \quad i=1, \ldots, n-1, \qquad e^{nn}=1 \tag{1.8}$$

and

$$\lambda |\xi|^2 \leqq e^{ij}(y)\, \xi^i \xi^j, \qquad \xi \in \mathbf{R}^n, \quad y \in G_\delta,$$

for some positive constant λ, where G_δ is the image of Ω_δ under the transformation $y=y(x)$. Λ will denote a constant such that

$$\lambda^{-1/2} + \left| \frac{\partial y}{\partial x} \right| + \left| \frac{\partial^2 y}{\partial x\, \partial x} \right| + \left| \frac{\partial^3 y}{\partial x\, \partial x\, \partial x} \right| \leqq \Lambda,$$

uniformly in Ω_δ.

If f is a function defined on Ω_δ then $\tilde{f}$ is defined on G_δ by $\tilde{f}(y)=f(x)$. For functions f, $g \in C^1(\Omega_\delta)$ we have

$$D^i_x f(x) . D^i_x g(x) = e^{ij}(y) . D^i_y \tilde{f}(y) . D^j_y \tilde{g}(y), \tag{1.9}$$

in view of the definition of the e^{ij}'s.

μ will denote the Jacobian of the transformation $y \to x$, i. e.:

$$\mu(y) = (\det(e^{ij}))^{-1/2}, \qquad y \in G_\delta.$$

We also introduce the following functions on G_δ:

$$\tilde{v}(y) = (1 + e^{ij}(y) . D^i_y \tilde{u} . D^j_y \tilde{u})^{1/2} \equiv v(x) \equiv (1 + |Du(x)|^2)^{1/2},$$

$$\tilde{v}_T(y) = \left(1 + \sum_{i,j=1}^{n-1} e^{ij}(y) . D^i_y \tilde{u} . D^j_y \tilde{u}\right)^{1/2} = v_T(x),$$

$$\hat{v}_T(y) = \left(1 + \sum_{i=1}^{n-1} |D^i_y \tilde{u}^2|\right)^{1/2},$$

$$\tilde{\nu}_i = e^{ij}(D^j_y \tilde{u}/\tilde{v}), \qquad i = 1, \ldots, n,$$

$$\chi = (\tilde{v}_T/\tilde{v})^2,$$

and

$$g^{ij} = e^{ij} - \tilde{\nu}_i . \tilde{\nu}_j, \qquad i, j = 1, \ldots, n.$$

Note the relations

$$\chi = g^{nn} = 1 - \tilde{\nu}_n^2. \tag{1.10}$$

In terms of the transformed coordinates (1.1) becomes

$$\left\{\begin{aligned} -D^i_y(\mu . \tilde{\nu}_i) + \mu . \tilde{H}(y, \tilde{u}) = 0 \quad &\text{on } G_\delta, \\ -\tilde{\nu}_n = \tilde{\beta}(y, \tilde{u} - \tilde{\varphi}) \quad &\text{on } \Gamma, \end{aligned}\right. \tag{1.11}$$

where

$$\Gamma = \tilde{G}_\delta \cap \{y \in \mathbf{R}^n : y^n = 0\}.$$

This can most easily be seen by writing equation (1.1) in integral form, namely,

$$\int_\Omega a_i . D^i \zeta \, dx + \int_\Omega H(x, u) . \zeta \, dx + \int_{\partial\Omega} \beta(x, u - \varphi) . \zeta \, d\mathscr{H}_{n-1} = 0, \tag{1.12}$$

for all Lipschitz continuous ζ with support in $B_\delta(x_0)$. Making the transformation $y = y(x)$ we obtain

$$\int_{G_\delta} \tilde{v}^{-1} . e^{ij} D^i_y \tilde{u} . D^j_y \tilde{\zeta} . \mu \, dy + \int_{G_\delta} \tilde{H}(y, \tilde{u}) . \tilde{\zeta} . \mu \, dy + \int_\Gamma \tilde{\beta}(y, \tilde{u} - \tilde{\varphi}) . \tilde{\zeta} . \mu \, d\mathscr{H}_{n-1} = 0, \tag{1.13}$$

or equivalently

$$(1.14)\qquad \int_{G_\delta} \mu.\tilde{\nu}_i.D_y^i \eta\, dy+\int_{G_\delta} \mu.\tilde{H}(y, \tilde{u}).\eta\, dy+\int_{\Gamma} \mu.\tilde{\beta}(y, \tilde{u}-\tilde{\varphi}).\eta\, d\mathscr{H}_{n-1}=0$$

for all Lipschitz continuous functions η with support in U_δ, $U_\delta=\{y(x): x\in B_\delta(x_0)\}$. The relations (1.11) then follow immediately.

We now present some inequalities which will be needed in the next section. First the Cauchy inequalities

$$(1.15)\qquad a_{ij}\xi^i\eta^j \leqq (a_{ij}\xi^i\xi^j)^{1/2}.(a_{ij}\eta^i\eta^j)^{1/2},$$

valid for all $\xi, \eta\in\mathbf{R}^n$ and any symmetric positive semi-definite matrix (a_{ij}), and

$$(1.16)\qquad ab\leqq \varepsilon a^2+(4\varepsilon)^{-1}b^2,$$

valid for real a, b and $\varepsilon>0$.

Next we have the identity

$$(1.17)\qquad \begin{cases} D_y^l\tilde{\nu}_i=D_y^l\gamma^{mi}\gamma^{mj}(D_y^j\tilde{u}/\tilde{v})+\tilde{v}^{-1}g^{ij}D_y^l D_y^j\tilde{u}+\gamma_{ks}g^{ik}D_y^l\gamma^{sr}(D_y^r\tilde{u}/\tilde{v}),\\ \qquad i, l=1, \ldots, n,\end{cases}$$

where (γ^{ij}) is any differentiable $n\times n$ matrix on G_δ satisfying

$$\gamma^{ij}=\gamma^{ji},\quad \gamma^{ti}\gamma^{tj}=e^{ij},\qquad i, j=1, \ldots, n,$$
$$\gamma^{ni}=0,\quad i=1, \ldots, n-1,\qquad \gamma^{nn}=1,$$

and where $(\gamma_{ij})=(\gamma^{ij})^{-1}$.

In view of (1.8) we may choose

$$(\gamma^{ij})=(e^{ij})^{1/2}.$$

The coefficients γ^{ij} will then be of class C^2 in G_δ and their derivatives up to order 2 will be bounded in terms of n and Λ.

The relation (1.17) can be easily derived from the following identities

$$(1.18)\qquad \begin{cases} D_y^l(\gamma^{mj}.D_y^j\tilde{u}/\tilde{v})=\tilde{v}^{-1}\{\delta_{ms}-\gamma^{mj}\gamma^{sr}.(D_y^j\tilde{u}/\tilde{v}).(D_y^r\tilde{u}/\tilde{v})\}.D_y^l(\gamma^{ss'}D_y^{s'}\tilde{u}),\\ \qquad l, m=1, \ldots, n,\end{cases}$$

and

$$(1.19)\qquad \gamma_{km}g^{kj}=\gamma^{mj}-\gamma^{mr}(D_y^r\tilde{u}/\tilde{v})\tilde{\nu}_j,\quad m, j=1, \ldots, n.$$

δ_{ms} denotes the Kronecker symbol.

Due to the fact that $D_y^k\gamma^{ij}=0$ if i or j are equal to n, we derive from (1.19) using (1.15):

$$(1.20)\qquad |D_y^\sigma\tilde{\nu}_i|<c.(\mathscr{C}+\chi^{1/2}),\qquad i=1, \ldots, n,\qquad \sigma=1, \ldots, n-1,$$

where the non-negative function $\mathscr{C}$ is defined on G_δ by

$$(1.21)\qquad \mathscr{C}^2=\tilde{v}^{-2}\sum_{\sigma=1}^{n-1} g^{ij}\,D_y^\sigma D_y^i\tilde{u}.D_y^\sigma D_y^j\tilde{u},$$

and where the constant c depends on n and Λ (in the following we shall denote varying constants with the same letter c).

Moreover, since

$$D_y^n\tilde{\nu}_n=D_y^n(\mu^{-1}.\mu.\tilde{\nu}_n)=(D_y^n\mu^{-1}).\mu.\tilde{\nu}_n+\mu^{-1}.D_y^n(\mu.\tilde{\nu}_n)$$
$$=(D_y^n\mu^{-1}).\mu.\tilde{\nu}_n-\mu^{-1}\sum_{i=1}^{n-1}D_y^i(\mu.\tilde{\nu}_i)+\tilde{H}(y,\tilde{u})$$

we obtain from (1.20):

$$(1.22)\qquad |D_y^n\tilde{\nu}_n|<c(\mathscr{C}+1),$$

where c depends on n, M, and Λ.

The quantity $\mathscr{C}^2$ of (1.21) satisfies

$$(1.23)\qquad \tilde{v}^{-1}g^{ij}D_y^j\hat{v}_T.D_y^i\hat{v}_T\leqq\tilde{v}.\mathscr{C}^2,$$

as is easily calculated.

2. Gradient estimates

We are going to prove that the gradient of u or equivalently the gradient of $\tilde{u}$ is uniformly bounded in some boundary neighbourhood Ω_δ resp. G_δ. First we prove that the tangential derivatives of u are uniformly bounded.

THEOREM 2.1. — *Let $u\in C^2(\bar{\Omega})$ be a solution of the boundary value problem* (1.1). *Then, the tangential gradient of u, $(Du)_T$, is uniformly bounded in a suitable boundary neighbourhood Ω_δ in terms of the quantities* L, M, N, Λ, δ, *and* $\varkappa$.

Proof. — We use the identity (1.14) with $\eta=\mu^{-1}\zeta$, where ζ is of class $C_c^{0,1}(U_\delta)$ and obtain

$$(2.1)\qquad \int_{G_\delta}\{\tilde{\nu}_i D_y^i\zeta+\mu.D_y^i\mu^{-1}.\tilde{\nu}_i\zeta+\tilde{H}(y,\tilde{u}).\zeta\}\,dy+\int_\Gamma\tilde{\beta}(y,\tilde{u}-\tilde{\varphi})\zeta\,d\mathscr{H}_{n-1}=0.$$

Replacing ζ by $-\sum_{\sigma=1}^{n-1}D_y^\sigma(\zeta.D_y^\sigma\tilde{u})$, where $\zeta\in C_c^{0,1}(U_\delta)$ is arbitrary, and integrate partially we get

$$(2.2)\qquad \int_{G_\delta}\Big\{D_y^\sigma\tilde{\nu}_i.D_y^i(\zeta.D_y^\sigma\tilde{u})+D_y^\sigma(\mu.D_y^i\mu^{-1}).\tilde{\nu}_i.D_y^\sigma\tilde{u}.\zeta$$
$$+\mu.D_y^i\mu^{-1}.D_y^\sigma\tilde{\nu}_i.D_y^\sigma\tilde{u}.\zeta+\frac{\partial\tilde{H}}{\partial y^\sigma}.D_y^\sigma\tilde{u}.\zeta+\frac{\partial\tilde{H}}{\partial t}|D_y^\sigma\tilde{u}|^2.\zeta\Big\}\,dy$$
$$=-\int_\Gamma\Big\{\frac{\partial\tilde{\beta}}{\partial y^\sigma}.D_y^\sigma\tilde{u}.\zeta+\frac{\partial\tilde{\beta}}{\partial t}.D_y^\sigma(\tilde{u}-\tilde{\varphi}).D_y^\sigma\tilde{u}.\zeta\Big\}\,d\mathscr{H}_{n-1},$$

where we sum over Greek indices from 1 to $n-1$.

Inserting $\zeta=\max(\hat{v}_T \eta^2-h, 0)$ in this identity, where $\eta\in C_c^1(U_\delta)$, $0\leqq\eta\leqq 1$, is a cut-off function and h is a positive number greater than $\max(N, 1)$, and using the relations (1.2), (1.3), (1.17), and (1.20) we derive the inequality

$$(2.3)\quad \int_{G_\delta}\{\varkappa.|D_y^\sigma\tilde{u}|^2.\zeta+\tilde{v}^{-1}g^{ij}D_y^\sigma D_y^j\tilde{u}.D_y^i(\zeta.D_y^\sigma\tilde{u})\}\,dy$$
$$\leqq\int_{G_\delta}-\{D_y^\sigma\gamma^{mi}\gamma^{mj}(D_y^j\tilde{u}/\tilde{v})+\gamma_{ks}g^{ik}D_y^\sigma\gamma^{sr}(D_y^r\tilde{u}/\tilde{v})\}$$
$$\times D_y^i(\zeta.D_y^\sigma\tilde{u})\,dy+c_1.\int_{G_\delta}\hat{v}_T.\zeta\,dy$$
$$+c_1.\int_{G_\delta}(\mathscr{C}+\chi^{1/2}).\hat{v}_T.\zeta dy+c_1.\int_\Gamma\hat{v}_T.\zeta\,d\mathscr{H}_{n-1}.$$

Note that the second term of the boundary integral in (2.2) is nonnegative in view of (1.3) and the definition of h.

The constant c_1 in (2.3) depends on L, M, and Λ.

Moreover, we remark that no boundary integral occurs in (2.3) if β only depends on t.

Denoting the first integral on the right-hand side of (2.3) by I, and taking the relations (1.15), (1.16), (1.21),

$$(2.4)\qquad 2|D_y^\sigma\tilde{u}|^2.\zeta\geqq\hat{v}_T^2.\zeta,$$

and

$$D_y^i\hat{v}_T=\hat{v}_T^{-1}.D_y^iD_y^\sigma\tilde{u}.D_y^\sigma\tilde{u},\qquad i=1,\ \ldots,\ n,$$

into account we deduce from (2.3):

$$(2.5)\quad \int_{G_\delta}\{\hat{v}_T^2.\zeta+\tilde{v}.\mathscr{C}^2.\zeta\}\,dy+\int_{A(h,\eta)}\chi^{1/2}.g^{ij}D_y^i\hat{v}_T.D_y^j\hat{v}_T.\eta^2\,dy$$
$$\leqq c_1.I+c_1.\int_\Gamma\hat{v}_T.\zeta\,d\mathscr{H}_{n-1}+c_1.\int_{G_\delta}\hat{v}_T.\zeta\,dy+c_2.\int_{A(h,\eta)}\hat{v}_T.\chi.\tilde{v}\,dy,$$

where the constant c_1 depends on $\varkappa$, L, M, Λ, where c_2 depends on the same quantities and on $|D\eta|$, and where

$$A(h,\eta)=\{y\in G_\delta:\ \hat{v}_T.\eta^2>h\}.$$

Dividing the integral I into the parts

$$I_1=-\int_{G_\delta}D_y^\sigma\gamma^{mi}\gamma^{mj}(D_y^j\tilde{u}/\tilde{v}).D_y^i(\zeta.D_y^\sigma\tilde{u})\,dy$$

and

$$I_2=-\int_{G_\delta}\gamma_{ks}g^{ik}D_y^\sigma\gamma^{sr}(D_y^r\tilde{u}/\tilde{v}).D_y^i(\zeta.D_y^\sigma\tilde{u})\,dy,$$

and noting particularly the relations (1.18), (1.19), and

$$D_y^\sigma \gamma^{ij}=0, \qquad \text{if} \quad i \text{ or } j \text{ are equal to } n,$$

we transform the term I_1 *via* integration by parts as follows

$$\begin{aligned}(2.6)\quad I_1=\sum_{i=1}^{n-1}\int_{G_\delta}\{&D_y^i D_y^\sigma \gamma^{mi}\gamma^{mj}(D_y^j \tilde{u}/\tilde{v}).D_y^\sigma \tilde{u}.\zeta\\ &+D_y^\sigma \gamma^{mi}.[\delta_{ms}-\gamma^{mj}\gamma^{sr}.(D_y^j \tilde{u}/\tilde{v}).(D_y^r \tilde{u}/\tilde{v})]\\ &\times(D_y^t \tilde{u}/\tilde{v}).D_y^i \gamma^{st}.D_y^\sigma \tilde{u}.\zeta+\tilde{v}^{-1}D_y^\sigma \gamma^{mi}.\gamma_{km}g^{kj}.D_y^i D_y^j \tilde{u}.D_y^\sigma \tilde{u}.\zeta\}\,dy.\end{aligned}$$

Then, using (1.15), (1.16), and (1.21) we obtain

$$(2.7)\qquad I_1 \leqq \varepsilon.\int_{G_\delta}\tilde{v}\mathscr{C}^2\zeta\,dy+c_1.\int_{G_\delta}\hat{v}_T.\zeta\,dy,$$

where ε is an arbitrary positive constant and where c_1 depends on Λ and ε.

Similarly I_2 is estimated by

$$\begin{aligned}(2.8)\quad I_2\leqq \varepsilon.\int_{G_\delta}\tilde{v}\mathscr{C}^2\zeta\,dy+c_1.\int_{G_\delta}\hat{v}_T\zeta\,dy&\\ +\varepsilon.\int_{A(h,\eta)}\chi^{1/2}g^{ij}D_y^i\hat{v}_T D_y^j\hat{v}_T\eta^2\,dy+c_2.&\int_{A(h,\eta)}\hat{v}_T.\chi.\tilde{v}\,dy,\end{aligned}$$

with constants c_1, c_2 depending on ε, Λ, resp. on ε, Λ, and $|D\eta|$.

Combining the relations (2.5), (2.7), and (2.8), and using the estimate

$$(2.9)\qquad g^{ij}D_y^i\zeta D_y^j\zeta \leqq 8\,g^{ij}D_y^i\hat{v}_T D_y^j\hat{v}_T.\eta^2+8\,g^{ij}D_y^i\eta D_y^j\eta.\hat{v}_T^2.\eta^2,$$

we then conclude

$$\begin{aligned}(2.10)\quad \int_{G_\delta}\hat{v}_T^2.\zeta\,dy+\int_{G_\delta}\tilde{v}\mathscr{C}^2.\zeta\,dy+\int_{G_\delta}\chi^{1/2}g^{ij}D_y^i\zeta D_y^j\zeta\,dy&\\ \leqq c_1.\int_{G_\delta}\hat{v}_T.\zeta\,dy+c_1.\int_\Gamma\hat{v}_T.\zeta\,d\mathscr{H}_{n-1}+c_2.&\int_{A(h,\eta)}\hat{v}_T.\chi\,\tilde{v}\,dy,\end{aligned}$$

where c_1 depends on $\varkappa$, L, M, N, Λ, δ, and c_2 in addition on $|D\eta|$, and where we note that the boundary term vanishes if $\beta=\beta(t)$.

To estimate the boundary integral in (2.10) in the general case when $\beta=\beta(x,t)$ we use (1.4) and the inequality

$$(2.11)\qquad \int_\Gamma\chi^{1/2}.f.\tilde{v}\,d\mathscr{H}_{n-1}\leqq c_1.\int_{G_\delta}\{\chi.f+\chi(g^{ij}D_y^i f D_y^j f)^{1/2}+\chi^{1/2}f.\mathscr{C}\}\tilde{v}\,dy,$$

valid for all non-negative functions $f \in C_c^{0,1}(U_\delta)$, where c_1 depends on n, Λ, M, and a. Note that in view of (1.4) the estimates

$$\hat{v}_T \leqq \tilde{v} \leqq c_1 . \hat{v}_T, \tag{2.12}$$

hold on Γ, where the constant c_1 depends on a. A proof of (2.11) is given in [25], formula (2.13).

Inserting $f=\zeta$ in (2.11) we conclude

$$\int_\Gamma \hat{v}_T . \zeta \, d\mathscr{H}_{n-1} \leqq c_1 . \int_{G_\delta} \zeta \chi \tilde{v} \, dy + \varepsilon . \int_{G_\delta} \chi^{1/2} g^{ij} D_y^i \zeta D_y^j \zeta \, dy + \varepsilon . \int_{G_\delta} \tilde{v} \mathscr{C}^2 \zeta \, dy + c_1 . \int_{A(h,\eta)} \hat{v}_T . \chi \tilde{v} \, dy. \tag{2.13}$$

Thus, we finally deduce from (2.10):

$$\int_{G_\delta} \hat{v}_T^2 . \zeta \, dy + \int_{G_\delta} \tilde{v} \mathscr{C}^2 . \zeta \, dy + \int_{G_\delta} \chi^{1/2} g^{ij} D_y^i \zeta D_y^j \zeta \, dy \leqq c_1 \int_{G_\delta} \hat{v}_T . \zeta \, dy + c_2 . \int_{A(h,\eta)} \hat{v}_T \chi \tilde{v} \, dy, \tag{2.14}$$

where the constants c_1, c_2 depend on the quantities mentioned above, and where we note that only c_2 depends on $|D\eta|$.

Since the support of ζ is contained in the set where $\hat{v}_T$ is greater or equal to h, we may drop the first integral on the right-hand side of (2.14) provided that $h \geqq 2.c_1$.

Assuming this in the following we obtain

$$\int_{G_\delta} \hat{v}_T^2 \zeta \, dy + \int_{G_\delta} \tilde{v} \mathscr{C}^2 . \zeta \, dy + \int_{G_\delta} \chi^{1/2} g^{ij} D_y^i \zeta D_y^j \zeta \, dy \leqq c_2 . \int_{A(h,\eta)} \hat{v}_T . \chi . \tilde{v} \, dy. \tag{2.15}$$

We shall use this inequality twice. First we observe that

$$\int_{G_\delta} \zeta \hat{v}_T^2 \, dy \leqq c_2 . \int_{A(h,\eta)} \hat{v}_T^2 \, dy, \tag{2.16}$$

since

$$\hat{v}_T . \chi . \tilde{v} \leqq c . \hat{v}_T^2.$$

From the definition of $\zeta = \max(\hat{v}_T . \eta^2 - h, 0)$ we then conclude

$$(k-h) . \int_{A(k,\eta)} \hat{v}_T^2 \, dy \leqq c_2 . \int_{A(h,\eta)} \hat{v}_T^2 \, dy, \tag{2.17}$$

for all $k \geqq h \geqq h_0 = \max\{1, N, 2.c_1\}$.

But this implies in view of a lemma due to Stampacchia (*cf.* [30], Lemma 4.1) that $\hat{v}_T . \eta^2$ is p-summable over Ω with respect to the measure $\hat{v}_T^2 dy$ for any finite p provided that $\int_{A(h_0,\eta)} \hat{v}_T^2 \, dy$ is bounded.

Hence we obtain

$$\hat{v}_T \in L^p(G_{\delta/2}), \qquad \forall\, 1 \leqq p < \infty, \tag{2.18}$$

provided

$$\hat{v}_T \in L^2(G_{(3/4)\,\delta}). \tag{2.19}$$

To prove (2.19) we use (2.2) with

$$\zeta = \frac{\max(\hat{v}_T - h,\, 0)}{\hat{v}_T} . \eta^2 \equiv w . \eta^2,$$

where η is a cut-off function, and where h is a fixed number greater than $\max(1, N)$. Let $A(h) = \{x \in G_\delta : \zeta(x) > 0\}$. We then deduce using (1.23):

$$\begin{aligned} \int_{G_\delta} \hat{v}_T^2 . \zeta\, dy + \int_{G_\delta} \tilde{v}\mathscr{C}^2 . \zeta\, dy + \int_{A(h)} \hat{v}_T^2 . \chi^{1/2} g^{ij} D_y^i w . D_y^j w . \eta^2\, dy \\ \leqq c . I + c . \int_\Gamma \hat{v}_T . \zeta\, d\mathscr{H}_{n-1} + c . \int_{G_\delta} \hat{v}_T . \zeta\, dy + c . \int_{A(h)} \hat{v}_T . dy, \end{aligned} \tag{2.20}$$

where the constant c depends on h, $|D\eta|$, and known quantities. The symbol $I = I_1 + I_2$ has the same meaning as in (2.5). I_1 can be estimated by

$$I_1 \leqq c . \int_{G_\delta} \hat{v}_T . \zeta\, dy + \varepsilon . \int_{G_\delta} \tilde{v}\mathscr{C}^2 . \zeta\, dy \tag{2.21}$$

and I_2 by

$$\begin{aligned} I_2 \leqq \varepsilon . \int_{G_\delta} \tilde{v}\mathscr{C}^2 . \zeta\, dy + \varepsilon . \int_{A(h)} \hat{v}_T^2 . \chi^{1/2} g^{ij} D_y^i w . D_y^j w\, \eta^2\, dy \\ + c \int_{G_\delta} \hat{v}_T . \zeta\, dy + c . \int_{A(h)} \hat{v}_T\, dy. \end{aligned} \tag{2.22}$$

The boundary integral in (2.20) can be estimated by applying (2.11) with ζ in place of f to obtain

$$\begin{aligned} \int_\Gamma \hat{v}_T . \zeta\, d\mathscr{H}_{n-1} \leqq c . \int_{G_\delta} \hat{v}_T . \zeta\, dy + c . \int_{A(h)} dy \\ + \varepsilon . \int_{G_\delta} \tilde{v}\mathscr{C}^2 . \zeta\, dy + \varepsilon \int_{A(h)} \hat{v}_T^2 . \chi^{1/2} . g^{ij} D_y^i w . D_y^j w . \eta^2\, dy. \end{aligned} \tag{2.23}$$

Combining the inequalities (2.20)-(2.23) we conclude

$$\int_{G_\delta} \hat{v}_T^2 . \zeta\, dy \leqq c . \int_{G_\delta} \hat{v}_T\, dy \leqq c . \int_{G_\delta} \tilde{v}\, dy. \tag{2.24}$$

Hence

$$\int_{G_\delta} \hat{v}_T^2 . \eta^2 \, dy$$

will be bounded provided that

$$\int_{G_\delta} \tilde{v} \, dy$$

is bounded.

But this result follows immediately from the fact that

$$(2.25) \qquad \int_\Omega (1+|Du|^2)^{1/2} \, dx \leqq \text{const.},$$

for any surface we consider. Assuming u to be bounded, (2.25) can easily be derived by multiplying equation (1.1) with u and integrating by parts.

Now, we return to inequality (2.15). In order to deduce the boundedness of $\hat{v}_T$ we need some kind of Sobolev inequality.

LEMMA 2.1. — *For each non-negative function* $f \in C_c^{0,1}(U_\delta)$ *we have*

$$(2.26) \quad \left\{ \int_{G_\delta} f^{2\alpha} . \chi^{2\alpha-1} . \chi . \tilde{v} \, dy \right\}^{1/\alpha} \leqq c_1 . \int_{G_\delta} f^2 . \chi . \tilde{v} \, dy$$
$$+ c_1 . \int_{G_\delta} \chi^{1/2} . (g^{ij} D_y^i f . D_y^j f)^{1/2} . f . \chi^{1/2} . \tilde{v} \, dy + c_1 . \int_{G_\delta} \mathscr{C} . f^2 . \chi^{1/2} . \tilde{v} \, dy,$$

where $\alpha = n/(n-1)$ *and where the constant* c_1 *depends on* n, Λ, *and* M.

Lemma 2.1 is proved in [25], *cf.* the formula following (2.16).

Applying (2.26) with ζ in place of f, we conclude from (2.15):

$$(2.27) \quad \int_{G_\delta} \hat{v}_T^2 . \zeta \, dy + \left\{ \int_{G_\delta} \zeta^{2\alpha} . \chi^{2\alpha-1} . \chi . \tilde{v} \, dy \right\}^{1/\alpha} \leqq c_2 . \int_{A(h,\eta)} \{ \hat{v}_T + \hat{v}_T^2 + \hat{v}_T^3 \} \chi . \tilde{v} \, dy,$$

or, if we express every integral in terms of the measure $d\mu = \chi . \tilde{v} \, dy$, and if we use the trivial estimate $\zeta \leqq \hat{v}_T$, we finally obtain

$$(2.28) \quad \int_{G_\delta} \chi^{-1/2} \zeta^2 \, d\mu + \left\{ \int_{G_\delta} \zeta^{2\alpha} . \chi^{2\alpha-1} \, d\mu \right\}^{1/\alpha} \leqq c_2 . \int_{A(h,\eta)} \{ \hat{v}_T + \hat{v}_T^2 + \hat{v}_T^3 \} \, d\mu$$

On the other hand, we have with $q = 2(3n+2)/(3n+1)$:

$$(2.29) \qquad \zeta^q = \zeta^{4(n+1)/(3n+1)} . \chi^{-(n+1)/(3n+1)} . \zeta^{2n/(3n+1)} . \chi^{(n+1)/(3n+1)},$$

from which we deduce

$$(2.30) \quad \int_{G_\delta} \zeta^q \, d\mu \leqq \left(\int_{G_\delta} \zeta^2 . \chi^{-1/2} \, d\mu \right)^{2(n+1)/(3n+1)} . \left(\int_{G_\delta} |\zeta|^{2\alpha} . \chi^{2\alpha-1} \, d\mu \right)^{(n-1)/(3n+1)},$$

where we used the Hölder inequality with

$$p=\frac{3n+1}{2n+2},\qquad p'=\frac{3n+1}{n-1},\qquad \frac{1}{p}+\frac{1}{p'}=1.$$

Moreover, noting that

$$\frac{n-1}{3n+1}=\frac{n-1}{n}\cdot\frac{n}{3n+1}=\frac{1}{\alpha}\cdot\frac{n}{3n+1},$$

and that

$$r=\frac{2(n+1)}{3n+1}+\frac{n}{3n+1}=\frac{3n+2}{3n+1}>1,$$

we derive from (2.28) and (2.30):

$$\int_{G_\delta}\zeta^q\,d\mu\leqq c_2^r\cdot\left\{\int_{A(h,\eta)}(\hat{v}_T+\hat{v}_T^2+\hat{v}_T^3)\,d\mu\right\}^r. \tag{2.31}$$

We have proved before that $\hat{v}_T$ belongs to $L^p(G_{\delta/2})$ for any finite p. Thus, choosing the support of η sufficiently small, $\operatorname{supp}\eta\subset G_{\delta/2}$, we conclude

$$\int_{G_\delta}\zeta^q\,d\mu\leqq c\left(\int_{A(h,\eta)}d\mu\right)^{((p-1)/p).r}, \tag{2.32}$$

where the constant c depends on r, p, $|D\eta|$ and on known quantities.

Denoting

$$|A(h,\eta)|=\int_{A(h,\eta)}d\mu$$

and $\gamma=((p-1)/p).r$, we get for $k>h>h_0$:

$$(k-h)^2.|A(k,\eta)|\leqq c.|A(h,\eta)|^\gamma, \tag{2.33}$$

where γ is greater than 1 if we choose p sufficiently large.

The boundedness of ζ and hence of $\hat{v}_T$ now follows immediately from a lemma due to Stampacchia (*cf.* [30], Lemma 4.1), which we have already used before.

Theorem 2.1 is thus proved. Let us note the following remark.

REMARK 2.1. — *The result of Theorem* 2.1 *is also valid if we replace the boundary condition* $-a_i\rho_i=\beta(x,u-\varphi)$ *by*

$$-a_i\rho_i=\beta_1(x,u-\varphi_1)+\beta_2(x,u-\varphi_2), \tag{2.34}$$

where β_i *and* φ_i, $i=1,2$, *are Lipschitz continuous functions satisfying* $\partial\beta_i/\partial t\geqq 0$ *for* $i=1,2$, *and*

$$|\beta_1(x,u-\varphi_1)+\beta_2(x,u-\varphi_2)|\leqq(1-a),\qquad a>0.$$

The tangential gradient of u is then bounded in terms of a, $|D\varphi_i|$, $|(\partial/\partial x)\beta_i|$, $i=1, 2$, *and the quantities mentioned in Theorem* 2.1. *We note particularly that the estimate does not depend on* a *if* $\beta_i=\beta_i(t)$ *for* $i=1, 2$.

An immediate corollary of Theorem 2.1 and of Remark 2.1 is the following:

COROLLARY 2.1. — *Let the assumptions of Theorem* 2.1 *or of Remark* 2.1 *hold. Then*

$$u\in C^{0,\alpha}(\overline{\Omega}_{\delta/2}), \tag{2.35}$$

for some suitable Hölder exponent α, $0<\alpha<1$, *where* α *and the Hölder norm of u depend on the quantities mentioned in Theorem* 2.1 *or in Remark* 2.1, *respectively.*

This follows from the results in [23].

To bound the *gradient* of u in $\Omega_{\delta/2}$ we take the boundary condition in (1.1) or in (2.34) into account yielding that

$$|a_i\rho_i|\leqq(1-a),\qquad a>0, \tag{2.36}$$

which together with the estimates for the tangential derivatives of u implies that the normal derivative of u is bounded on $\partial\Omega\cap\overline{\Omega}_{\delta/2}$. Thus, the gradient of u is bounded on $\partial\Omega\cap\overline{\Omega}_{\delta/2}$, from which we deduce a gradient bound for u in $\overline{\Omega}_{\delta/3}$ (*cf.* [8], Th. A 1). We state this as a Theorem.

THEOREM 2.2. — *Under the assumptions mentioned above the gradient of u is bounded in* $\overline{\Omega}_{\delta/3}$, *the estimate depending on* L, M, N, Λ, δ, $\varkappa$, a, *and* n.

3. Existence of a solution

We shall prove the existence of a solution to the boundary value problem

$$\begin{cases} Au+H(x, u)=0 & \text{in } \Omega,\\ -a_i\rho_i\in\beta(x, u-\varphi) & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where H, φ, and Ω satisfy the conditions stated in Section 1, and where for fixed $x\in\partial\Omega$, $\beta(x, .)$ is a *maximal monotone* graph such that

$$|\beta(x, t)|\leqq 1-a,\qquad a>0 \tag{3.2}$$

and

$$\left|\frac{\partial\beta}{\partial x}(x, t)\right|\leqq L, \tag{3.3}$$

uniformly in x and t. Moreover, we assume that φ is the trace of a function $\tilde{\varphi}\in H^{2,2}(\Omega)$. For brevity we identify φ and $\tilde{\varphi}$ and set

$$\sum_{i,j=1}^{n}\int_{\Omega}|D^iD^j\varphi|^2\,dx=L_0. \tag{3.4}$$

For the proof we use the *a priori* estimates of the preceding section together with a continuity argument (*cf.* [7], Section 2). Thus, we must know that the solution of (3.1) is uniquely determined. This will be derived from the following lemma.

LEMMA 3.1. — *Let the functions* u, $v \in H^{2,2}(\Omega) \cap H^{1,\infty}(\Omega)$ *satisfy the inequalities*

$$Au + H(x, u) \geqq 0 \quad \text{in } \Omega, \tag{3.5}$$

$$Av + H(x, v) \leqq 0 \quad \text{in } \Omega, \tag{3.6}$$

together with the boundary conditions

$$-a_i(Du)\,\rho_i \in \beta_1(x, u-\varphi_1) + \beta_2(x, u-\varphi_2) \tag{3.7}$$

and

$$-a_i(Dv)\,\rho_i \in \beta_1(x, v-\psi_1) + \beta_2(x, v-\psi_2), \tag{3.8}$$

where $\beta_i(x, .)$ *is a maximal monotone graph for* $i=1, 2$. *Then*

$$v - u \leqq \max\{\sup_{\partial\Omega} |\varphi_1 - \psi_1|, \sup_{\partial\Omega} |\varphi_2 - \psi_2|\}. \tag{3.9}$$

Proof. — Denote the right-hand side of (3.9) by c and let

$$\eta = \max(v - u - c, 0).$$

Multiplying the inequality

$$0 \leqq Au - Av + H(x, u) - H(x, v)$$

with η and integrating by parts in the first term yield

$$0 \leqq \int_{\{\eta>0\}} \{[a_i(Du) - a_i(Dv)]\, D^i(v-u) + [H(x, u) - H(x, v)].(v-u-c)\}\, dx \\ + \int_{\partial\Omega \cap \{\eta>0\}} [a_i(Dv)\,\rho_i - a_i(Du)\,\rho_i].(v-u-c)\, d\mathscr{H}_{n-1}. \tag{3.10}$$

In view of the relations (3.7), (3.8) and in view of the definitions of c and the β_i's the boundary term in this inequality is non-positive, so that the result

$$\eta = 0 \quad \textit{or equivalently} \quad v - u \leqq c,$$

follows from the properties of the coefficients a_i and from the strict monotonicity of $H(x, .)$.

To prove the existence of a solution, let us first assume that H, β, and φ are smooth functions, e. g. of class $C^{1,\lambda}$ for some $0<\lambda<1$. Then, for any number τ, $0 \leqq \tau \leqq 1$, let us consider the boundary value problems

$$\begin{cases} Au_\tau + \tau H(x, u_\tau) + (1-\tau).\varkappa.u_\tau = 0 & \text{in } \Omega, \\ -a_i\,\rho_i = \tau\beta(x, u_\tau - \varphi) & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Let T be the set

$$T=\{\tau\text{: there exists a solution } u_\tau\in C^2(\overline{\Omega})\}.$$

T is obviously not empty for $u_0=0$ belongs to it, and we shall show that it is both open and closed.

As the mean curvature term now has the form $\tau . H(x, t)+(1-\tau)\varkappa . t$ we deduce an *a priori* bound of $|u_\tau|_\Omega$ for any $\tau\in T$ independent of τ (*cf.* [4]). Furthermore, let us remark that any solution $u_\tau\in C^2(\overline{\Omega})$ is of class $C^{2,\alpha}(\overline{\Omega})$ with some fixed α, $0<\alpha<1$, such that the norm of u_τ in $C^{2,\alpha}(\overline{\Omega})$ is bounded independently of τ.

To prove this, we first deduce from Theorem 2.1 that $|Du_\tau|_\Omega$ is uniformly bounded

$$|Du_\tau|_\Omega\leqq c. \tag{3.12}$$

Then, we choose a smooth vector field $\tilde{a}_i$ such that $\partial\tilde{a}_i/\partial p^j$ is uniformly elliptic, and such that

$$\tilde{a}_i(p)=a_i(p) \qquad \text{for} \quad |p|\leqq 3.c. \tag{3.13}$$

From [29], Chap. 10, Th. 2.2, we conclude that the problem

$$\left\{\begin{array}{ll} \tilde{A}\tilde{u}_\tau+\tau H(x, \tilde{u}_\tau)+(1-\tau)\varkappa\tilde{u}_\tau=0 & \text{in } \Omega, \\ -\tilde{a}_i\rho_i=\tau\beta(x, \tilde{u}_\tau-\varphi) & \text{on } \partial\Omega, \end{array}\right. \tag{3.14}$$

has a solution $\tilde{u}_\tau\in C^{2,\alpha}(\overline{\Omega})$ for any τ.

Moreover, in view of (3.12) and (3.13) we derive

$$\tilde{A}u_\tau=Au_\tau.$$

Hence, we obtain from the uniqueness of the solution to the boundary value problem (3.14):

$$u_\tau=\tilde{u}_\tau.$$

Thus, we finally conclude that $|u_\tau|_{2,\alpha,\Omega}$ is uniformly bounded

$$|u_\tau|_{2,\alpha,\Omega}\leqq\tilde{c}, \tag{3.15}$$

where the constant is determined by known quantities.

From the estimate (3.15) it follows immediately that T is closed.

On the other hand, let $\tau_0\in T$. Then, we consider the boundary value problem (3.14) as before. Since $|D\tilde{u}_\tau|_\Omega$ depends continuously on τ, it turns out that

$$|D\tilde{u}_\tau|_\Omega\leqq 2.c \qquad \text{for} \quad |\tau-\tau_0|<\delta. \tag{3.16}$$

This yields $\tilde{u}_\tau=u_\tau$ for those τ's. Thus, T is open and we obtain a solution $u\in C^{2,\alpha}(\overline{\Omega})$ of the boundary value problem

$$\left\{\begin{array}{ll} Au+H(x, u)=0 & \text{in } \Omega \\ -a_i\rho_i=\beta(x, u-\varphi) & \text{on } \partial\Omega. \end{array}\right. \tag{3.17}$$

Next we note

REMARK 3.1. — *The* $H^{2,2}(\Omega)$ *norm of* u *can be estimated by a constant depending on* $|Du|$, $|\partial\beta/\partial x|$, L_0, *the* C^2*-norm of* $\partial\Omega$, *and other known quantities, but not on* $|\partial\beta/\partial t|$.

For a proof we transform a boundary neighbourhood $\Omega_\delta=\Omega\cap B_\delta(x_0)$ *via* a C^2-diffeomorphism $y=y(x)$ into an open set G_δ, such that

$$\Gamma=y(\partial\Omega\cap\bar{\Omega}_\delta)=\{y\in G_\delta:\ y^n=0\}.$$

The boundary value problem (3.17) is then transformed to

$$(3.18)\qquad -D^k_y(\tilde{a}_i).D^i_x\,y^k+\tilde{H}(y,\tilde{u})=0\quad\text{in } G_\delta,\qquad -\tilde{a}_i.D^i_y\,y^n=\beta.|D_x y^n|=\tilde{\beta}\quad\text{on } \Gamma.$$

Let $U_\delta=\{y(x):\ x\in B_\delta(x_0)\}$ and η any Lipschitz continuous function with $\operatorname{supp}\eta\subset U_\delta$. From (3.18) we then deduce

$$(3.19)\qquad \int_{G_\delta}\{\tilde{a}_i.D^i_x\,y^k\,D^k_y.\eta+\tilde{a}_i\,D^i_x\,D^k_y\,y^k.\eta+\tilde{H}(y,\tilde{u}).\eta\}\,dy+\int_\Gamma\tilde{\beta}.\eta\,d\mathscr{H}_{n-1}=0.$$

Inserterting in this identity $\eta=-D^\sigma_y(D^\sigma_y(\tilde{u}-\tilde{\varphi})\zeta^2)$, where ζ, $0\leqq\zeta\leqq1$, is a cut-off function with support in U_δ and where σ runs from 1 to $n-1$, we obtain in view of the monotonicity of β and the ellipticity of the coefficients $\tilde{a}_i$ a bound for

$$(3.20)\qquad \sum_{i=1}^{n}\sum_{\sigma=1}^{n-1}\int_{G_\delta}|D^i_y\,D^\sigma_y\tilde{u}|^2.\zeta^2\,dy,$$

depending on ζ and known quantities. Then, a bound for

$$\int_{G_\delta}|D^n_y\,D^n_y\tilde{u}|^2\zeta^2\,dy,$$

follows from the equation in view of the ellipticity. The interior bounds are trivial.

Thus, approximating the maximal monotone graph β by smooth monotone graphs β_ε and taking the *a priori* estimates for $|Du_\varepsilon|$ and $|u_\varepsilon|_{2,2}$ into account, where u_ε is the corresponding solution, we conclude

THEOREM 3.1. — *The boundary value problem* (3.1) *has a unique solution* $u\in H^{1,\infty}(\Omega)\cap H^{2,2}(\Omega)$. *The estimates for* $|u|$, $|Du|$, *and* $|u|_{2,2}$ *are of local nature depending only on local quantities. Moreover, if* β *does not depend on* x, *then the* $C^{0,\alpha}$*-norm of* u *for some* α, $0<\alpha<1$, *can be bounded independently of* a; *the (local) estimate only depends on* $\varkappa$ *and the other quantities mentioned in Theorem* 2.1.

4. Plateau's problem for H-surfaces

We consider the weak formulation of Plateau's problem for H-surfaces, namely, we look at the variational problem

$$(4.1)\quad J(v)=\int_\Omega(1+|Dv|^2)^{1/2}\,dx+\int_\Omega\int_0^v H(x,t)\,dt\,dx+\int_{\partial\Omega}|v-\varphi|\,d\mathscr{H}_{n-1}\to\min,\qquad\forall v\in H^{1,1}(\Omega).$$

For simplicity we shall first assume $\partial\Omega \in C^4$ and $\varphi \in C^2(\partial\Omega)$, while H is Lipschitz satisfying the monotonicity condition

$$\frac{\partial H}{\partial t} \geqq \varkappa > 0. \tag{4.2}$$

Let β be the maximal monotone graph

$$\beta(t) = \begin{cases} -1, & t<0, \\ [-1, 1], & t=0, \\ 1, & t>0 \end{cases}$$

and let $\beta_\lambda(t) = \lambda . \beta(t)$ for $0<\lambda<1$.

Then we deduce from Theorem 3.1 the existence of a solution $u_\lambda \in H^{1,\infty}(\Omega) \cap H^{2,2}(\Omega)$ of the boundary value problem

$$\begin{cases} A u_\lambda + H(x, u_\lambda) = 0 & \text{in } \Omega \\ -a_i \rho_i \in \beta_\lambda(u-\varphi) & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where

$$|u_\lambda|_{0, \alpha, \Omega} \leqq \text{const.}, \tag{4.4}$$

uniformly in λ for some $0<\alpha<1$.

It is not hard to verify that the solution of (4.3) also solves the variational problem

$$\begin{aligned} J_\lambda(v) = \int_\Omega (1+|Dv|^2)^{1/2}\, dx + \int_\Omega \int_0^v H(x, t)\, dt\, dx \\ + \lambda . \int_{\partial\Omega} |v-\varphi|\, d\mathscr{H}_{n-1} \to \min, \qquad \forall v \in H^{1,1}(\Omega), \end{aligned} \tag{4.5}$$

e. g. we can approximate the monotone Lipschitz continuous function $j(t) = |t|$ by smooth monotone functions $j_\varepsilon(t)$ with corresponding smooth monotone graphs $\beta^\varepsilon(t) = j'_\varepsilon(t)$ and $\beta^\varepsilon_\lambda(t) = \lambda\beta^\varepsilon(t)$. Then, the solutions of the approximating variational problems (which exist, *cf.* [9]) coincide with the solutions of the corresponding boundary value problems. If ε goes to zero we get the desired result in view of the *a priori* estimates which hold independently of ε.

Hence, the u_λ's are a minimizing sequence of the variational problem (4.1) with uniformly bounded $C^{0,\alpha}(\overline{\Omega})$-norm. Therefore, a subsequence converges (in fact the sequence will do it) to a solution $u \in H^{1,1}(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) \cap C^{0,1}(\Omega)$. The fact that $u \in C^{0,1}(\Omega)$ follows from the interior gradient estimates for the u_λ's (*cf.* [1], [8], [17], [27]).

In general the solution u of (4.1) will not take on the prescribed boundary values φ, and the question arises when this will be the case. A sufficient answer has been given in [8], [20], namely: if in a neighbourhood of a boundary point x_0 the mean curvature H_{n-1} of $\partial\Omega$ satisfies the inequality

$$|H(x, \varphi(x)| \leqq (n-1) H_{n-1}(x), \tag{4.6}$$

then $u=\varphi$ in that neighbourhood. The proof, at least that in [8], uses the variational property of u. By the method we described above it will be possible to get this information directly from the approximating sequence u_λ.

Precisely, let Ω_δ be a boundary neighbourhood of x_0 whose boundary $\partial\Omega_\delta$ is decomposed into the parts $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \subset \Omega$. We assume that (4.6) is valid on some open connected subset Γ_0 of $\partial\Omega$ with $\Gamma_1 \subset\subset \Gamma_0$. Then, as in ([8], Chap. 4,) we can find upper and lower barriers δ^+ and δ^- in $C^2(\overline{\Omega}_\delta)$ satisfying

$$A\delta^+ + H(x, \delta^+) \geqq 0 \quad \text{in } \Omega_\delta, \tag{4.7}$$

$$A\delta^- + H(x, \delta^-) \leqq 0 \quad \text{in } \Omega_\delta, \tag{4.8}$$

$$\delta^- \leqq u_\lambda \leqq \delta^+ \quad \text{on } \Gamma_2 \cup (\Gamma_1 - \Gamma_3), \tag{4.9}$$

for all λ,

$$\delta^- \leqq \varphi \leqq \delta^+ \quad \text{on } \Gamma_3, \tag{4.10}$$

and

$$\delta^- = \varphi = \delta^+ \quad \text{on } \Gamma_4, \tag{4.11}$$

where

$$\Gamma_4 \subset\subset \Gamma_3 \subset\subset \Gamma_1.$$

Let

$$\lambda_0 = \max\left\{ \sup_{\Omega_\delta} \frac{|D\delta^-|}{(1+|D\delta^-|^2)^{1/2}}, \quad \sup_{\Omega_\delta} \frac{|D\delta^+|}{(1+|D\delta^+|^2)^{1/2}} \right\}.$$

Then, $\lambda_0 < 1$, and we claim that

$$\delta^- \leqq u_\lambda \leqq \delta^+ \quad \text{in } \Omega_\delta \tag{4.12}$$

for all λ with $\lambda_0 < \lambda < 1$. Hence,

$$u_\lambda = \varphi \quad \text{on } \Gamma_4, \tag{4.13}$$

and therefore

$$u = \varphi \quad \text{on } \Gamma_4. \tag{4.14}$$

Let us remark, that if the inequality (4.6) will hold on $\partial\Omega$ then the relation (4.13) will also be valid on $\partial\Omega$ for $\lambda_0 < \lambda < 1$. Thus, the u_λ' , for $\lambda_0 < \lambda < 1$, are all identical in this case.

We only prove the second inequality in (4.12). The proof of the lower estimate will be similar.

Let $\eta = \max(u - \delta^+, 0)$. Then η vanishes on $\partial\Omega_\delta - \Gamma_3$ and on Γ_3 there holds

$$-a_i(D\delta^+).\rho_i \leqq \beta_\lambda(\delta^+ - \varphi), \tag{4.15}$$

[i. e. there exists in element in $\beta_\lambda(\delta^+ - \varphi)$ s. t. the inequality is valid] for $\lambda_0 < \lambda < 1$, in view of (4.10) and the definition of β_λ. Thus, the proof of Lemma 3.1 yields the result.

In general the mean curvature function H is not assumed to be strictly monotone, but it has to satisfy the isoperimetric inequality (0.9), which will be the right condition for solving the variational problem (4.1). Thus, the way we here proposed to solve it is not applicable.

But considering the mean curvature functions

$$H_\varkappa(x, t) = H(x, t) + \varkappa t,$$

for $\varkappa > 0$, we see that to each $\varkappa$ there corresponds a solution $u_\varkappa$ of the perturbed variational problem, for which we can derive *a priori* estimates in $L^\infty(\Omega) \cap C^{0,1}(\Omega)$ independent of $\varkappa$, *cf.* [8]. If H (not $H_\varkappa$!) satisfies the relation (4.6) on certain boundary parts, then by similar arguments as above we can conclude that $u_\varkappa = \varphi$ on those boundary parts. The construction of appropriate barrier functions is still possible in this case; for details we refer to [22].

Moreover, let Ω_δ be a boundary neighbourhood with $u_\varkappa = \varphi$ on $\partial\Omega \cap \overline{\Omega}_\delta$, and assume φ to be of class C^2. Then, we have $u_\varkappa \in C^1(\overline{\Omega}_{\delta/2})$ with a uniform bound for $|Du_\varkappa|_{\Omega_{\delta/2}}$ (*cf.* [8], Th. 2).

If $\varkappa$ tends to zero the $u_\varkappa$'s will converge to a solution $u \in C^2(\Omega) \cap L^\infty(\Omega) \cap C^{0,1}(\overline{\Omega}_{\delta/2})$ of the variational problem (4.1) satisfying $u = \varphi$ on $\partial\Omega \cap \overline{\Omega}_{\delta/2}$, provided $\varphi \in C^2(\partial\Omega \cap \overline{\Omega}_\delta)$, and provided (4.6) is valid on $\partial\Omega \cap \overline{\Omega}_\delta$.

5. Variational problems with constraints on the boundary

In this section we consider variational problems whose classical formulation is

$$\begin{cases} \displaystyle\int_\Omega (1 + |Dv|^2)^{1/2}\,dx + \int_\Omega \int_0^v H(x, t)\,dt\,dx \to \min, \\ \forall v \in K = \{ v \in H^{1,1}(\Omega) : \varphi_1 \leqq v|_{\partial\Omega} \leqq \varphi_2 \}. \end{cases} \tag{5.1}$$

If $\varphi_1 = \varphi_2$ then we have Plateau's problem.

The weak formulation of (5.1) is

$$\begin{aligned} J(v) = &\int_\Omega (1 + |Dv|^2)^{1/2}\,dx + \int_\Omega \int_0^v H(x, t)\,dt\,dx \\ &+ \int_{\partial\Omega} \{ -\min(v - \varphi_1, 0) + \max(v - \varphi_2, 0) \}\,d\mathscr{H}_{n-1} \to \min, \qquad \forall v \in H^{1,1}(\Omega). \end{aligned} \tag{5.2}$$

This is justified by the fact, that any solution of (5.1) also solves (5.2). Another motivation is, that writing the side conditions

$$\varphi_1 \leqq v\big|_{\partial\Omega} \leqq \varphi_2,$$

as an *isoperimetric* constraint, namely, as

$$(5.3)\qquad \int_{\partial\Omega} \{-\min(u-\varphi_1, 0)+\max(v-\varphi_2, 0)\}\, d\mathscr{H}_{n-1}=0,$$

the Lagrange multiplier method, formally applied, leads to the problem

$$(5.4)\qquad \mathrm{J}_\lambda(v)=\int_\Omega (1+|\mathrm{D}v|^2)^{1/2}\,dx+\int_\Omega\int_0^v \mathrm{H}(x, t)\,dt\,dx$$
$$+\lambda.\int_{\partial\Omega} \{-\min(u-\varphi_1, 0)+\max(u-\varphi_2, 0)\}\, d\mathscr{H}_{n-1}\to \min, \qquad \forall v\in \mathrm{H}^{1,1}(\Omega),$$

with some unknown Lagrange multiplier $\lambda\in\mathbf{R}$. In order that the functional remains convex λ has to be nonnegative, and setting the first variation of the functional to be zero, it is clear that $|\lambda|\leqq 1$. Hence, we seek λ in the interval $[0, 1]$.

Let us consider the problem (5.4) for $0<\lambda<1$. The corresponding boundary value problem would be

$$(5.5)\qquad \begin{cases} \mathrm{A}u_\lambda+\mathrm{H}(x, u_\lambda)=0 & \text{in } \Omega \\ -a_i.\rho_i\in\lambda\{\beta_1(u-\varphi_1)+\beta_2(u-\varphi_2)\} & \text{on } \partial\Omega, \end{cases}$$

where β_1 and β_2 are the maximal monotone graphs

$$(5.6)\qquad \beta_1(t)=\begin{cases} -1, & t<0, \\ [-1, 0], & t=0, \\ 0, & t>0 \end{cases}$$

and

$$(5.7)\qquad \beta_2(t)=\begin{cases} 0, & t<0, \\ [0, 1], & t=0, \\ 1, & t>0. \end{cases}$$

Again we first assume $\partial\Omega\in \mathrm{C}^4$, $\varphi_i\in \mathrm{C}^2(\partial\Omega)$, and H to be strictly monotone, i. e.:

$$\frac{\partial \mathrm{H}}{\partial t}\geqq \varkappa>0.$$

Moreover, we suppose at the beginning

$$(5.8)\qquad \varphi_1<\varphi_2 \quad \text{on } \partial\Omega.$$

Then approximating the β_i's by Lipschitz continuous monotone functions β_i^ε's we deduce from the existence theorem in Section 3, which is still applicable in this case, the existence of functions $u_\lambda^\varepsilon \in H^{2,p}(\Omega)$ solving the corresponding perturbed boundary value problems. We note that

$$(5.9) \qquad |\beta_1(t-\varphi_1)+\beta_2(t-\varphi_2)| \leqq 1,$$

for all t, and that the approximations β_i^ε can be chosen such as to satisfy the same estimate.

From the *a priori* estimates in Section 2 we know

$$(5.10) \qquad |u_\lambda^\varepsilon|_{0,\alpha,\Omega} \leqq \text{const.},$$

for some α, $0<\alpha<1$, uniformly in ε and λ, and

$$(5.11) \qquad |D u_\lambda^\varepsilon|_\Omega \leqq \text{const.},$$

uniformly in ε.

As in Section 4 we can conclude that the u_λ^ε's converge uniformly to a Lipschitz continuous solution of the variational problem (5.4). We call it u_λ, since we shall see that it will also be the solution of (5.5).

Indeed, since u_λ is uniformly continuous and since the convergence of the u_λ^ε's to u_λ is uniform, we deduce that the coincidence sets

$$E_1^\varepsilon = \{ x \in \partial\Omega : u_\lambda^\varepsilon(x) = \varphi_1(x) \}$$

and

$$E_2^\varepsilon = \{ x \in \partial\Omega : u_\lambda^\varepsilon(x) = \varphi_2(x) \},$$

can be separated by open sets uniformly in ε. Let U_i, for $i=1, 2$, be open sets in $\mathbf{R}^n$ separating them, and let η, $0 \leqq \eta \leqq 1$, be a cut-off function with support in U_1, where

$$E_1^\varepsilon \subset U_1 \qquad \text{and} \qquad E_2^\varepsilon \subset U_2,$$

for all ε. We shall show show that the u_λ^ε's are uniformly bounded in $H^{2,2}(\Omega)$, uniformly with respect to ε. To prove this, we suppose the support of η to be small enough to ensure the existence of some C^2-diffeomorphism $y=y(x)$ flattening $\partial\Omega$ in some boundary neighbourhood Ω_δ with supp $\eta \subset \overline{\Omega}_\delta$. Denoting the transformed cut-off function η with the same letter and the image of Ω_δ with G_δ we derive an estimate for

$$(5.12) \qquad \sum_{i=1}^n \int_{G_\delta} |D_y^i D_y^r \tilde{u}_\lambda^\varepsilon|^2 \eta^2 \, dy,$$

for any r, $1 \leqq r \leqq n-1$, in terms of η, $|\varphi_1|_{2,2,\Omega}$ and known quantities. The proof is identical to the first part of the proof of Remark 3.1: first we observe that there exists $\gamma>0$ such that

$$(5.13) \qquad u_\lambda^\varepsilon \leqq \varphi_2 - \gamma \quad \text{on } U_1,$$

uniformly in ε and λ, and hence that

$$\beta_2^\varepsilon(u_\lambda^\varepsilon-\varphi_2)=0 \quad \text{on } U_1, \tag{5.14}$$

for small values of ε. Indeed, if we choose

$$\beta_2^\varepsilon(t)=\begin{cases} 0, & t\leqq 0, \\ t/\varepsilon, & 0\leqq t\leqq\varepsilon, \\ 1, & t\geqq\varepsilon, \end{cases} \tag{5.15}$$

then the relation (5.14) is valid for all ε for which (5.13) holds. Second, we note that the test function $\zeta=-D_y^r(D_y^r(\tilde{u}_\lambda^\varepsilon-\tilde{\varphi}_1).\eta^2)$ corresponding to that in the proof of Remark 3.1 has support in $y(U_1)$, so that in view of (5.13) the boundary term $\beta_2^\varepsilon(\tilde{u}_\lambda^\varepsilon-\tilde{\varphi}_2)$ is not involved in the derivation of an estimate for (5.12). Similarly, we get a bound of the integral in (5.12) for cut-off functions η with $\operatorname{supp}\eta\subset y(U_2)$. Hence, we finally derive:

THEOREM 5.1. — *If $\varphi_1<\varphi_2$ on $\partial\Omega$ then the boundary value problem* (5.5) *has a unique solution $u_\lambda\in C^{0,1}(\Omega)\cap H^{2,2}(\Omega)$, where*

$$|u_\lambda|_{0,\alpha,\Omega}, \tag{5.16}$$

can be bounded independently of λ and $\inf\limits_{\partial\Omega}(\varphi_2-\varphi_1)$,

$$|Du_\lambda|_\Omega, \tag{5.17}$$

is bounded independently of $\inf\limits_{\partial\Omega}(\varphi_2-\varphi_1)$, and where

$$|u_\lambda|_{2,2,\Omega}, \tag{5.18}$$

depends on λ and $\inf\limits_{\partial\Omega}(\varphi_2-\varphi_1)$. All estimates also hold locally. From the approximation procedure it is clear that u_λ also solves the variational problem (5.4).

In the limit case when λ tends to 1, *the u_λ's converge uniformly to the unique solution $u\in C^{0,\alpha}(\bar{\Omega})\cap H^{1,1}(\Omega)\cap C^2(\Omega)$ of the variational problem* (5.2).

If the obstacles φ_i, $i=1, 2$, are only Lipschitz continuous satisfying the weak inequality $\varphi_1\leqq\varphi_2$ on $\partial\Omega$, then, via *approximation, we can still find a unique solution $u\in C^{0,\alpha}(\bar{\Omega})\cap H^{1,1}(\Omega)\cap C^2(\Omega)$ of the variational problem* (5.2).

REMARK 5.1. — (i) *We note that the preceding results only hold under the general assumption that* H *satisfies $\partial H/\partial t\geqq\varkappa>0$, though it is possible to bound $|u_\lambda|$ or $|u|$ independent of $\varkappa$ provided* H *satisfies the isoperimetric inequality* (0.9). *We shall see below how to get rid of this restriction in some special cases.*

(ii) *If* H *is not strictly monotone but satisfies the isoperimetric inequality* (0.9), *then by looking at the perturbed problems where* H *is replaced by $H_\varkappa(x, t)=H(x, t)+\varkappa.t$, $\varkappa>0$, we* *get the existence of solutions* $u_\varkappa \in H^{1,1}(\Omega) \cap C^2(\Omega) \cap L^\infty(\Omega)$ *which converge uniformly on compact subsets of* Ω *to a solution* $u \in H^{1,1}(\Omega) \cap C^2(\Omega) \cap L^\infty(\Omega)$ *of the variational problem* (5.2).

Again the question arises under which assumptions on the data a solution u of (5.2) satisfies the relation $\varphi_1 \leqq u \leqq \varphi_2$ (locally) on the boundary. This inequality will be valid if, as in the case of Plateau's problem, the mean curvature of the boundary and the mean curvature function H are (locally) related by the inequalities

$$-H(x, \varphi_2) \leqq (n-1) H_{n-1}(x) \tag{5.19}$$

and

$$H(x, \varphi_1) \leqq (n-1) H_{n-1}(x), \tag{5.20}$$

for these relations ensure the existence of barrier functions δ^+ and δ^- in some boundary neighbourhood Ω_δ where (5.19) and (5.20) are valid on $\partial\Omega_{2\delta} \cap \partial\Omega$ such that

$$\delta^- \leqq u \leqq \delta^+ \quad \text{in } \Omega_\delta \tag{5.21}$$

and

$$\varphi_1 = \delta^- \leqq \delta^+ = \varphi_2 \quad \text{on } \partial\Omega_\delta \cap \partial\Omega, \tag{5.22}$$

provided

$$\varphi_i \in C^2(\Omega_{2\delta}) \qquad \text{for} \quad i = 1, 2. \tag{5.23}$$

If the φ_i's are only continuous then a modified version of proof still leads to the result

$$\varphi_1 \leqq u \leqq \varphi_2 \quad \text{on } \partial\Omega_\delta \cap \partial\Omega, \tag{5.24}$$

where the inequalities in (5.24) only hold $\mathscr{H}_{n-1}$ — a. e. on $\partial\Omega_\delta \cap \partial\Omega$, for we do not know if u is continuous up to the boundary in the general case of non-strictly monotone H. (5.24) can be proved using the variational property of u (*cf.* [8], the methods developed there can also be applied in this case).

Now, we shall show that in the case when $\Omega \subset \mathbf{R}^2$, the solutions of the variational problem (5.4) are uniformly Lipschitz continuous in those boundary neighbourhoods Ω_δ, where the inequalities (5.19) and (5.20) are valid on $\partial\Omega_{2\delta} \cap \partial\Omega$.

THEOREM 5.2. — *Let* $\Omega \subset \mathbf{R}^2$ *be a bounded domain with Lipschitz boundary* $\partial\Omega$ *and let* Γ_0 *be an open subset of* $\partial\Omega$ *being of class* C^2. *Assume that the functions* $\varphi_1, \varphi_2 \in L^1(\partial\Omega)$, *with* $\varphi_1 \leqq \varphi_2$, *belong to* $C^2(\Gamma_0)$ *and that* $H = H(x, t)$ *satisfies besides the conditions* (0.8) *and* (0.9) *the inequalities* (5.19) *and* (5.20) *in* Γ_0. *Then the variational problem* (5.2) *has a solution* $u \in H^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega}'_\delta) \cap C^2(\Omega)$ *such that* (5.24) *holds.* Ω_δ *is a suitable boundary neighbourhood with* $\partial\Omega_\delta \cap \partial\Omega \subset\subset \Gamma_0$. *The gradient of* u *is bounded in* Ω_δ *by a constant depending on* $\varepsilon_0, \delta, \|\varphi_i\|_{L^1(\partial\Omega)}, |D^2\varphi_i|_{\Gamma_0}, |(\partial/\partial x) H(x, u(x))|_{\Omega_\delta}, |H(., 0)|_\Omega$, *and the* C^2-*norm of* Γ_0.

Proof. — First we observe that without loss of generality we may assume that $\partial\Omega \in C^4$, $\partial H/\partial t \geqq \varkappa > 0$, and $\varphi_1 < \varphi_2$ on Γ_0. For otherwise we approximate Ω by smooth domains Ω_ε and φ_i by functions φ_i^ε such that

$$(5.25)\qquad -H(x, \varphi_2^\varepsilon(x)) \leqq (n-1) H_{n-1}^\varepsilon(x) + \gamma$$

and

$$(5.26)\qquad H(x, \varphi_1^\varepsilon(x)) \leqq (n-1) H_{n-1}^\varepsilon(x) + \gamma,$$

for all $x \in \Gamma_0^\varepsilon$, where γ is an arbitrarily small positive constant and the relations hold uniformly in ε for all $\varepsilon \leqq \varepsilon^*(\gamma)$. Replacing H by $H_\varkappa(x, t) = H(x, t) + \varkappa . t$ if necessary the inequalities (5.25) and (5.26) are also valid for $H_\varkappa$, if $\varkappa$ is sufficiently small. Thus, the construction of barrier functions δ_ε^+, δ_ε^- is still possible for the perturbed problems provided γ is sufficiently small depending on the C^2-norms of the φ_i's. The estimate for $|Du_\varkappa^\varepsilon|_{\Omega_\delta^\varepsilon}$, where $u_\varkappa^\varepsilon$ is a solution of the perturbed problem will hold uniformly in ε and $\varkappa$, since $|u_\varkappa^\varepsilon|_{\Omega_\delta^\varepsilon}$ is bounded independently of ε and $\varkappa$ (*cf.* the proof of [8], Lemma 1).

Let us therefore assume that $\partial\Omega$, H, and the φ_i's satisfy the stronger assumptions. Then, from the results of [4] we know that any solution of the equation

$$Au + H(x, u) = 0,$$

in Ω is bounded by some constant m depending on $\varkappa$ and the C^2-norm of $\partial\Omega$,

$$(5.27)\qquad |u|_\Omega \leqq m.$$

Let Ω_δ be a boundary neighbourhood such that $\partial\Omega_\delta \cap \partial\Omega \subset\subset \Gamma_0$, and decompose $\partial\Omega_\delta$ into the parts Γ_1 and Γ_2 such that $\Gamma_2 \subset \Omega$ and $\Gamma_1 \subset \Gamma_0$. Then, there exist upper and lower barriers δ^+ and $\delta^- \in C^2(\overline{\Omega}_\delta)$ satisfying

$$(5.28)\qquad A\delta^+ + H(x, \delta^+) \geqq 0 \quad \text{in } \Omega_\delta,$$

$$(5.29)\qquad A\delta^- + H(x, \delta^-) \leqq 0 \quad \text{in } \Omega_\delta,$$

$$(5.30)\qquad \delta^- \leqq -m \leqq m \leqq \delta^+ \quad \text{on } \partial\Omega_\delta - \Gamma_3,$$

$$(5.31)\qquad \delta^- \leqq \varphi_1 \leqq \varphi_2 \leqq \delta^+ \quad \text{on } \Gamma_3,$$

and

$$(5.32)\qquad \varphi_1 = \delta^- \leqq \delta^+ = \varphi_2 \quad \text{on } \Gamma_4,$$

where $\Gamma_4 \subset\subset \Gamma_3 \subset\subset \Gamma_1$. The C^2-norms of the barriers only depend on δ, the C^2-norm of Γ_0, $|\varphi_i|_{2,0,\Gamma_0}$, m, and on other local quantities.

Let

$$\lambda_0 = \max\left\{ \sup_{\Omega_\delta} \frac{|D\delta^+|}{(1+|D\delta^+|^2)^{1/2}},\ \sup_{\Omega_\delta} \frac{|D\delta^-|}{(1+|D\delta^-|^2)^{1/2}} \right\},$$

and let λ_ε be a sequence of smooth functions such that

$$\lambda_0 \leqq \lambda_\varepsilon(x) < 1, \qquad x \in \partial\Omega, \tag{5.32}$$

$$\lambda_\varepsilon(x) = \lambda_0, \qquad x \in \Gamma_4, \tag{5.33}$$

and

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon(x) = 1, \qquad x \in \partial\Omega - \Gamma^4. \tag{5.34}$$

Then, we consider the boundary value problem

$$\left\{\begin{array}{l} A u_\varepsilon + H(x, u_\varepsilon) = 0 \quad \text{in } \Omega, \\ -a_i \rho_i \in \lambda_\varepsilon \{ \beta_1(u_\varepsilon - \varphi_1) + \beta_2(u_\varepsilon - \varphi_2) \} \quad \text{on } \partial\Omega, \end{array}\right. \tag{5.35}$$

where β_1 and β_2 are the maximal monotone graphs in (5.6) and (5.7).

From our preceding results we know, that for each ε the boundary value problem (5.35) has a solution $u_\varepsilon \in C^{0,1}(\overline{\Omega}) \cap H^{2,2}(\Omega_\delta)$ such that

$$| D u_\varepsilon |_{\Omega_{\delta/2}} \leqq M_1 = M_1(\lambda_0) \tag{5.36}$$

and

$$| u_\varepsilon |_{2, 2, \Omega_{\delta/2}} \leqq M_2 = M_2(\lambda_0, \inf_{\Gamma_0}(\varphi_2 - \varphi_1)), \tag{5.37}$$

uniformly in ε provided $\partial\Omega_{\delta/2} \cap \partial\Omega \subset \Gamma_4$. We shall always assume this.

From the definitions of the barrier functions δ^+, δ^- and from the definition of λ_ε we deduce as in Section 4:

$$\delta^- \leqq u_\varepsilon \leqq \delta^+ \quad \text{in } \Omega_\delta. \tag{5.38}$$

Note that u_ε satisfies (5.27)

Hence, we obtain

$$\varphi_1 \leqq u_\varepsilon \leqq \varphi_2 \quad \text{on } \Gamma_4, \tag{5.39}$$

so that in the limit case, $\varepsilon \to 0$, we conclude that the u_ε's converge uniformly on compact subsets of Ω and uniformly in $\overline{\Omega}_{\delta/2}$ to a solution

$$u \in H^{1,1}(\Omega) \cap C^2(\Omega) \cap C^{0,1}(\overline{\Omega}_{\delta/2}) \cap H^{2,2}(\Omega_{\delta/2})$$

of the variational problem (5.2). Indeed, to prove that u is a solution it is sufficient to show that the u_ε's are a minimizing sequence of (5.2): let

$$\tilde{\lambda}_{\varepsilon'}(x) = \left\{\begin{array}{ll} \lambda_\varepsilon(x) & \text{if } x \in \partial\Omega - \Gamma_4, \\ 1, & \text{if } x \in \Gamma_4 \end{array}\right.$$

and let

$$J_\mu(v) = \int_\Omega (1 + |Dv|^2)^{1/2}\, dx + \int_\Omega \int_0^v H(x, t)\, dt\, dx$$
$$+ \int_{\partial\Omega} \mu \{ -\min(v - \varphi_1, 0) + \max(v - \varphi_2, 0) \}\, d\mathcal{H}_{n-1},$$

for $\mu=\tilde{\lambda}_\varepsilon$ or $\mu=\lambda_\varepsilon$. Then, we have

$$J_{\lambda_\varepsilon}(u_\varepsilon)\leqq J_{\lambda_\varepsilon}(v), \qquad \forall v\in H^{1,1}(\Omega), \tag{5.40}$$

$$J_{\tilde{\lambda}_\varepsilon}(u_\varepsilon)=J_{\lambda_\varepsilon}(u_\varepsilon)\leqq J_{\lambda_\varepsilon}(v)\leqq J_{\tilde{\lambda}_\varepsilon}(v), \qquad \forall v\in H^{1,1}(\Omega), \tag{5.41}$$

in view of (5.39). The conclusion that u_ε is a minimizing sequence for the functional J is now immediate.

Moreover, on $\Gamma=\partial\Omega_{\delta/2}\cap\partial\Omega$ u satisfies the boundary condition

$$-a_i\rho_i\in\lambda_0\{\beta_1(u-\varphi_1)+\beta_2(u-\varphi_1)\}, \tag{5.42}$$

in view of (5.36), (5.37), and the maximal monotonicity of the β_i's. Indeed, regarded as a multivalued operator in $L^2(\Gamma)$ each $\beta_i(\,.\,-\varphi_i)$ is maximal monotone; furthermore we deduce from (5.36) and (5.37) that u_ε converges strongly and $-a_i(Du_\varepsilon).\rho_i$ weakly in $L^2(\Gamma)$ to u and $-a_i(Du).\rho_i$ respectively. The conclusion (5.42) then follows from well-known results on maximal monotone operators (*cf.* e.g. [3], Chap. I, Prop. 2.5).

REMARK 5.2. — *The preceding considerations are valid for* $\Omega\subset\mathbf{E}^n$, $n\geqq 2$.

But, in the case $n=2$ Frehse [6] has proved that u is not only Lipschitz continuous in $\bar{\Omega}_{\delta/2}$ but also of class $C^1(\bar{\Omega}_{\delta/2})$. This result will enable us to estimate $|Du|_{\Omega_{\delta/3}}$ independent of $\varkappa$ and independent of the further assumptions we supposed $\partial\Omega$ and the φ_i's to satisfy.

THEOREM 5.3. — *Let* $u\in C^1(\bar{\Omega}_{\delta/2})\cap H^{2,2}(\Omega_{\delta/2})$ *satisfy the relations*

$$Au+H(x,u)=0 \quad \text{in } \Omega_{\delta/2}, \tag{5.43}$$

$$-a_i\rho_i\in\lambda_0\{\beta_1(u-\varphi_1)+\beta_2(u-\varphi_2)\} \quad \text{on } \Gamma, \tag{5.44}$$

and

$$\varphi_1\leqq u\leqq\varphi_2 \quad \text{on } \Gamma, \tag{5.45}$$

where $0<\lambda_0<1$.

Then,

$$|Du|_{\Omega_{\delta/3}}\leqq M_3, \tag{5.46}$$

where the constant depends on δ, λ_0, $|D\varphi_1|_\Gamma$, $|D\varphi_2|_\Gamma$, $|u|_{\Omega_{\delta/2}}$, $|(\partial/\partial x)H(x,u(x)|_{\Omega_{\delta/2}}$, $|H(x,u(x))|_{\Omega_{\delta/2}}$, *and on the* C^2*-norm of* Γ.

Proof of Theorem 5.3. — Let $v=(1+|Du|^2)^{1/2}$, and ζ, $0\leqq\zeta\leqq 1$ a cut-off function such that $\operatorname{supp}\zeta\subset\bar{\Omega}_{(3/8)\delta}$. Furthermore, let h be a positive number such that

$$h\geqq\max\{|D\varphi_1|_\Gamma,\ |D\varphi_2|_\Gamma\}+1=h_0. \tag{5.47}$$

The idea is to estimate

$$\eta=\max\{v.\zeta^2-h,\ 0\}. \tag{5.48}$$

Let $A(h, \zeta)$ be the set where η is positive, and let E_1 and E_2 be defined by

$$E_1 = \{x \in \Gamma : u(x) = \varphi_1(x)\}$$

and

$$E_2 = \{x \in \Gamma : u(x) = \varphi_2(x)\}.$$

Since u and the φ_i's are of class C^1 we know that the tangential derivatives of u and φ_i coïncide on E_i for $i=1, 2$. From (5.44) we then conclude

$$|Du| \leqq h_1 = h_1(h_0, \lambda_0) \quad \text{on } E_i \text{ for } i=1, 2. \tag{5.49}$$

Thus, we obtain

$$A(h, \zeta) \cap E_i = \emptyset, \tag{5.50}$$

for $i=1, 2$, if $h > h_1$. But this implies the important result, that

$$-a_i \rho_i = 0 \quad \text{on } A(h, \zeta) \cap \Gamma, \tag{5.51}$$

for those values of h.

Now, we are ready to apply the *a priori* estimates of [7] to conclude that $\eta = 0$ if h is large enough depending on the quantities mentioned in the theorem. We proved in [7] *a priori* estimates for the gradient of solutions to the capillarity equation

$$Au + H(x, u) = 0 \quad \text{in } \Omega, \tag{5.52}$$

$$a_i \rho_i = \beta \quad \text{on } \partial\Omega, \tag{5.53}$$

where $\beta = \beta(x)$ was assumed to be Lipschitz such that $|\beta| \leqq 1 - a$, $a > 0$. But since the estimates are of local nature and since the calculations are performed on the set $A(h, \zeta)$, we can use these estimates in our special case setting formally $\beta = 0$ in (5.53) in view of (5.51), (5.44), (5.45), and the definition of β_1 and β_2.

Strictly speaking there is a formal difficulty to apply the results of [7] directly, namely, we had there to estimate the integral

$$\int_{A(h, \zeta)} w^2 . W . dx, \tag{5.54}$$

for large h, where $w = \log v$, and $W = (1 + |Du|^2)^{1/2}$ (i.e. $v = W$ in our special case), and for simplicity we proved a stronger estimate, namely, we gave a bound for

$$\int_{B(h_0)} w^2 W \, dx, \tag{5.55}$$

where $B(h_0) = \{x \in \Omega : v(x) > h_0\}$ and h_0 is sufficiently large (*cf.* [7]; formula (1.62)).

It is not difficult to bound the integral in (5.54) using only local informations. Indeed we shall show in the Appendix, that if U is a local boundary neighbourhood, then the gradient of a solution u of the boundary value problem (5.52), (5.53) can be estimated in U in terms of

$$|D\beta|_{U \cap \{|Du| > h\}}, \tag{5.56}$$

where h is large, a, and some other *local* quantities. Applied to our case we conclude that the gradient of u is bounded in $\Omega_{\delta/3}$ in terms of the quantities mentioned in Theorem 5.3.

Thus, Theorem 5.2 is also proved.

6. Mixed boundary value problems

In [14] Giusti considered the variational problem

$$(6.1)\quad J(v)=\int_\Omega (1+|Dv|^2)^{1/2}+\int_\Omega\int_0^v H(x,t)\,dt\,dx + \int_{\Gamma_1}|v-\varphi|\,d\mathscr{H}_{n-1}+\int_{\Gamma_2}\beta_0 . v\,d\mathscr{H}_{n-1}\to\min,\qquad \forall v\in H^{1,1}(\Omega),$$

where $\partial\Omega$ is decomposed into the disjoint subsets Γ_1 and Γ_2, and where $\varphi\in L^1(\Gamma_1)$ and $\beta_0\in L^\infty(\Gamma_2)$ are prescribed. Imposing some natural conditions on H, β, and Γ_2 he could prove the existence of a solution $u\in H^{1,1}(\Omega)$ of this variational problem.

We do not know any physical problem where a variational problem of this type occurs, but, nevertheless, it will be of mathematical interest to study those problems.

Giusti raised the question if the solution is smooth near Γ_2 if β_0 and Γ_2 are smooth, but he could not solve it.

Using the preceding results it is very easy to give an affirmative answer. We shall prove that in every boundary neighbourhood of Γ_2, where Γ_2 is of class C^2, and where β_0 is Lipschitz and strictly less than 1, the solution is of class $H^{2,p}$ for any finite p, provided u is bounded in that neighbourhood. We might also treat the case where $\beta_0=\beta_0(x,t)$, and $\beta_0(x,.)$ is a maximal monotone graph. Then, if (locally) Γ_2 is of class C^4, $|\beta_0|<1$, $\partial H/\partial t\geqq\varkappa>0$, and $\beta_0(.,t)$ is Lipschitz, the solution would be Lipschitz continuous and of class $H^{2,2}$ up to those boundary parts, where the assumptions are satisfied. Moreover, if (locally) β_0 is of the form $\beta_0=\beta_0(t)$, β_0 a maximal monotone graph, then the solution would be Hölder continuous up to the boundary even in the most general case where $|\beta_0|\leqq 1$, provided H and Γ_2 locally satisfy appropriate assumptions.

We could also generalize the Dirichlet data on Γ_1 to boundary constraints of the kind: $\varphi_1\leqq v\leqq\varphi_2$. But we shall not treat the most general cases. Instead we assume $\partial\Omega\in C^4$, that $H=H(x,t)$ and $\beta_0=\beta_0(x)$ are Lipschitz continuous functions such that

$$(6.2)\qquad \frac{\partial H}{\partial t}\geqq\varkappa>0$$

and

$$(6.3)\qquad |\beta_0|\leqq 1-a,\qquad a>0.$$

φ is assumed to be of class C^2.

Let $\Gamma_2^\varepsilon=\{x\in\partial\Omega:\ \mathrm{dist}(x,\ \Gamma_2)<\varepsilon\}$, and let λ_ε, $0\leqq\lambda_\varepsilon\leqq 1$, be a sequence of smooth functions such that

$$\lambda_\varepsilon(x)=\begin{cases} 1, & x\in\Gamma_2, \\ 0, & x\in\Gamma_2^\varepsilon-\Gamma_2^{\varepsilon/2}, \\ 1-\varepsilon, & x\in\partial\Omega-\Gamma_2^\varepsilon, \end{cases} \tag{6.4}$$

Then, we define

$$\beta_\varepsilon(x,\ t)=\begin{cases} \lambda_\varepsilon(x)\,\beta(t), & x\in\partial\Omega-\Gamma_2^\varepsilon, \\ 0, & x\in\Gamma_2^\varepsilon-\Gamma_2^{\varepsilon/2}, \\ \lambda_\varepsilon(x)\,\beta_0(x), & x\in\Gamma_2^{\varepsilon/2}, \end{cases} \tag{6.5}$$

where $\beta(t)$ is the maximal monotone graph

$$\beta(t)=\begin{cases} -1, & t<0, \\ [-1,\ 1], & t=0, \\ 1, & t>0 \end{cases} \tag{6.6}$$

and consider the boundary value problem

$$\begin{cases} \mathrm{A}u_\varepsilon+\mathrm{H}(x,\ u_\varepsilon)=0 & \text{in } \Omega, \\ -a_i\,\rho_i\in\beta_\varepsilon(x,\ u_\varepsilon-\varphi) & \text{on } \partial\Omega. \end{cases} \tag{6.7}$$

From Theorem 3.1 we conclude that there exist solutions $u_\varepsilon\in\mathrm{H}^{1,\infty}(\Omega)\cap\mathrm{H}^{2,2}(\Omega)$. Let x_0 be a boundary point interior to Γ_2. Then, for sufficiently small δ, $|\mathrm{D}u_\varepsilon|$ can be estimated in $\Omega_\delta=\Omega\cap\mathrm{B}_\delta(x_0)$ in terms of local quantities involving δ, $|u_\varepsilon|_{\Omega_\delta}$ and the C^2-norm of $\partial\Omega\cap\Omega_{2\delta}$.

Since the estimate is independent of ε, we can go to the limit obtaining a solution u of (6.1) satisfying the same estimate in Ω_δ.

Appendix

In the proof of Theorem 5.3 we needed a completely local version of the *a priori* estimates given in [7], Th. 1.1. Unfortunately, some of the estimates in [7] are not of local nature, so that the results can not be applied directly. We now indicate how to prove a completely local version.

In the following we shall use the notations in [7]. Let Ω be a bounded domain of $\mathbf{R}^n$, $n\geqq 2$, with C^2-boundary $\partial\Omega$, and let $u\in\mathrm{C}^2(\overline{\Omega})$ be a solution of the boundary value problem

$$\mathrm{A}u+\mathrm{H}(x,\ u)=0 \quad \text{in } \Omega,$$
$$a_i\gamma_i=\beta \quad \text{on } \partial\Omega,$$

where $\gamma=(\gamma_1, \ldots, \gamma_n)$ is the exterior normal to $\partial\Omega$ and where $\beta=\beta(x)\in C^{0,1}(\partial\Omega)$ satisfying

$$|\beta|<1-a, \qquad a>0. \tag{A 1}$$

We extend γ and β as Lipschitz continuous functions inside Ω such that the estimate (A 1) remains valid for β. H is Lipschitz continuous satisfying

$$\frac{\partial H}{\partial t}\geqq 0.$$

Let $x_0\in\partial\Omega$ and $\Omega_{\delta_0}=\Omega\cap B_{\delta_0}(x_0)$, $\delta_0>0$.

Then we shall prove:

THEOREM A 1. — *Under the assumptions stated above* $|Du|_{\Omega_{\delta_0/2}}$ *can be estimated by a constant depending on* $|u|_{\Omega_{\delta_0}}$, $|H(x, u(x))|_{\Omega_{\delta_0}}$, $|(\partial/\partial x)H(x, u(x))|_{\Omega_{\delta_0}}$, *the* C^2*-norm of* $\partial\Omega\cap\partial\Omega_{\delta_0}$, n, a, δ_0, *and on the supremum of* $|D\beta|$ *in* Ω_{δ_0} *with respect to the set where the gradient of u is sufficiently large.*

Proof. — We introduce the function

$$v=W-\beta D^i u.\gamma_i, \qquad W=(1+|Du|^2)^{1/2},$$

and we shall show that $|v|_{\Omega_{\delta_0/2}}$ is bounded in terms of the above quantities, and hence $|Du|_{\Omega_{\delta_0/2}}$ in view of (A 1).

We denote by $\mathscr{S}$ the graph of u:

$$\mathscr{S}=\{(x, u(x)):x\in\Omega\}$$

and by $\delta=(\delta^1, \ldots, \delta^{n+1})$ the usual differential operators on $\mathscr{S}$, i. e. for $g\in C^1(\mathbf{R}^{n+1})$ we have

$$\delta^i g=D^i g-\nu_i\sum_{k=1}^{n+1}\nu_k D^k g, \qquad i=1, \ldots, n+1,$$

where $\nu=(\nu_1, \ldots, \nu_{n+1})$ is the exterior normal to $\mathscr{S}$:

$$\nu=W^{-1}(-D^1 u, \ldots, -D^n u, 1).$$

Furthermore, let $a_{ij}=\partial a_i/\partial p^j$, then, the following relations are valid

$$a_{ij}D^i g D^j g=W^{-1}|\delta g|^2, \qquad g\in C^1(\overline{\Omega}), \tag{A 2}$$

$$|a_{ij}D^i g D^j\varphi|\leqq W^{-1}|\delta g|.|D\varphi|, \qquad g, \varphi\in C^1(\overline{\Omega}), \tag{A 3}$$

$$a_{ij}p^i q^j\leqq\frac{\varepsilon}{2}a_{ij}p^i p^j+\frac{1}{2\varepsilon}a_{ij}q^i q^j, \tag{A 4}$$

and

$$a.W\leqq v\leqq 2.W. \tag{A 5}$$

Let $w = \log v$. In view of the results in [7], p. 167.
Theorem A 1 would be proved provided the integral

(A 6) $$\int_{\Omega_{\delta_0/2} \cap \{v > h\}} w^2 \, \mathrm{W} \, dx$$

could be estimated in terms of local quantities for large values of h. Equivalently we could ask to estimate

(A 7) $$\int_{\mathrm{B}(h, \zeta)} w^2 . \mathrm{W} . \zeta^2 \, dx,$$

where ζ, $0 \leqq \zeta \leqq 1$, is any smooth function with $\operatorname{supp} \zeta \subset \mathrm{B}_{\delta_0}(x_0)$ and where

$$\mathrm{B}(h, \zeta) = \{ x \in \Omega : \max(v - h, 0) . \zeta^2 > 0 \}.$$

To estimate the integral in (A 7) we prove (*cf.* [7], Lemma 1.5):

LEMMA A 1. — *Suppose the assumptions of Theorem* A 1 *to be satisfied. Then we have*

(A 8) $$\int_{\mathrm{B}(h, \zeta)} [\mathrm{W}^{-3} | \mathrm{D} v |^2 + v] \, dx \leqq \text{const.},$$

where the constant depends on local well-known quantities.

Proof of Lemma A 1. — We use the crucial inequality (*cf.* [7], (1.49)):

(A 9) $$\int_{\Omega} a_{ij} [\mathrm{D}^j v + \mathrm{D}^j (\beta \gamma_k) . \mathrm{D}^k u] . \mathrm{D}^i \eta \, dx \leqq c \int_{\partial \Omega} \eta \, d\mathscr{H}_{n-1} + c \int_{\Omega} \left[\frac{|\delta v|}{\mathrm{W}} + 1 \right] \eta \, dx,$$

where c is a suitable constant and η any nonnegative Lipschitz function with $\operatorname{supp} \eta \subset \{ v > h \}$, h sufficiently large. Inserting $\eta = \max(v - h, 0) \, \zeta^2$ in this inequality and using (A 2)-(A 5) we obtain

(A 10) $$\int_{\mathrm{B}(h, \zeta)} \frac{|\delta v|^2}{\mathrm{W}} \zeta^2 \, dx \leqq c \int_{\mathrm{B}(h, \zeta)} \mathrm{W} \, dx + c \int_{\partial \Omega} v . \zeta^2 \, d\mathscr{H}_{n-1},$$

where the constant c depends on $|\mathrm{D}\zeta|$ and known quantities.

On the other hand we know from ([7], Lemma 1.4) that

$$\int_{\partial \Omega} v . \zeta^2 \, d\mathscr{H}_{n-1} \leqq c \int_{\Omega} [|\delta \zeta| . \zeta + \zeta^2] \, \mathrm{W} \, dx.$$

Now, look at the identity

(A 11) $$\int_{\Omega} a_i \mathrm{D}^i \eta \, dx + \int_{\Omega} \mathrm{H} . \eta \, dx - \int_{\partial \Omega} \beta . \eta \, d\mathscr{H}_{n-1} = 0, \qquad \forall \eta \in \mathrm{C}^1(\overline{\Omega}).$$

Choosing $\eta = u . \zeta^2$ and taking the estimate

(A 12) $$\int_{\partial \Omega} |\beta \eta| \, d\mathscr{H}_{n-1} \leqq (1 - a) \int_{\Omega} |\mathrm{D} \eta| \, dx + c \int_{\Omega} |\eta| \, dx,$$

into account ([10], Lemma 1) we deduce

$$\text{(A 13)} \qquad \int_{B(h,\zeta)} W^{-3}|Dv|^2.\zeta^2\,dx \leqq \int_{B(h,\zeta)} \frac{|\delta v|^2}{W}\zeta^2\,dx \leqq \text{const.}$$

To estimate the integral in (A 7) we use the relation (A 11) once more, this time with $\eta = u.\max(w^2-h,0).\zeta^2$, and we obtain in view of (A 12):

$$\text{(A 14)} \quad \int_{\{w^2>h\}} \{a_i D^i u(w^2-h).\zeta^2 + u.a_i D^i v.v^{-1}.\zeta^2.2w + u.a_i.2.D^i\zeta.\zeta(w^2-h) + H.u.(w^2-h).\zeta^2\}\,dx$$
$$\leqq (1-a)\int_{\{w^2>h\}} \{|Du|.(w^2-h).\zeta^2 + |u|2w.|D^i v|.v^{-1}.\zeta^2 + |u|.(w^2-h).2.\zeta|D^i\zeta|\}\,dx + c\int_{\{w^2>h\}} |u|(w^2-h)\zeta^2\,dx.$$

Using the inequality $|ab| \leqq (\varepsilon/2)a^2 + (1/2\varepsilon)b^2$ we deduce from (A 14):

$$\text{(A 15)} \qquad \int_{\{w^2>h\}} |Du|w^2\zeta^2\,dx \leqq c\int_{\{w^2>h\}\cap\operatorname{supp}\zeta} \{W^{-3}|Dv|^2.\zeta^2 + h.W + W\}\,dx,$$

where the constant depends on a, $|D\zeta|$, and known quantities.

Here we also used the estimate

$$w^2 \leqq c.W,$$

with some suitable constant c.

The result now follows in view of Lemma A 1.

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