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Existence and Regularity of Capillary Surfaces.

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Sunto. – *Si prova che il problema variazionale*

$$\int_{\Omega}(1+|Dv|^2)^{\frac{1}{2}}\,dx+\int_{\Omega}\int_0^v H(x,t)\,dt\,dx-\int_{\partial\Omega}\varkappa v\,dH_{n-1}\to\min$$

in $BV(\Omega)$ *ha una soluzione* $u\in C^{0,1}(\Omega)\cap L^{\infty}(\Omega)$ *nell'ipotesi che* H, $\varkappa$ *e* $\partial\Omega$ *verifichino opportune condizioni.*

0. – Introduction.

Let $\Omega\subset\mathbf{R}^n$, $n\geqslant 2$, be a bounded domain and let

$$A=-D^i(a_i(p))\ (^1)$$

be the *minimal surface operator*, i.e.

$$a_i(p)=p^i(1+|p|^2)^{-\frac{1}{2}}.$$

The classical problem of capillarity [9] consists in determining a function $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ such that

$$Au+cu=0 \quad \text{in } \Omega \tag{1}$$

and

$$D^iu\cdot\nu_i=\varkappa(1+|Du|^2)^{-\frac{1}{2}} \quad \text{on } \partial\Omega, \tag{2}$$

where ν_i are the components of the outward normal vector at $\partial\Omega$, and c and $\varkappa$ are given constants such that

$$c>0 \quad \text{and} \quad |\varkappa|\leqslant 1. \tag{3}$$

(1) Here and in the following repeated indices will denote summation over them from 1 to n.

If there exists a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of problem (1) and (2), then this solution also minimizes the functional

$$I(v):=\int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\,dx + c\int_\Omega |v|^2\,dx - \varkappa\int_{\partial\Omega} v\,d\mathcal{H}_{n-1} \tag{4}$$

in the function class $BV(\Omega)$ (see Appendix I for the definition of $BV(\Omega)$ and for some properties of its elements).

In a recent paper M. EMMER [4] has proved that the functional I has a minimum in $BV(\Omega)$ which is locally bounded provided that $|\varkappa|<(1+L^2)^{-\frac{1}{2}}$, where L is a constant which depends on the boundary of Ω. Moreover, by a result due to MASSARI [8] he could show that u is real analytic in the interior of Ω for $n\leqslant 6$.

In this paper we shall consider a slightly more general variational problem in $BV(\Omega)$, namely, we want to minimize the functional

$$J(v):=\int_\Omega (1+|Dv|^2)^{\frac{1}{2}}\,dx + \int_\Omega\int_0^v H(x,t)\,dt\,dx - \int_{\partial\Omega} \varkappa v\,d\mathcal{H}_{n-1} \tag{5}$$

where $\varkappa$ belongs to $L^\infty(\partial\Omega)$, $|\varkappa|_{\partial\Omega}\leqslant 1$, and $H \in C^{0,1}(\mathbb{R}^n\times\mathbb{R})$ satisfies the conditions

$$\frac{\partial H}{\partial t}>0 \tag{6}$$

and

$$\begin{cases} H(x,k_0)\geqslant 1+c_1\,, \\ H(x,-k_0)\leqslant -(1+c_1)\,, \end{cases} \tag{7}$$

for some $k_0>0$, where c_1 is a constant which appears in the following.

The main theorem which we shall prove is

THEOREM 1. – *Suppose that $\partial\Omega$ is of class C^2 and that $|\varkappa|_{\partial\Omega}\leqslant 1-a$, $a>0$. Then the variational problem*

$$J(v)\to\min \quad \text{in } BV(\Omega) \tag{8}$$

has a solution $u\in C^{0,1}(\Omega)\cap H^{1,1}(\Omega)\cap L^\infty(\Omega)$.

When writing the manuscript I heard from R. FINN that P. CONCUS and he have considered a similar problem. They have shown that a solution $u\in C^{0,1}(\Omega)$ of the equation

$$Au+H(x,u)=0 \qquad \text{in } \Omega$$

is bounded in $\bar{\Omega}$ provided that $\partial\Omega$ satisfies an internal sphere condition, i.e. for any boundary point $x_0 \in \partial\Omega$, there is a ball B of fixed radius such that $B \subset \Omega$ and $x_0 \in \partial B$. The result of Concus and Finn does not seem to apply in its present form to that step in the existence proof. Moreover, our method is also applicable in the case of Lipschitz domains, using Lemma 1.1 in [4], with the natural restriction on $|\varkappa|$, instead of Lemma 1 below.

Applying the results of Concus and Finn one might show by approximation that Theorem 1 remains valid under the natural condition $|\varkappa|_{\partial\Omega} \leqslant 1$.

1. – A priori estimate for $|u|$.

Since CONCUS and FINN have shown in [1] that in general one cannot expect a bounded solution of (8) if $\partial\Omega$ has vertices, the a priori estimate will depend on the fact that we assumed $\partial\Omega$ to be smooth. In this case we can prove the following lemma (compare [4]; Lemma 1.1).

LEMMA 1. – *Let $v \in BV(\Omega)$ and suppose that $\partial\Omega$ is of class C^2 satisfying an internal sphere condition of radius R. Then we have*

$$\int_{\partial\Omega} |v|\, d\mathcal{H}_{n-1} \leqslant \int_{\Omega} |Dv|\, dx + c_1 \int_{\Omega} |v|\, dx \tag{9}$$

where

$$c_1 = \frac{2n}{R}. \tag{10}$$

PROOF. – In view of our assumption the distance function d,

$$d(x) = \operatorname{dist}(x, \partial\Omega)\,,$$

belongs to $H^{2,\infty}(\Omega_{R/2})$, where $\Omega_{R/2}$ is the boundary strip $\{x \in \Omega: d(x) < R/2\}$. Moreover, we have the relation (cf. [13]; Chap. 1.3)

$$-\Delta d(x) = \sum_{i=1}^{n-1} \frac{k_i(y)}{1 - k_i(y)\, d(x)}\,, \tag{11}$$

where k_i are the components of the normal curvature vector of that point $y \in \partial\Omega$ having smallest distance to x. From the interior sphere condition we derive $k_i \leqslant 1/R$. Hence, we have

$$(12)\qquad \frac{k_i}{1-k_1 d} \leqslant \begin{cases} 0 & \text{if } k_i \leqslant 0\,, \\ \dfrac{2}{R} & \text{if } 0 \leqslant k_i \leqslant \dfrac{1}{R}\,. \end{cases}$$

Thus, we deduce the estimate

$$(13)\qquad -\Delta d(x) \leqslant \frac{2(n-1)}{R} \qquad \forall x \in \Omega_{R/2}\,.$$

In order to prove the inequality (9) we may restrict ourselves to the case $v \geqslant 0$. Assuming this, we get by partial integration (cf. Appendix I, Lemma A3)

$$(13a)\qquad -\int_{\Omega_{R/2}} \Delta d \cdot v \cdot (R/2 - d)\,dx = -\int_{\Omega_{R/2}} v\,dx + \\ + \int_{\Omega_{R/2}} (R/2 - d)\, D^i d \cdot D^i v\,dx + R/2 \int_{\partial\Omega} v\,d\mathcal{H}_{n-1}\,.$$

Hence,

$$(13b)\qquad \int_{\partial\Omega} v\,dH_{n-1} \leqslant \int_{\Omega} |Dv|\,dx + \int_{\Omega_{R/2}} \left\{\max(-\Delta d, 0) + \frac{2}{R}\right\} \cdot v\,dx$$

from which the assertion follows.

With the help of Lemma 1 we immediately get

LEMMA 2. – *Let $\partial\Omega$ be of class C^2 and $|\varkappa| \leqslant 1 - a$, $a > 0$. Then a solution $u \in BV(\Omega)$ of the variational problem* (8) *is absolutely bounded by some constant c_2, which depends on a, c_1, Ω, and the structure of H.*

PROOF. – Let $k \geqslant 0$ be a real number and set $A(k) := \{x \in \Omega : u(x) > k\}$. Since $v := \min(u, k)$ belongs to $BV(\Omega)$ we obtain from the minimum property of u

$$(14)\qquad J(u) \leqslant J(v)$$

and hence

$$(15)\qquad \int_{\Omega} |D \max(u - k, 0)|\,dx + \int_{A(k)} \int_k^u H(x, t)\,dt\,dx - \\ - \int_{\partial\Omega} \varkappa \max(u - k, 0)\,d\mathcal{H}_{n-1} \leqslant |A(k)| \text{ [2]}$$

(see Lemma A4 in Appendix I for the verification of (15).)

[2] For any measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes its Lebesgue measure in $\mathbb{R}^n$.

Setting $w := u - v = \max(u - k, 0)$, we get from Lemma 1

$$(16)\qquad \Big|\int_{\partial\Omega} \varkappa w\, d\mathcal{H}_{n-1}\Big| \leqslant (1-a)\int_\Omega |Dw|\, dx + (1-a)\, c_1 \int_\Omega w\, dx\,.$$

Combining the inequalities (15) and (16) we obtain

$$(17)\qquad a\int_\Omega |Dw|\, dx + \int_{A(k)} \int_k^u \{H(x,t) - c_1\}\, dt\, dx \leqslant |A(k)|$$

or finally

$$(18)\qquad a\int_\Omega |Dw| dx + \inf_{x\in\Omega} \{H(x,k) - c_1\}\cdot \int_{A(k)} (u-k)\, dx \leqslant |A(k)|\,.$$

Since $\partial\Omega$ is regular we have the following Sobolev imbedding result:

Any function $v \in BV(\Omega)$ belongs to $L^{n/n-1}(\Omega)$, and

$$(19)\qquad \Big(\int_\Omega |v|^{n/n-1}\, dx\Big)^{(n-1)/n} \leqslant c_3\cdot \Big\{\int_\Omega |Dv|\, dx + \int_\Omega |v|\, dx\Big\}\,.$$

Taking $v = w$ in (19) we get from (18)

$$(20)\qquad a/c_3 \Big(\int_\Omega |v|^{n/n-1}\, dx\Big)^{(n-1)/n} + \inf_{x\in\Omega}\{H(x,k) - c_1 - a\}\cdot$$
$$\cdot \int_{A(k)} (u-k)\, dx \leqslant |A(k)|$$

From the assumption (7) and using the Hölder inequalities we therefore obtain for $k \geqslant k_0$

$$(21)\qquad \int_{A(k)} (u-k)\, dx \leqslant c_4\cdot |A(k)|^{1+1/n}\,, \qquad c_4 > 0\,,$$

and hence

$$(22)\qquad |h-k|\cdot |A(h)| \leqslant c_4\cdot |A(k)|^{1+1/n} \qquad \text{for } h > k\,.$$

From a lemma due to STAMPACCHIA [14; Lemma 4.1] we now conclude that

$$u \leqslant k_0 + c_4\cdot |\Omega|^{1/n}\cdot 2^{(n+1)}\,.$$

In order to get a lower bound for u, we set $v := \max(u, -k)$ in (14). Then one could complete the proof of Lemma 2 by similar conclusions, which will be omitted.

2. – Existence of a solution in $BV(\Omega)$.

We shall show that under the assumptions of Theorem 1 the variational problem (8) has a solution in $BV(\Omega)$.

Let v_ε be a minimizing sequence

$$J(v_\varepsilon) \to \inf_{v \in BV(\Omega)} J(v) \leqslant J(0) =: c_5 . \tag{23}$$

From Lemma 1 we conclude that

$$a\int_\Omega |Dv_\varepsilon|\, dx + \int_\Omega \Big\{ \int_0^{v_\varepsilon} H(x, t)\, dt - c_1 |v_\varepsilon| \Big\}\, dx \leqslant c_5 . \tag{24}$$

Hence, we easily derive from the assumption (7)

$$\int_\Omega |Dv_\varepsilon|\, dx + \int_\Omega |v_\varepsilon|\, dx \leqslant c_6 . \tag{25}$$

From [12; Theorem XVI], the Sobolev imbedding theorem, and [11; Theorem 2.1.3] we then conclude that the sequence v_ε is precompact in any $L^p(\Omega)$, $1 \leqslant p < n/n-1$. Since the functional J is lower semicontinuous with respect to a minimizing sequence in $BV(\Omega)$ (see Appendix II), a subsequence of v_ε converges to some element $u \in BV(\Omega)$, which is a solution of problem (8).

REMARK 1. – *Since we assume $H(x, \cdot)$ to be strictly increasing* a.e., *the variational problem* (8) *has a unique solution. Or more precisely, if $\varkappa \leqslant \varkappa'$, and u, u' are the respective solutions of* (8) *according to the functionals J, J' then $u \leqslant u'$.*

PROOF OF REMARK 1. – From the strict monotonicity of $H(x, \cdot)$ we deduce

$$J(u) < J\big(\min(u, u')\big) \qquad \text{or} \qquad u = \min(u, u') \tag{26}$$

and

$$J'(u') < J'\big(\max(u, u')\big) \qquad \text{or} \qquad u' = \max(u, u') . \tag{27}$$

Combining these relations and using the fact that

$$(28)\qquad -\int_{\partial\Omega}\varkappa(u-\min(u,u'))\,d\mathcal{H}_{n-1}\geqslant\int_{\partial\Omega}\varkappa'(u'-\max(u,u'))\,d\mathcal{H}_{n-1}$$

it follows from (26) or (27) that $u=\min(u,u')$.

3. – Regularity of solutions in $BV(\Omega)$.

The regularity of u will follow from a general theorem concerning the regularity of solutions $w\in BV(\Omega)$ of the variational problem

$$(29)\qquad L(v):=\int_{\Omega}(1+|Dv|^2)^{1/2}\,dx+\int_{\Omega}\int_0^v H(x,t)\,dt\,dx+j(v)\ ,$$

where $j(v)$ denotes a boundary term continuous in $L^1(\partial\Omega)$, e.g.

$$(30)\qquad j(v)=\int_{\partial\Omega}|v-f|\,d\mathcal{H}_{n-1}\ ,\qquad f\in L^1(\partial\Omega)$$

or

$$(31)\qquad j(v)=\int_{\partial\Omega}\varkappa v\,d\mathcal{H}_{n-1}\ .$$

THEOREM 2. – *Let w be a bounded* (3) *solution in $BV(\Omega)$ of the variational problem* (29). *Suppose that $H\in C^{0,1}(\mathbb{R}^n\times\mathbb{R})$ is strictly increasing in t. Then w is locally Lipschitz in Ω.*

PROOF. – We shall use the results of SERRIN [13] concerning the existence of surfaces of prescribed mean curvature together with the a priori estimates of LADYZHENSKAYA and URALTSEVA [7].

Without loss of generality we may assume H to be bounded in Ω (4). Then for R sufficiently small, $R<R_0$, we can solve the Dirichlet problem in any Ball $B\subset\subset\Omega$ of radius R

$$(32)\qquad \begin{cases} Av_\varepsilon+H(x,v_\varepsilon)=0 & \text{in } B\ ,\\ \qquad v_{\varepsilon|\partial B}=w_{\varepsilon|\partial B}\ , \end{cases}$$

(3) If $H=H(x)$ or $H(x,t)=c\cdot t$, $c>0$, then it suffices to assume w to be locally bounded.

(4) Choose *e.g.* $H_k:=\min(H,k)+\max(H,-k)-H$. Then $H(x,u)=$ $=H_k(x,u)$ if k is large enough.

where w_ε is a mollification of w. From the results of Serrin we conclude that (32) has a solution $v_\varepsilon \in C^{\cdot}(\bar{B})$ such that

$$|v_\varepsilon|_B \leqslant c_7 = c_7(|w|_B, R, |H|) . \tag{33}$$

From the a priori estimates of Ladyzhenskaya and Uraltseva we then deduce

$$|Dv_\varepsilon|_{\Omega'} \leqslant c_8 = c_8(c_7, |DH|, \Omega') \qquad \text{for } \Omega' \subset\subset B . \tag{34}$$

Moreover, we know that v_ε minimizes the functional

$$I_\varepsilon(v) := \int_B (1 + |Dv|\ )^{1/2} + \int_B \int_0^v H(x, t)\, dt\, dx + \int_{\partial\Omega} |v - w_\varepsilon|\, d\mathcal{H}_{n-1} \tag{35}$$

in $BV(B)$.

Hence we have the inequality

$$\begin{aligned} \int_B (1 + |Dv_\varepsilon|^2)^{1/2} dx + \int_B \int_0^{v_\varepsilon} H(x, t)\, dt\, dx \leqslant \\ \leqslant \int_B (1 + |Dw_\varepsilon|^2)^{1/2} dx + \int_B \int_0^{w_\varepsilon} H(x, t)\, dt\, dx . \end{aligned} \tag{36}$$

Setting

$$\tilde{v}_\varepsilon := \begin{cases} v_\varepsilon & \text{in } B , \\ w_\varepsilon & \text{in } \Omega - B , \end{cases} \tag{37}$$

we derive in view of (36)

$$L(\tilde{v}_\varepsilon) \leqslant L(w_\varepsilon) . \tag{38}$$

From Appendix I, Lemma A1 and Lemma A2, and from Lebesgue's theorem of dominated convergence we conclude, that the right side of (38) tends to $L(w)$, if ε goes to zero. From the estimates (33), (34), and from the definition (37) we conclude, that the v_ε's converge in $BV(\Omega)$ to some element v_0 which is locally Lipschitz in B and agrees with w in $\Omega - \bar{B}$. Moreover, we immediately derive on account ot Lemma A2 in Appendix I that

$$L(v_0) \leqslant \liminf L(v_\varepsilon) \leqslant L(w) . \tag{39}$$

Hence, v_0 equals to w, since $v_{0|\partial\Omega} = w_{|\partial\Omega}$ and the variational problem (29) has no distinct solutions withe qual boundary valu es.

As we mentioned in the Introduction P. CONCUS and R. FINN proved a priori estimates for the modulus of solutions to the equation

$$Au + H(x, u) = 0 \qquad \text{in } \Omega$$

provided that $\partial\Omega$ satisfies an internal sphere condition.

In order to prove Lemma 1, which is the key lemma in the existence proof, we made the more restrictive assumption that $\partial\Omega$ should be of class C^2. However, we shall show that a conclusion similar to that of Lemma 1 is valid in the more general case.

REMARK 2. – *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ which satisfies an internal sphere condition of radius R. Then*

$$\int_{\partial\Omega} |v|\, d\mathcal{H}_{n-1} \leqslant \int_{\Omega_{R/2}} |Dv|\, dx + c_1 \cdot \int_{\Omega} |v|\, dx \qquad \forall v \in BV(\Omega) \tag{40}$$

where c_1 depends on n, R, and $\partial\Omega$.

PROOF OF REMARK 2. – Let Γ be an relatively open subset of $\partial\Omega$ which is representable as the graph of a Lipschitz function φ defined on some open subset V_r of $\mathbb{R}^{n-1}$, $V_r = \{x' \in \mathbb{R}^{n-1} \colon |x'| < r\}$, such that $0 < \varphi \leqslant \alpha$, and for $\beta > \alpha$ let

$$U_{r,\beta} = \{(x', x^n) \colon x' \in V_r,\ \varphi(x') < x^n < \beta\} \subset \Omega_{R/2}\,. \tag{41}$$

Furthermore, choose two positive numbers r_1, r_2 with $r_1 < r_2 < r$, and let φ^ε be a mollification of φ with a mollifier η. Then φ^ε is well defined in V_{r_2} if ε is sufficiently small and

$$U^\varepsilon_{r_2,\beta'} = \{(x', x^n) \colon x' \in V_{r_2},\ \varphi^\varepsilon(x') < x^n < \beta'\} \subset \Omega_{R/2} \tag{42}$$

for some β', $\alpha < \beta' < \beta$, independent of ε.

We shall show that the principal curvatures k_i^ε of $\Gamma_\varepsilon = \operatorname{graph} \varphi^\varepsilon{}_{|V_{r_1}}$ (with respect to the internal normal vector) are bounded from above by some constant c_0 which depends on R and the Lipschitz constant of φ.

Let $x_0' \in V_r$ be given and let L be the Lipschitz constant of φ. In view of the interior sphere condition there exists a ball B with radius R such that $B \subset \Omega$ and $x_0 = (x_0', \varphi(x_0')) \in \partial B$. Since φ is Lipschitz, a part of the sphere containing x_0 might be represented as the graph of a C^2 function f defined in a suitable neighbourhood V of x_0'. Thus, we obtain

$$\varphi(x_0') = f(x_0') \quad \text{and} \quad \varphi(x') \leqslant f(x') \quad \forall x' \in V\,. \tag{43}$$

Let $\xi \in \mathbb{R}^{n-1}$ be an arbitrary unit vector and let $t > 0$ be sufficiently small. Then, we deduce from (43)

$$f(x_0' + t\xi) - f(x_0') \geqslant \varphi(x_0' + t\xi) - \varphi(x_0') \geqslant -t \cdot L\,, \tag{44}$$

hence

$$Df(x_0') \cdot \xi \geqslant -L \tag{45}$$

which implies

$$|Df(x_0')| \leqslant L \tag{46}$$

since ξ was arbitrary.

Therefore, we conclude that the second derivatives of f are bounded by a constant depending on R and L, independent of x_0'.

Now, let $h \neq 0$ be any vector in $\mathbb{R}^{n-1}$ of sufficiently small norm and set for any function v

$$v_h(x') = \frac{1}{|h|^2} \cdot \{v(x' + h) - 2v(x') + v(x' - h)\}\,. \tag{47}$$

In view of (43) and (46) we deduce that there are some positive number h_0 and a constant c_0 such that for any $x' \in V_{r_2}$ we have

$$\varphi_h(x') \leqslant f_h(x') \leqslant c_0 \qquad \forall h \neq 0\,,\ |h| < h_0\,, \tag{48}$$

where only f depends on x' but not c_0.

Thus, for sufficiently small ε, we derive

$$\varphi_h^{\varepsilon}(x') = \int\limits_{|z| \leqslant 1} \eta(z) \varphi_h(x' + \varepsilon z)\, dz \leqslant c_0 \qquad \forall x' \in V_{r_1}. \tag{49}$$

Moreover, let $\xi \in \mathbb{R}^{n-1}$ be any unit vector and set

$$h = t \cdot \xi \qquad \text{with } 0 < t < h_0\,. \tag{50}$$

Inserting h into (49) and letting t go to zero this yields

$$D^2\varphi_\varepsilon(x')\xi^2 \leqslant c_0 \qquad \forall x' \in V_{r_1}\,. \tag{51}$$

Throughout the rest of the proof of Remark 2 let us observe that the indices which will appear run from 1 to $n-1$. To compute the principal curvature k_i^ε of Γ_ε at $x = (x', \varphi^\varepsilon(x'))$, we set

$$x_i = D^i x = \left(\delta_i, D^i\varphi^\varepsilon(x')\right)\,, \tag{52}$$

where $\delta_i = (\delta^{ij})_{j=1,\ldots,n-1}$ and

$$g_{ik} = x_i \cdot x_k = \delta^{ik} + D^i\varphi^\varepsilon \cdot D^k\varphi^\varepsilon\,. \tag{53}$$

Furthermore, let $\boldsymbol{n} = (\alpha_1, \ldots, \alpha_n)$ be the internal normal vector at x and define L_{ik} by

$$L_{ik} = \boldsymbol{n}\cdot D^k x_i = \alpha_n D^i D^k \varphi^\varepsilon \tag{54}$$

where we observe that $\alpha_n \geqslant \alpha^* > 0$, since Γ^ε is a Lipschitz graph.

Then, the principal curvatures k_i^ε of Γ^ε at x are the extrema of the quadratic form

$$\xi \to L_{ik}\xi^i\xi^k \tag{55}$$

subject to the constraints

$$g_{ik}\cdot\xi^i\xi^k = 1\,. \tag{56}$$

Therefore, since

$$g_{ik}\xi^i\xi^k = |\xi|^2 + |D\varphi^\varepsilon\cdot\xi|^2 \geqslant |\xi|^2 \tag{57}$$

and

$$0 < \alpha_n \leqslant 1 \tag{58}$$

we derive in view of (51) and (54)

$$k_i^\varepsilon \leqslant c_0 \qquad \forall x' \in V_{r_1}\,. \tag{59}$$

To complete the proof of the remark, we observe that in view of (59) and [13; Chapter 1.3] there is some positive number γ,

$\alpha < \gamma < \beta'$, where γ is independent of ε, such that the distance function $d_\varepsilon(x) = \operatorname{dist}(x, \Gamma_\varepsilon)$ is of class C^2 in

$$U^\varepsilon_{r_1,\gamma} = \{(x', x^n) \colon x' \in V_{r_1},\ \varphi^\varepsilon(x') < x^n < \gamma\} \tag{60}$$

for sufficiently small values of ε and that

$$-\Delta d_\varepsilon \leqslant c = c(n, c_0, \gamma) \tag{61}$$

taking the relation (13) into account.

Now, let $U \subset \mathbb{R}^n$ be open such that $U \cap \Omega \subset U_{r_1,\gamma'}$, for some fixed γ', $\alpha < \gamma' < \gamma$, and let $v \geqslant 0$ be a smooth function whose support is contained in U. Then, using integration by parts we obtain

$$\int_{U^\varepsilon_{r_1,\gamma}} -\Delta d_\varepsilon \cdot v\,dx = \int_{U^\varepsilon_{r_1,\gamma}} \sum_{i=1}^n D^i d_\varepsilon \cdot D^i v\,dx + \int_{\Gamma_\varepsilon} v\,d\mathcal{H}_{n-1} \tag{62}$$

for small values of ε. Hence, we conclude from (61)

$$\int_{\Gamma_\varepsilon} v\,d\mathcal{H}_{n-1} \leqslant \int_{U^\varepsilon_{r_1,\gamma}} |Dv|\,dx + c \cdot \int_{U^\varepsilon_{r_1,\gamma}} v\,dx\,. \tag{63}$$

Taking the limit on both sides this yields

$$\int_{\partial\Omega} v\,d\mathcal{H}_{n-1} = \int_{\Gamma} v\,d\mathcal{H}_{n-1} \leqslant \int_{\Omega_{R/2}} |Dv|\,dx + c \cdot \int_{\Omega} v\,dx\,. \tag{64}$$

Finally, we let U_k, $k = 1, \ldots, m$ be a finite covering of $\partial\Omega$ by open sets of the kind we described above, and we let φ_k, $k = 1, \ldots, m$, be a subordinate partition of unity by smooth functions φ_k such that

$$\sum_{k=1}^m \varphi_k(x) = 1 \qquad \forall x \in \partial\Omega\,. \tag{65}$$

Let $v \geqslant 0$ be an arbitrary smooth function. Then, applying the estimate (64) to $\varphi_k \cdot v$ and summing over k we obtain

$$\int_{\partial\Omega} v\,dH_{n-1} \leqslant \int_{\Omega_{R/2}} |Dv|\,dx + \left\{\sum_{k=1}^m \sup|D\varphi_k| + c\right\} \cdot \int_{\Omega} v\,dx\,. \tag{66}$$

The estimate (40) now follows by approximation in view of the Lemmata A1 and A2 in Appendix I.

REMARK 3. – *If we take* $H(x,t) = c\cdot t$, $c > 0$, *then we may also solve the variational problem* (8) *when the volume* V *is prescribed*

$$V = \int_\Omega v\,dx = \text{const}\,.$$

THEOREM 3. – *Under the assumptions stated above the variational problem*

$$J(v) \to \min \qquad \textit{in } BV(\Omega) \cap \left\{ \int_\Omega v\,dx = V \right\} \tag{67}$$

has a unique solution $u^* \in C^{0,1}(\Omega) \cap H^{1,1}(\Omega) \cap L^\infty(\Omega)$.

PROOF. – Let u be the solution of (8). Define u^* by

$$u^* := u + \lambda \tag{68}$$

where λ is a real number such that the volume of u^* equals to V.
Let v be a function in $BV(\Omega)$ with volume V. Then

$$\begin{aligned} J(u^*) &= J(u) + c/2\int_\Omega (2u\lambda + \lambda^2)\,dx - \lambda\cdot\int_{\partial\Omega} \varkappa\, d\mathcal{H}_{n-1} \leqslant \\ &\leqslant J(v-\lambda) + c/2\int_\Omega (2u\lambda + \lambda^2)\,dx - \lambda\cdot\int_{\partial\Omega} \varkappa\, d\mathcal{H}_{n-1} = J(v) \end{aligned} \tag{69}$$

as one easily checks from the definition of λ. The uniqueness of the solution follows from the Remark 1.

Appendix I.

We present here the definition of $BV(\Omega)$ and some properties of its elements. We assume throughout the following that Ω is a bounded, open set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$.

DEF. – $BV(\Omega) := \{v \in L^1(\Omega) \colon D^i v$ is a bounded Radon measure on Ω, $i = 1, \ldots, n\}$.

1) Every $v \in BV(\Omega)$ has a trace $t(v)$ on $\partial\Omega$, such that $t(v) \in L^1(\partial\Omega)$. For brevity we shall write v instead of $t(v)$.

If (A, w) is a local boundary neighbourhood of $\partial\Omega$, such that A is an open set in $\mathbb{R}^{n-1}$ and $w\colon A \to \mathbb{R}_+$ is a Lipschitz map with $\inf w(A) > 0$, and such that

$$\Omega' := \{(x, y) \colon x \in A,\ 0 < y < w(x)\} \subset \Omega$$

and

$$S := \{(x, w(x)) : x \in A\} \subset \partial\Omega ,$$

then we have the following Green's formula for $v \in BV(\Omega)$

$$\text{(A1)} \qquad \int_{\Omega'} v D^i g_i \, dx + \int_{\Omega'} g_i D^i v \, dx = \int_S v g_i \nu_i \, d\mathcal{H}_{n-1} \qquad \forall g_i \in C_c^1(A \times \mathbb{R}_+) ,$$

where ν_i are the components of the outward normal vector at S, and where we have written $\int_{\Omega'} g_i D^i v \, dx$ instead of $\int_{\Omega'} g_i \, d(D^i v)$ (see [10; Theorem 1]).

2) By a trace theorem due to GAGLIARDO [5] every $v \in BV(\Omega)$ can be extended to $\tilde{v} \in BV(B)$, where B is a ball, $\Omega \subset\subset B$, such that $\tilde{v}_{|\partial B} = 0$ and $\tilde{v}_{|\partial\Omega} = v$. Hence

$$\text{(A2)} \qquad \int_{\partial\Omega} |Dv| \, dx = 0 .$$

$|Dv|$ is the total variation of $(D^1 v, \ldots, D^n v)$. For any open set $A \subset \mathbb{R}^n$

$$\int_A |Dv| \, dx := \sup \left\{ \int_A v D^i g_i \, dx : \; g_i \in C_c^1(A), \; |(g_1, \ldots, g_n)| \leqslant 1 \right\} .$$

(see [10; Theorem 2]).

3) *For any $v \in BV(\Omega)$ $D^i v$, $i = 1, \ldots, n$, is a measure on $C^0(\bar{\Omega})$.*

PROOF. – From [3; (13.9.17)] it follows that

$$\text{(A3)} \qquad \left| \int_\Omega g D^i v \, dx \right| \leqslant \|g\|_\infty \cdot \int_\Omega |D^i v| \, dx$$

for any $g \in C^0(\Omega) \cap L^\infty(\Omega)$. The conclusion is now evident in view of (A2).

LEMMA A1. – *Let $v \in BV(\Omega)$ and v_ε be a mollification* [5]. *Then*

$$\text{(A4a)} \qquad \int_\Omega (1 + |Dv_\varepsilon|^2)^{1/2} \, dx \to \int_\Omega (1 + |Dv|^2)^{2/1} \, dx ,$$

$$\text{(A4b)} \qquad \int_\Omega |Dv_\varepsilon| \, dx \to \int_\Omega |Dv| \, dx .$$

[5] The mollification is possible in view of point 2).

PROOF. – We shall only prove (A4*b*). From the definition of the total variation we immediately get

$$\text{(A5)} \qquad \int_{\Omega} |Dv_\varepsilon|\,dx \leqslant \int_{\Omega+\varepsilon} |Dv|\,dx\,.$$

The assertion now follows from the lower semicontinuity of the total variation and from (A2).

LEMMA A2. – *Let* $v \in BV(\Omega)$ *and* v_ε *be a mollification. Then*

$$\text{(A6)} \qquad v_\varepsilon \to v \qquad \text{in } L^1(\partial\Omega)\,.$$

PROOF. – From the proof of Lemma 1.1 in [4] we may conclude that

$$\text{(A7)} \qquad \int_{\partial\Omega} |v_\varepsilon - v|\,d\mathcal{H}_{n-1} \leqslant (1+L^2)^{1/2}\cdot\int_{\Omega_\delta} |D(v_\varepsilon - v)|\,dx + \\ + \gamma(\delta, \partial\Omega)\int_{\Omega} |v_\varepsilon - v|\,dx\,,$$

where Ω_δ is a boundary strip of width δ and L is a constant depending on $\partial\Omega$. Hence we obtain from (A5) and from known properties of the mollification

$$\text{(A8)} \qquad \limsup \int_{\partial\Omega} |v_\varepsilon - v|\,d\mathcal{H}_{n-1} \leqslant (1+L^2)^{1/2}\cdot 2\cdot\int_{\bar{\Omega}_\delta} |Dv|\,dx\,.$$

In view of (A2) the right side of (A8) converges to zero.

LEMMA A3. – *The following generalization of formula* (A1) *is valid for* $v \in BV(\Omega)$

$$\text{(A9)} \qquad \int_{\Omega} vD^i g_i\,dx + \int_{\Omega} g_i D^i v\,dx = \int_{\partial\Omega} v g_i \nu_i\,d\mathcal{H}_{n-1} \qquad \forall g_i \in C^{0,1}(\bar{\Omega})\,.$$

PROOF. – Let v_ε be a mollification of v, then

$$\text{(A10)} \qquad \int_{\Omega} v_\varepsilon D^i g_i\,dx + \int_{\Omega} g_i D^i v_\varepsilon\,dx = \int_{\partial\Omega} v_\varepsilon g_i \nu_i\,d\mathcal{H}_{n-1} \qquad \forall g_i \in C^{0,1}(\bar{\Omega})\,.$$

From point 3) we know that for each i, $i = 1, \ldots, n$, $D^i v_\varepsilon$ is a bounded sequence of measures on $C^0(\bar{\Omega})$. Hence, a subsequence converges weakly to some Radon measure μ_i on $C^0(\bar{\Omega})$ (compare

[3; (13.4.2)]), so that in view of Lemma A2

$$\int_{\Omega} v D^i g_i \, dx + \int_{\bar{\Omega}} g_i \, d\mu_i = \int_{\partial\Omega} v g_i \nu_i \, d\mathcal{H}_{n-1} . \tag{A11}$$

Let us show that $\mu_i \equiv D^i v$. If we choose g_i as in (A1) then the combination of (A1) and (A11) gives

$$\int_{\Omega} g_i D^i v \, dx = \int_{\bar{\Omega}} g_i \, d\mu_i . \tag{A12}$$

Moreover, an easy calculation shows that the $D^i v$'s agree with the measures μ_i on $C_c^1(\Omega)$. Thus

$$\int_{\Omega} g_i D^i v \, dx = \int_{\Omega} g_i \, d\mu_i \tag{A13}$$

for all g_i's which appear in formula (A1) (see [3; (13.9.19)]). From (A12) and (A13) we finally obtain

$$\int_{\partial\Omega} g_i \, d\mu_i = 0 , \tag{A14}$$

hence

$$\int_{\partial\Omega} |d\mu| = 0 \tag{A15}$$

which implies $\mu_i = D^i v$ in view of (A13).

LEMMA A4. – *Let $u \in BV(\Omega)$ and k a real number. Then* $\min(u, k)$ *belongs to $BV(\Omega)$ and the following relations are valid*

$$\begin{aligned} \int_{\Omega} |D \max(u-k, 0)| \, dx - |A(k)| \leqslant \\ \leqslant \int_{\Omega} (1 + |Du|^2)^{1/2} \, dx - \int_{\Omega} (1 + |D \min(u, k)|^2)^{1/2} \, dx \end{aligned} \tag{A15a}$$

and

$$u - \min(u, k) = \max(u - k, 0) \qquad \textit{in } L^1(\partial\Omega) . \tag{A15b}$$

PROOF. – Let u_ε be a mollification of u. Then it follows from Lemma A1 and from the lower semicontinuity of the total variation

$$\text{(A15}c\text{)}\qquad \int_\Omega (1+|Du|^2)^{1/2}dx - \int_\Omega (1+|D\min(u,k)|^2)^{1/2}dx \geqslant$$

$$\geqslant \liminf\left\{\int_\Omega (1+|Du_\varepsilon|^2)^{1/2}dx - \int_\Omega (1+|D\min(u_\varepsilon,k)|^2)^{1/2}dx\right\} \geqslant$$

$$\geqslant \liminf\left\{\int_{u_\varepsilon>k} (1+|Du_\varepsilon|^2)^{1/2}dx - |\{u_\varepsilon>k\}|\right\} \geqslant$$

$$\geqslant \liminf\left\{\int_\Omega |D\max(u_\varepsilon-k,0)|\,dx - |\{u_\varepsilon>k\}|\right\} \geqslant$$

$$\geqslant \int_\Omega |D\max(u-k,0)|\,dx - |A(k)|$$

by which the first relation is proved.

To prove the second one, we use the triangle inequality

$$\text{(A15}d\text{)}\qquad \|\max(u-k,0)-(u-\min(u,k))\|_{L^1(\partial\Omega)} \leqslant$$

$$\leqslant \|\max(u_\varepsilon-k,0)-(u_\varepsilon-\min(u_\varepsilon,k))\|_{L^1(\partial\Omega)} +$$

$$\|u_\varepsilon-u\|_{L^1(\partial\Omega)} + \|\min(u_\varepsilon,k)-\min(u,k)\|_{L^1(\partial\Omega)} +$$

$$\|\max(u_\varepsilon-k,0)-\max(u-k,0)\|_{L^1(\partial\Omega)}\,.$$

The first term on the right side of this inequality is identically zero, while the other ones converge to zero. This is a consequence of (A6), the proof of Lemma A2, and of the estimates

$$\int_{\Omega_\delta} |D\min(u_\varepsilon,k)|\,dx \leqslant \int_{\Omega_\delta} |Du_\varepsilon|\,dx$$

and

$$\int_{\Omega_\delta} |D\max(u_\varepsilon-k,0)|\,dx \leqslant \int_{\Omega_\delta} |Du_\varepsilon|\,dx\,.$$

REMARK A1. – *By the same method of proof one can show that*

$$\text{(A15}e\text{)}\qquad u-\min(u,v)=\max(u-v,0) \qquad \textit{in } L^1(\partial\Omega)$$

for any functions $u, v \in BV(\Omega)$.

Appendix II.

Here we want to prove that the functional J in (5) is lower semicontinuous with respect to a minimizing sequence in $BV(\Omega)$.

Let v_ε be a minimizing sequence and suppose for simplicity that $v_\varepsilon \to v$ in $L^1(\Omega)$ (compare the considerations at the beginning of Section 2). Assume by contradiction that $J(v)$ is strictly greater than $\liminf J(v_\varepsilon)$. Then there exist a positive constant γ and a number ε_0 such that

$$J(v_\varepsilon) \leqslant J(v) - \gamma \qquad \forall \varepsilon \leqslant \varepsilon_0 . \tag{A16}$$

In view of (40) we have the relation

$$\int_{\partial\Omega} |v - v_\varepsilon|\, d\mathcal{H}_{n-1} \leqslant \int_{\Omega_\delta} |D(v - v_\varepsilon)|\, dx + c(\delta, \Omega) \cdot \int_\Omega |v - v_\varepsilon|\, dx , \tag{A17}$$

where Ω_δ is a boundary strip of width δ, and δ is sufficiently small. Hence

$$\begin{aligned} \int_{\Omega - \bar{\Omega}_\delta} (1 + |Dv_\varepsilon|^2)^{1/2}\, dx + \int_\Omega \int_0^{v_\varepsilon} H(x, t)\, dt\, dx \leqslant \int_\Omega (1 + |Dv|^2)^{1/2}\, dx \\ + \int_\Omega \int_0^{v} H(x, t)\, dt\, dx + \int_{\Omega_\delta} |Dv|\, dx + c(\delta, \Omega) \cdot \int_\Omega |v - v_\varepsilon|\, dx - \gamma . \end{aligned} \tag{A18}$$

If ε tends to zero, then we obtain in view of the lower semicontinuity of the integrals on the left side of (A18)

$$\int_{\Omega - \bar{\Omega}_\delta} (1 + |Dv|^2)^{1/2}\, dx \leqslant \int_\Omega (1 + |Dv|^2)^{1/2}\, dx + \int_{\Omega_\delta} |Dv|\, dx - \gamma . \tag{A19}$$

To complete the proof, we let δ converge to zero which gives the contradiction.

REMARK A2. – *By the same method one could show that the functional*

$$\int_\Omega (1 + |Dv|^2)^{1/2}\, dx + \int_\Omega \int_0^{v} H(x, t)\, dt\, dx + \int_{\partial\Omega} |v - f|\, d\mathcal{H}_{n-1}, \qquad f \in L^1(\partial\Omega) , \tag{A20}$$

is lower semicontinuous with respect to a minimizing sequence in $BV(\Omega)$ the elements of which are bounded in the norm

$$\int_\Omega |Dv|\,dx + \int_\Omega |v|\,dx\,, \tag{A21}$$

provided that $\partial\Omega$ satisfies an internal sphere condition.

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